Real solvability of the equation $\partial_{\bar z}^2\omega = \rho g$ and the topology of isolated umbilics

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Abstract

The geometric form of a conjecture associated with the names of Loewner and Carathéodory states that near an isolated umbilic in a smooth surface in $\mathbb{R}^3$, the principal line fields must have index ≤ 1. Real solutions of the differential equation $\partial_{\bar z}^2\omega = g$, where the complex function g is given only up to multiplication by a positive function, are intimately related to umbilics. We determine necessary and sufficient conditions of an integral nature for real solvability of this equation, which is really a system of two wave equations. We then construct germs of line fields of every index $j \in \dfrac{1}{2}\mathbb{Z}$ on S^2 that cannot be realised as the Gauss image of the principal line fields near an isolated umbilic of positive curvature on any smooth surface in $\mathbb{R}^3$. These include the standard dipole line field of index two and controlled distortions of it.

Introduction

For certain questions in differential geometry and hydrodynamics relating to umbilic points and stagnation points the following problem is of prime interest: Given a smooth *complex* function g with $g(o) = 0$, find a *real* function w and a *positive* function ρ , vanishing perhaps at o , such that $\partial_{\bar z}^2\omega = \rho g$; we write $\partial_{\bar z}^2 = \frac{1}{4}\{(\partial_x^2 - \partial_y^2) + 2i\partial_x\partial_y\}$ for the square of the Cauchy-Riemann operator.

We emphasize that the compatibility condition $Im\ \partial_z^2(\rho g) = 0$ for *real* solvability of $\partial_{\bar z}^2\omega = \rho g$ is not useful for the problem at hand because it involves the unknown function ρ . Neither is the *singular* equation $Im[\bar g \partial_{\bar z}^2\omega] = 0$. Theorem 2 gives a new necessary and sufficient condition on g for real solvability of the equation $\partial_{\bar z}^2\omega = g$, the criterion being the vanishing of a certain family of integrals in which g appears multiplicatively. It is precisely this feature that makes Theorem 2 useful in the problem $\partial_{\bar z}^2\omega = \rho g$, $g(o) = 0$, described above.

Before proceeding to explain the geometric results, we sketch the main ideas of our work on the real solvability of $\partial_{\bar z}^2\omega = g$. After a few simple observations we see that the existence of real solutions of the above equation on a neighbourhood of $o \in \mathbb{R}^2$ is equivalent to the existence of real solutions of $\partial_{\bar z}^2\omega = g - g(o)$ which are $0(r^3)$ on a neighbourhood of o. Thus the right-hand side of the equation may be assumed to vanish at o to begin with. The real and imaginary parts of the equation $\partial_{\bar z}^2\omega = g$ give two standard inhomogeneous wave equations to be satisfied by ω. The crux of the

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proof is that there is another pair of hyperbolic equations that is better adapted to this study and these are obtained from the real and imaginary parts of the equation $\frac{\bar{z}^4}{|z|^6}\partial_{\bar{z}}^2\omega = \frac{\bar{z}^4}{|z|^6}g$ after the substitution $u = \omega/r^2$.

What is altogether remarkable about the resulting hyperbolic equations — and decisive for the analysis of real solvability — is that, although they are singular, have variable coefficients and lower-order terms, most terms appearing in the classical method of characteristics are zero on algebraic grounds; the characteristics are orthogonal families of circles passing through o. Because of this, the contribution from the integration of the Cauchy data, along the non-characteristic side of the triangle used in this method, can be made arbitrarily small by letting this side approach o. In this way one obtains for each of these equations a representation of the solution in terms of a single integral involving g — and not four terms as might be expected from using the standard wave equations. The equality of these two integral representations is therefore necessary for the existence of solutions. The co-area formula is used to prove sufficiency.

The geometric source of our interest in the above equation is the principal foliation defined on a neighbourhood of an isolated umbilic on a smooth surface in $\mathbb{R}^3$; these foliations satisfy the conditions of the following definition, in suitable local coordinates (see §2).

Definition. A smooth one-dimensional foliation $\mathcal{F}$ of a neighbourhood Ω of $o \in \mathbb{R}^2$ with an isolated essential singularity at o is called a *singular Hessian foliation* if there exists a smooth real-valued function ω on Ω whose Hessian operator

$$\text{Hess } \omega = \begin{bmatrix} \omega_{xx} & \omega_{xy} \\ \omega_{xy} & \omega_{yy} \end{bmatrix}$$

has the following properties:

i) Hess ω is not a multiple of the identity for any $p \in \Omega - \{o\}$.

ii) The eigenspace corresponding to the large (or small) eigenvalue of Hess ω is tangent to $\mathcal{F}$ for each $p \in \Omega - \{o\}$.

Clearly if $\mathcal{F}$ is a singular Hessian foliation so also is the orthogonal foliation $\mathcal{F}^\perp$. Moreover it can be shown that any foliation is Hessian on a neighbourhood of a nonsingular point. As examples of singular Hessian foliations we give, in §2, the flow foliation of a steady irrotational flow near an isolated stagnation point in hydrodynamics and the principal foliations (in a canonical set of coordinates) near an isolated umbilic on a smooth surface in $\mathbb{R}^3$.

Beginning with a foliation $\mathcal{F}$ on a punctured neighbourhood $\Omega - \{o\}$ of o, we may represent it locally by a smooth unit vector field $\zeta = a + ib$ and note that ζ^2 is a well-defined unit vector field on $\Omega - \{o\}$. Let $J\zeta = -b + ia$. The condition that $\mathcal{F}$ be Hessian is equivalent to

$$< (\text{Hess } \omega)\zeta, J\zeta > \equiv 0 \text{ on } \Omega - \{o\}$$

for some smooth real-valued function ω on Ω, such that Hess ω is not a multiple of the identity on $\Omega-\{o\}$. With a little calculation these conditions may be rewritten $\text{Im}[\omega_{\bar{z}\bar{z}}\bar{\zeta}^2] \equiv 0$ on $\Omega-\{o\}$ and $\omega_{\bar{z}\bar{z}} \neq 0$ on $\Omega-\{o\}$; hence $\omega_{\bar{z}\bar{z}} = \rho\zeta^2$ for some smooth non-vanishing real function ρ on $\Omega-\{o\}$. After a change in sign of ω, if necessary, we have $\rho = |\omega_{\bar{z}\bar{z}}| > 0$ and, assuming the singularity at o to be essential, the function ρ extends continuously to be zero at o.

Thus the foliation represented by ζ is a singular Hessian foliation if and only if, for some neighbourhood Ω of o, there exists a function $\omega \in C^\infty(\Omega, \mathbb{R})$ and a non-negative function $\rho \in C^\infty(\Omega - \{o\}, \mathbb{R}) \cap C^0(\Omega, \mathbb{R})$ vanishing only at o such that

$$\omega_{\bar{z}\bar{z}} = \rho\zeta^2 \text{ on } \Omega - \{o\}$$

In fact $\rho = |\omega_{\bar{z}\bar{z}}|$. The number $j \in \frac{1}{2}\mathbb{Z}$ which is half of the index of the vector field ζ^2 at o is called the *index of the foliation $\mathcal{F}$ at o.* Note that the Hessian foliations of a smooth real function ω are given by the line fields $\sqrt{\omega_{\bar{z}\bar{z}}}$ and $i\sqrt{\omega_{\bar{z}\bar{z}}}$.

Checking that a foliation is Hessian involves the determination of real solvability of one of infinitely many singular equations of the above form — one for each function ρ which is positive on a punctured neighbourhood of o — so that it is not even clear that there are any singular foliations which are not Hessian. We use Theorem 2 to explicitly construct singular foliations of every index $j \in \frac{1}{2}\mathbb{Z}$ which are not Hessian; this is Lemma 4 of § 3. A glance at the proof of Lemma 4 makes it clear that these foliations may be chosen so that all nearby (in a suitable sense) singular foliations are not Hessian.

If, on the other hand, we begin with a Hessian foliation it can be seen — but only from the geometric content of Lemma 2 — that there is an abundance of nearby Hessian foliations. This comes about as follows; a Hessian foliation determines, via Lemma 2, a smooth surface of positive curvature with an isolated umbilic in $\mathbb{R}^3$; any deformation of this surface in $\mathbb{R}^3$ preserving the isolated umbilic produces, via Lemma 2 again, a deformation of the original Hessian foliation through Hessian foliations. This is reminiscent of how Bäcklund transformations are used to generate solutions of the Sine-Gordon equation [18].

Near an isolated umbilic p on a smooth surface M in $\mathbb{R}^3$ there are smooth orthogonal foliations (the principal foliations, see §2) with a singularity at p. The Gauss curvature at p is necessarily non-negative and, if positive, the Gauss map Γ carries the principal foliations to orthogonal foliations on S^2 with a singularity at $s = \Gamma(p)$ (the south pole, say). In the canonical coordinates on M, given by stereographic projection, the principal foliations appear as Hessian foliations. This is classical and due to Bonnet ([5] , or [6], p. 295) but we give our own proof of this as part of Lemma 2. What is new about this lemma is the observation that every Hessian foliation arises in this way from an umbilic in classical surface theory.

Umbilics of every index $j \leq 1$ are known to occur in smooth surface theory [17] and it has been conjectured that there are none of higher index (local Carathéodory Conjecture). We now know from Lemma 2, that this is entirely equivalent to the conjecture that all smooth Hessian foliations have index ≤ 1 (Loewner Conjecture). In our paper [17] we have shown that an isolated umbilic with $|j| \geq 1$ is rather special in that not only the gradients of the mean curvature H and the Gauss curvature K must vanish there, but even the 3-jet of $H^2 - K$; it follows that any immersion of S^2 in $\mathbb{R}^3$, for which the critical sets of H and K are disjoint, must have at least four umbilic points.

Motivated by this problem, and using our criterion for real solvability of $\partial_{\bar{z}}^2 \omega = g$ we have obtained the following geometric result which is proved in §3.

Theorem 1. *For every $j \in \frac{1}{2}\mathbb{Z}$ there are germs of singular foliations of index j on S^2 which cannot be realized as the Gauss image of the principal foliations near an isolated umbilic of positive curvature on any smooth surface in $\mathbb{R}^3$.*

If either of the conjectures of the previous paragraph hold, then by the Euler-Poincaré theorem every smooth immersion of S^2 would have at least two umbilics (Carathéodory Conjecture); a good reference is Hamburger ([9], cf. p. 63), where this conjecture is explicitly stated for smooth immersions. The first results on the smooth case are in our recent paper [17]. The earlier papers [10], [4], [11] and [19] provide a formidable analysis of the Carathéodory Conjecture in the real analytic case but there has been some reticence in recent times about these results and the reader is referred to the problem section compiled in Yau ([20], cf. p. 684) and also to Lang ([13], cf. p. 19).

The interested reader will find from the extensive bibliography in Lang [13], that contemplation of the umbilic is to be found in a variety of fields, from optics to dynamical systems, since the late nineteenth century.

§1 Hyperbolic differential equations and real solvability of $\partial_{\bar{z}}^2 \omega = g$.

The geometric motivation for the study of this equation is given in the introduction and §2. We refer to [16] for the basics on hyperbolic equations.

A linear partial differential operator of second order

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y$$

is hyperbolic if $b^2 - ac > 0$; the coefficient functions are understood to be smooth. Letting $\rho = \sqrt{b^2 - ac}$, the characteristic directions of the operator L are defined by

$$\begin{aligned} C^+ :\quad & (-b + \rho)dy \ + \ c\ dx \ = 0, \\ C^- :\quad & (-b - \rho)dy \ + \ c\ dx \ = 0. \end{aligned}$$

The adjoint operator L^* of L is defined by

$$L^*[v] = (av)_{xx} + 2(bv)_{xy} + (cv)_{yy} - (dv)_x - (ev)_y.$$

For any smooth function u it is straightforward to check that

$$(L[u] - uL^*[1])dxdy = d(\sigma + u\tau),$$

where

$$\sigma = u_x(-bdx + ady) + u_y(-cdx + bdy)$$

and

$$\tau = (b_x + c_y - e)dx - (a_x + b_y - d)dy.$$

Let ABz_0 be a positively-oriented curvilinear triangle (which together with its interior lies in the region of hyperbolicity of L) such that Bz_0 and z_0A are respectively parts of a C^--characteristic and a C^+-characteristic, as shown here. Fig

By Stoke's Theorem

$$\iint_{ABz_0}(L[u] - uL^*[1])dxdy = \oint_{ABz_0}(\sigma + u\tau)$$
$$= \int_{Bz_0}(\sigma + u\tau) \quad + \int_{z_0A}(\sigma + u\tau) \quad + \int_{AB}(\sigma + u\tau).$$

It is easily calculated that $\sigma = \pm\rho du$ along $C^\pm$; hence $\sigma + u\tau = \pm\rho du + u\tau = \pm[d(\rho u) - (d\rho)u] + u\tau = \pm d(\rho u) + u(\tau \mp d\rho)$ along $C^\pm$. Hence

$$\iint_{ABz_0}(L[u] - uL^*[1])dxdy = -2\rho u(z_0) + \rho u(A) + \rho u(B)$$
$$+ \int_{Bz_0} u(\tau + d\rho) + \int_{z_0A} u(\tau - d\rho) + \int_{AB}(\sigma + u\tau)$$

or

$$(1) \qquad 2\rho u(z_0) = \rho u(A) + \rho u(B) - \iint_{ABz_0}(L[u] - uL^*[1])dxdy$$
$$+ \int_{Bz_0} u(\tau + d\rho) + \int_{z_0A} u(\tau - d\rho) + \int_{AB}(\sigma + u\tau).$$

This formula will be used in the geometric situation which we now discuss.

Denote by $\mathcal{F}_0^+$ (resp. $\mathcal{F}_0^-$) the foliation of $\mathbb{R}^2 - \{o\}$, whose leaves are the circles of all radii — including ∞ — passing through o tangent to the x-axis (resp. y-axis). These foliations are orthogonal. For $z_0 \neq o$, denote the leaves of these foliations through z_0 by $C^+(z_0)$ and $C^-(z_0)$. Among the closures of the bounded components of the complement of $C^+(z_0) \cup C^-(z_0)$ denote by $R(z_0)$ *any* one of these components for which the positively oriented boundary of $R(z_0)$ is $\{R(z_0) \cap C^-(z_0)\} \cup \{R(z_0) \cap C^+(z_0)\}$ in that order. There are two such regions $R(z_0)$, neither convex, for each z_0 in the first and third quadrants and a unique one, convex, for each z_0 in the second and fourth quadrants. As z_0 moves from the first to the second quadrant, one of the two regions $R(z_0)$ becomes infinite and the other deforms continuously into the unique $R(z_0)$ of the second quadrant.

Figure 2.

Similarly for the foliations $\mathcal{F}_1^{\pm} = e^{-i\frac{\pi}{4}} \mathcal{F}_0^{\pm}$ of circles tangent to the lines $y = \pm x$ at the origin, we denote the corresponding regions by $S(z_0)$. Clearly $S(z_0) = e^{-i\frac{\pi}{4}} R(e^{i\frac{\pi}{4}} z_0)$. Similar remarks apply to the regions $S(z_0)$, depending on the location of z_0 relative to these lines.

For each z_0 in a ball of radius c about o, at least one of the regions $R(z_0)$ and at least one of the regions $S(z_0)$ is contained in a ball of radius $c\sqrt{2}$.

Lemma 1. *Let ω be a smooth real function which is $O(r^3)$ on a neighbourhood of o. Then for $z_0 \neq o$ we have*

$$\omega(z_0) = 2|z_0|^2 \iint_{R(z_0)} \operatorname{Im}\left[\frac{\bar{z}^4 \omega_{\bar{z}\bar{z}}}{|z|^6}\right] dxdy$$
$$= -2|z_0|^2 \iint_{S(z_0)} \operatorname{Re}\left[\frac{\bar{z}^4 \omega_{\bar{z}\bar{z}}}{|z|^6}\right] dxdy$$

when $R(z_0)$ and $S(z_0)$, as defined above, lie in the domain of ω.

Proof. Set $\omega_{\bar{z}\bar{z}} = g$. Writing $\omega = z\bar{z}u$ and differentiating this equation with respect to $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ we obtain

$$\omega_{\bar{z}\bar{z}} = z\bar{z}u_{\bar{z}\bar{z}} + 2zu_{\bar{z}}.$$

Then the earlier identity multiplied by $\bar{z}^4/|z|^6$ becomes

$$\frac{\bar{z}^4}{|z|^4}u_{\bar{z}\bar{z}} + \frac{2\bar{z}^3}{|z|^4}u_{\bar{z}} = \frac{\bar{z}^4 g}{|z|^6}$$

and, writing $\eta = \eta_1 + i\eta_2$ for the expression on the right, the imaginary part of this equation is

$$(2) \qquad au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = -4\eta_2,$$

where $a = -c = \sin 4\theta, b = -\cos 4\theta, d = \dfrac{4\sin 3\theta}{r}$ and $e = -\dfrac{4\cos 3\theta}{r}$. By a remarkable circumstance, the operator L defined by the left-hand side of this equation enjoys the following properties:

$$\rho \equiv 1, \tau \equiv 0 \text{ and } L^*[1] \equiv 0 \text{ since } d\tau \equiv 0.$$

Furthermore the $\pm$ characteristics through $z_0 \neq o$ of the operator L, are respectively, the leaves $C^{\pm}(z_0)$ of the foliations $\mathcal{F}_0^{\pm}$. Now choose $A \in C^+(z_0)$ and $B \in C^-(z_0)$ on the boundary of $R(z_0)$ and apply Eq. (1) with A and B approaching o. The relevant term is the line integral over AB. Since $\omega = O(r^3)$ the gradient of u is bounded and this line integral tends to zero as A and B approach o. In the limit we have

$$u(z_0) = 2 \iint_{R(z_0)} \eta_2 dxdy.$$

This gives the formula

$$\omega(z_0) = 2\ |z_0|^2 \iint_{R(z_0)} \text{Im}\ \left[\frac{\bar{z}^4\omega_{\bar{z}\bar{z}}}{|z|^6}\right] dxdy.$$

If instead we had worked with the real part of the complex equation, the corresponding differential equation for u would be

$$(3) \qquad au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = 4\eta_1,$$

where $a = -c = \cos 4\theta, b = \sin 4\theta, d = \dfrac{4\cos 3\theta}{r}$ and $e = \dfrac{4\sin 3\theta}{r}$. For this new operator $\tilde{L}$ the associated quantities are $\tilde{\rho} \equiv 1, \tilde{\tau} \equiv 0$ and $\tilde{L}^*[1] \equiv 0$ since $d\tilde{\tau} \equiv 0$; the associated $\pm$ characteristics of $\tilde{L}$ are the leaves of the foliations $\mathcal{F}_1^{\pm}$ respectively. Arguing from Eq. (1) as above we obtain

$$u(z_0) = -2 \iint_{S(z_0)} \eta_1 dxdy$$

or

$$\omega(z_0) = -2|z_0|^2 \iint_{S(z_0)} \text{Re}\,\left[\frac{\bar{z}^4 \omega_{\bar{z}\bar{z}}}{|z|^6}\right] dxdy.$$

This completes the proof of Lemma 1.

Suppose we are given a smooth complex-valued function g on an open neighbourhood Ω of o satisfying the additional assumption $g(o) = 0$. Let N be the set of $z_0 \in \Omega$ such that at least one of the regions $R(z_0)$ and at least one of the regions $S(z_0)$ is contained in Ω. N is a punctured neighbourhood of o.

We stress that if the equation $\omega_{\bar{z}\bar{z}} = g$ has a real solution ω on a neighbourhood of o then, since $g(o) = 0$, the function $\omega - \{\text{2-jet of } \omega \text{ at } o\}$ is necessarily another solution of this equation with the additional property that it is $O(r^3)$ near o. Thus by Lemma 1 a necessary condition for real solvability of $\omega_{\bar{z}\bar{z}} = g$ on Ω is the condition:

$$(*) \qquad \iint_{R(z_0)} \text{Im}\,\left[\frac{\bar{z}^4 g}{|z|^6}\right] dxdy = \iint_{S(z_0)} \text{Re}\,\left[-\frac{\bar{z}^4 g}{|z|^6}\right] dxdy$$

for all $z_0 \in N$. Suppose conversely (*) holds for all $z_0 \in N$. Then for each $z_0 \in N$ the common value $\frac{1}{2}u(z_0)$ of the two integrals defines a smooth real-valued function u on N which extends continuously to zero. Expressing u by the left-hand integral over $R(z_0)$, we will use the co-area formula to show that u satisfies the differential equation (2) on N. Similarly, by using the right-hand integral over $S(z_0)$, u is seen to satisfy the differential equation (3) on N. From the proof of Lemma 1 we then see that the smooth real-valued function $\omega = r^2 u$ on N satisfies the equation $\omega_{\bar{z}\bar{z}} = g$ on N.

However, the function g being smooth, the equation $\omega_{\bar{z}\bar{z}} = g$ always has a smooth *complex* solution on a neighbourhood of o. It follows easily that the function ω constructed in the previous paragraph, extends to be smooth on a neighbourhood of o. In what follows χ_A will denote the characteristic function of the set A.

Theorem 2. *Let g be a smooth complex-valued function defined on a neighbourhood of o. Then the equation $\omega_{\bar{z}\bar{z}} = g$ has a smooth real-valued solution on a neighbourhood of o if and only if*

$$\iint_{\mathbb{R}^2} \text{Im}\left[\left(\chi_{R(z_0)} + i\chi_{S(z_0)}\right)\frac{\bar{z}^4}{|z|^6}\left(g(z) - g(o)\right)\right] dxdy = 0$$

for all z_0 in a punctured disk about o.

Proof. Since $\omega_{\bar{z}\bar{z}} = g(o)$ has a real solution $\omega = \dfrac{1}{2}\,\text{Re}\,[\overline{g(o)}z^2]$, we may effectively assume $g(o) = 0$. The necessity follows from Lemma 1. Then by the remarks preceding the statement, in order to prove sufficiency we must prove that the function

$$u(z_0) = 2\iint_{R(z_0)} \text{Im}\,\left[\frac{\bar{z}^4}{|z|^6} g(z)\right] dxdy$$

satisfies the differential equation (2).

The co-area formula (cf. [8], p. 249 or [7], p. 118) in dimension 2 states that

$$\iint_{\{f>T\}} h \; dxdy \;=\; \int_T^{\infty} \left(\int_{\{f=k\}} \frac{h}{\| \nabla f \|} d\mathcal{H} \right) dk$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is integrable and $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function with ess.inf.$\| \nabla f \| > 0$ and $d\mathcal{H}$ is Hausdorff 1-dimensional measure; as a consequence

$$\frac{d}{dT} \iint_{\{f>T\}} h \; dxdy \;=\; -\int_{\{f=T\}} \frac{h}{\| \nabla f \|} d\mathcal{H}$$

for almost all T. We will be interested in such a formula for the case where h is bounded with compact support and

$$f(x,y) = f^+(x,y) = \frac{-y}{x^2+y^2}.$$

For each $\delta > 0$ the function

$$f_\delta^+(x,y) = -\min\left\{ \frac{1}{\delta^2}, \frac{1}{x^2+y^2} \right\} y$$

satisfies the conditions of the co-area formula on the support of h. Applying the formula to f_δ^+ and letting $\delta \to 0$ we obtain the co-area formula for f^+. Similar considerations apply to

$$f^-(x,y) = \frac{-x}{x^2+y^2}.$$

The vector fields

$$X = \frac{\nabla f^-}{\| \nabla f^- \|^2} = r^2(\cos 2\theta, \sin 2\theta)$$

and

$$Y = \frac{\nabla f^+}{\| \nabla f^+ \|^2} = r^2(\sin 2\theta, -\cos 2\theta)$$

are tangent to the leaves C^+ and C^-, respectively.

Furthermore, it is easily checked that the operator YX coincides with the operator $\frac{1}{2} r^4 L$, where L is as defined in the proof of Lemma 1. Applying both operators to the function $\frac{1}{2}u$ introduced above and using the co-area formula, we have

$$\frac{1}{4}|z_0|^4(Lu)|_{\zeta=z_0} = Y_{z_0}X \iint\limits_{R(\zeta)} \text{Im } \left[\frac{\bar{z}^4}{|z|^6}g\right] dxdy$$
$$= -Y_{z_0} \int_{C^-(\zeta)} \frac{1}{\| \nabla f^- \|} \text{ Im } \left[\frac{\bar{z}^4}{|z|^6}g\right] ds$$
$$= - \| Y_{z_0} \| \frac{Y_{z_0}}{\| Y_{z_0} \|} \int_{C^-(\zeta)} |z|^2 \text{ Im } \left[\frac{\bar{z}^4}{|z|^6}g\right] ds$$
$$= - \| Y_{z_0} \| \left(|z|^2 \text{ Im } \left[\frac{\bar{z}^4}{|z|^6}g\right]\right)_{z_0}$$
$$= -|z_0|^4 \text{ Im } \left[\frac{\bar{z}^4}{|z|^6}g\right]_{z_0}$$

Hence $Lu = -4 \text{ Im } \left[\frac{\bar{z}^4}{|z|^6}g\right]$, which is $Eq.(2)$.

Similarly, from the other integral representation of u, we find

$$\tilde{L}u = 4 \text{ Re} \left[\frac{\bar{z}^4}{|z|^6}g\right],$$

which is $Eq.$ (3). The construction of ω, and the verification that it is smooth, now follows the lines given in the preamble to Theorem 2. This ends the proof of Theorem 2.

Noting the construction in the previous proof, and the fact that any two real-valued solutions of $\omega_{\bar{z}\bar{z}} = g(0)$ differ by a function of the form $\alpha + \beta x + \gamma y + \delta(x^2 + y^2)$ where α, β, γ and δ are real, we have the following result:

Corollary 1. *Let g be a smooth complex-valued function on a neighbourhood of o which satisfies the condition of Theorem 2. Then any real-valued solution of the equation $\omega_{\bar{z}\bar{z}} = g$ is of the following form:*

$$\omega(x, y) = \alpha + \beta x + \gamma y + \delta(x^2 + y^2) + \frac{1}{2} \text{ Re } [\overline{g(o)}z^2]$$
$$+ 2(x^2 + y^2) \iint\limits_{R(z)} \text{ Im } \left[\frac{\bar{w}^4}{|w|^6}(g(w) - g(o))\right] dudv.$$

§2 Hessian foliations in hydrodynamics and geometry.

Let $V = ue_1 + ve_2$ be the velocity vector field of a steady irrotational flow of a compressible fluid in a simply-connected region Ω in $\mathbb{R}^2$. The pressure p, density ρ and the external force $G = G_1 e_1 + G_2 e_2$ per unit mass of fluid are all independent of time. Irrotationality of the flow means V is the gradient $\nabla\phi$ of some function ϕ and so

$$u_y - v_x = 0$$

and invariance of mass under the flow is expressed by $div(\rho V) = 0$ or

$$(\rho u)_x + (\rho v)_y = 0.$$

The equations of motion are

$$\frac{1}{2}\nabla(\| V \|^2) = G - \frac{1}{\rho}\nabla p.$$

In the absence of external forces the first component of the equations of motion is

$$(\rho u)u_x + p_x + (\rho v)v_x = 0$$

or

$$(\rho u^2 + p)_x - (\rho u)_x u + (\rho v)v_x = 0$$

which, by the earlier two equations, can be written

$$(\rho u^2 + p)_x + (\rho uv)_y = 0.$$

Thus

$$dg = -\rho uv dx + (\rho u^2 + p)dy$$

for some smooth function g defined in a neighbourhood of o. Similarly the second equation of motion means that

$$dh = (\rho v^2 + p)dx - \rho uv dy$$

for some smooth function h defined near o. Now the differential $hdx + gdy$ is also closed and so equal to $d\omega$ for some smooth function ω on Ω. The function ω is uniquely determined to within a linear function and a little computation shows that

$$\omega_{\bar{z}\bar{z}} = -\frac{1}{4}\rho(u + iv)^2$$

or, if preferred,

$$\omega_{\bar{z}\bar{z}} = -\frac{1}{2}k\zeta^2,$$

where $\zeta = (u + iv)/\sqrt{u^2 + v^2}$ is the unit vector field determined by the flow foliation and $k = \rho(u^2 + v^2)/2$ is the kinetic energy density of the flow. Thus the flow lines in this example define a Hessian foliation. The *stagnation points* of the flow are those points where $\omega_{\bar{z}\bar{z}} = 0$.

Let $\mathcal{F}$ be an orientable foliation defined by a smooth unit vector field ζ on a neighbourhood Ω of o with an isolated singularity of o. Assuming that $\mathcal{F}$ is a singular Hessian foliation, there exists a smooth real function ω defined on Ω which solves

$$\omega_{\bar{z}\bar{z}} = -\frac{1}{2}k\zeta^2$$

with k a smooth non-vanishing real function on $\Omega - \{o\}$. Clearly o must be an isolated zero of $\omega_{\bar{z}\bar{z}}$ and after a possible sign change in ω we may assume $2k = |\omega_{\bar{z}\bar{z}}| > 0$. It would be of interest to know under what conditions a solution ω of the above equation arises from hydrodynamics in the manner of the previous paragraph.

This discussion can be found in Loewner [15] who, according to Titus [19], made the following conjecture (see p. 75 of [19] for remarks on the smooth case).

Loewner's Conjecture. *Let ω be a smooth real-valued function defined on a neighbourhood of o in $\mathbb{R}^2$ such that $\omega_{\bar{z}\bar{z}}$ has an isolated zero at o. Then the index of $\omega_{\bar{z}\bar{z}}$ at o is at most two.*

In fact Loewner conjectured that an isolated zero of $\partial_{\bar{z}}^n \omega$ has index $\leq n$, if ω is real-valued [19] (see also Bol[4] p.418).

Differential geometry also provides examples of singular Hessian foliations. A piece of smooth surface in $\mathbb{R}^3$ may be thought of as an orientable 2-manifold M together with a smooth map $f : M \to \mathbb{R}^3$ which is an immersion, i.e., the differential f_{*_p} is injective for each $p \in M$. The induced Riemannian metric g on M is defined by $g_p(X, Y) = < f_{*_p}X, f_{*_p}Y >$ for each pair $X, Y \in T_p(M)$; here $<,>$ denotes the inner product on $\mathbb{R}^3$. Let ξ be a unit normal field to the immersion f, i.e., ξ is a unit vector-valued function on M such that $\xi(p)$ is orthogonal to $f_{*_p}(T_p(M))$ for each $p \in M$. Since ξ is a unit vector field, its derivative $X\xi$ with respect to any $X \in T_p(M)$ is orthogonal to $\xi(p)$ and so can be written $f_{*_p}(A_p(X))$ for some endomorphism A_p of $T_p(M)$. This operator A on M is called the second fundamental form of the immersion f with respect to the unit normal field ξ.

It can be easily seen that A is symmetric with respect to g. Away from the points where $A_p = \lambda I_p$, the eigenspaces of A corresponding to the large and small eigenvalues of A determine smooth foliations, which we call the *principal foliations*. The points $p \in M$ where $A_p = \lambda I_p$ for some real number λ are called *umbilic points* of the immersion f. If $\lambda \neq 0$ then $\xi : M^2 \to S^2$ is a diffeomorphism of a neighbourhood D of p in M onto a neighbourhood Δ of $\xi(p)$ in S^2; after a change of frame in $\mathbb{R}^3$ we may assume $\xi(p) = (0, 0, -1)$. Then, under the stereographic projection from $(0, 0, 1)$, the coordinate $z = x + iy$ on a neighbourhood D_0 of the point $o \in \mathbb{C}$ gives local coordinates on the neighbourhood D of $p \in M$. These we call *Bonnet coordinates* on M. In fact we now go on to see that there is no loss of generality in taking $\lambda \neq 0$.

Even if $\lambda = 0$ above we may replace f by $F = \sigma \circ f$, where σ is inversion in the unit sphere about $o \in \mathbb{R}^3$ which may be taken not to lie in $f(M)$. Then it can be easily verified that F is a smooth immersion and that

$$N = \xi - 2\frac{< f, \xi >}{< f, f >} f$$

is a unit normal field to the immersion F. It is then a simple matter to verify that the second fundamental form of F is

$$B = < f, f > A - 2 < f, \xi > I \ .$$

In particular p is also an umbilic of the immersion F and the principal foliations of F on M even coincide with those of f. If the origin of $\mathbb{R}^3$ is chosen not to lie in the tangent plane $f_*(T_p(M))$, then $B_p = \lambda I_p$, where $\lambda \neq 0$.

Returning to the Bonnet coordinates introduced in the first paragraph we have

(1) $$\xi(z) = (z + \bar{z}, \frac{z - \bar{z}}{i}, z\bar{z} - 1)/(1 + z\bar{z}) \ .$$

A vector X tangent to M is principal if and only if $< f_*(X) \times \xi_*(X), \xi > = 0$, where $\times$ denotes the vector cross product. Suitably interpreted, the equation for principal lines is therefore $< df \times d\xi, \xi > = 0$ or in Bonnet coordinates

$$< (f_z dz + f_{\bar{z}} d\bar{z}) \times (\xi_z dz + \xi_{\bar{z}} d\bar{z}), \xi > = 0$$

or

$$< f_z \times \xi_z, \xi > dz^2 + < (f_z \times \xi_{\bar{z}} + f_{\bar{z}} \times \xi_z), \xi > dz d\bar{z} + < f_{\bar{z}} \times \xi_{\bar{z}}, \xi > d\bar{z}^2 = 0.$$

This simplifies to

$$< f_z, \xi_z \times \xi > dz^2 + \{< f_z, \xi_{\bar{z}} \times \xi > + < f_{\bar{z}}, \xi_z \times \xi >\} dz d\bar{z} + < f_{\bar{z}}, \xi_{\bar{z}} \times \xi > d\bar{z}^2 = 0 \ .$$

Direct computation gives $\xi_z \times \xi = i\xi_z$ and, from this and its conjugate, the previous equation reduces to

$$< f_z, \xi_z > dz^2 + \{< f_{\bar{z}}, \xi_z > - < f_z, \xi_{\bar{z}} >\} dz d\bar{z} - < f_{\bar{z}}, \xi_{\bar{z}} > d\bar{z}^2 = 0 \ .$$

Since ξ is normal to the immersion we know $< f_z, \xi > = 0$, and so

$$< f_{z\bar{z}}, \xi > + < f_z, \xi_{\bar{z}} > = 0.$$

¿From this and its conjugate, the previous equation reduces to

$$< f_z, \xi_z > dz^2 - < f_{\bar{z}}, \xi_{\bar{z}} > d\bar{z}^2 = 0.$$

Now consider the *Bonnet function*

$$\omega(z) = (1 + z\bar{z}) < f, \xi > .$$

On differentiation we have

$$\begin{aligned}\omega_{zz} &= (1 + z\bar{z}) < f_z, \xi_z > + < f, 2\bar{z}\xi_z + (1 + z\bar{z})\xi_{zz} > \\ &= (1 + z\bar{z}) < f_z, \xi_z >,\end{aligned}$$

since direct differentiation of Eq. (1) shows

$$\xi_{zz} = -2\bar{z}\xi_z/(1 + z\bar{z}).$$

Now the equation for principal lines takes the simple form

$$\omega_{zz}dz^2 - w_{\bar{z}\bar{z}}d\bar{z}^2 = 0$$

or

$$\mathrm{Im}[\omega_{\bar{z}\bar{z}}d\bar{z}^2] = 0.$$

The umbilics correspond to points where $\omega_{\bar{z}\bar{z}} = 0$. Since $p \in M$ is an isolated umbilic it follows that o is an isolated zero of $\omega_{\bar{z}\bar{z}}$ and that $\omega_{\bar{z}\bar{z}} = \rho\zeta^2$ on a punctured neighbourhood of o for some smooth non-vanishing function ρ, ζ being a local unit vector field representing either of the two principal foliations.

Hence in a Bonnet coordinate neighbourhood of an isolated umbilic the principal foliations are singular Hessian foliations (of the Bonnet function).

Lemma 2.

(i) *Let $f : M^2 \to \mathbb{R}^3$ be a smooth surface. Then on some neighbourhood U of each $p \in M$ there exist local coordinates $z = x + iy$ with $z(p) = o$, and a smooth real function $\omega(z)$ such that the Hessian foliations of ω coincide with the principal foliations of f on U.*

(ii) *Let $\omega : U \to \mathbb{R}$ be a smooth real function on a neighbourhood of o in $\mathbb{R}^2$. Then there exists an immersion $f : U \to \mathbb{R}^3$ of some neighbourhood U of $p = o$ in $\mathbb{R}^2$ such that the principal foliations of f coincide with the Hessian foliations of ω on U.*

(iii) *In (i) and (ii) the point p is an isolated umbilic of f if and only if o is an isolated zero of the vector field $\omega_{\bar{z}\bar{z}}$, and the index of p as an umbilic of f is equal to half of the index of the vector field $\omega_{\bar{z}\bar{z}}$ at o.*

Proof.

(i) As we saw above, beginning with an immersion f we can always use inversion to obtain an immersion F whose Gauss map is non-degenerate at p. Choosing Bonnet coordinates z on a neighbourhood U of p for the immersion F, the principal foliations of F are the Hessian foliations of its Bonnet function ω. As we saw above, the principal foliations of f and F agree so that (i) follows.

(ii) Conversely let ω be a smooth real function defined on a neighbourhood of $o \in \mathbb{R}^2$. Then $\psi = \omega/(1+z\bar{z})$ may be considered, via the stereographic projection (1), as a function of the unit position vector ξ on S^2 in a neighbourhood of $(0,0,-1) \in S^2$. Let $f_c = \nabla\psi + (\psi + c)\xi$, where ∇ denotes the standard connexion on S^2. The differential of f_c is Hess $\psi + (\psi + c)I$, so that f_c is an immersion in a neighbourhood of o, for almost all c, and has support function $< f_c, \xi >= \psi + c$. By definition, the associated Bonnet function of this immersion is

$$\omega^c = (1+z\bar{z})(\psi + c) = \omega + c(1+z\bar{z}).$$

In particular $\omega^c_{\bar{z}\bar{z}} = \omega_{\bar{z}\bar{z}}$. Hence the principal foliations of f coincide with the Hessian foliations of ω.

(iii) This is clear from the remarks in the introduction, since $\omega_{\bar{z}\bar{z}} = \rho\zeta^2$, where ρ is non-vanishing and ζ represents one of the principal foliations on a punctured neighbourhood of o. This ends the proof of Lemma 2.

We record here, from the early work of Hamburger ([9], p. 63) on the subject, the original form of

Carathéodory's Conjecture. *Any smooth immersion of S^2 in $\mathbb{R}^3$ has at least two umbilics.*

Such attempts as have been made on this problem (all in the analytic case) have been through the following stronger conjecture (in its analytic form).

Local Carathéodory Conjecture. *An isolated umbilic on a smooth surface in $\mathbb{R}^3$ has index ≤ 1.*

¿From (i) and (ii) of Lemma 2 we have:

Corollary. *The Loewner conjecture and the local Carathéodory conjecture are equivalent.*

§3 Applications to foliations and geometry.

The first application gives a criterion for a singular foliation to be Hessian.

Representing the foliation $\mathcal{F}$ locally on a punctured neighbourhood of its singularity o by a smooth unit vector field ζ, we have seen that $\eta = \zeta^2$ is a well-defined smooth unit vector field with a singularity at o; we call η the *square* of the foliation $\mathcal{F}$. The result below follows immediately from the discussion in the introduction and Theorem 2 , with $g = \rho\eta$. Observe the relevance of vanishing at o .

Lemma 3. *Let $\mathcal{F}$ be a singular foliation on a punctured neighbourhood of o and let η denote its square. Then $\mathcal{F}$ is a singular Hessian foliation if and only if there exists* *a smooth positive function ρ on a punctured neighbourhood of o such that $\rho\eta$ extends smoothly to o and*

$$\iint_{\mathbb{R}^2} \rho \ \mathrm{Im}\left[(\chi_{R(z_0)} + i\ \chi_{S(z_0)})\frac{\bar{z}^4}{|z|^6}\ \eta\right] dxdy = 0$$

for all z_0 in a punctured disk about o.

Example. *For the singular dipole foliation of index two, given by the field z^2 on $\mathbb{C}$, we have $\eta = \dfrac{z^4}{|z|^4}$. Therefore it is not Hessian, by Lemma 3. This is the simplest of all index 2 foliations of $\mathbb{C}$ and is obtained by transforming by $\dfrac{1}{z}$ a foliation of $\mathbb{C}$ by parallel lines.*

The criterion of Lemma 3 is the basis for the next result.

Lemma 4. *For each $j \in \frac{1}{2}\mathbb{Z}$ there exists a smooth singular foliation of index j on a punctured disk about $o \in \mathbb{R}^2$ which is not Hessian on any neighbourhood of o.*

Proof. If a smooth foliation $\mathcal{F}$ with an isolated singularity at o is Hessian then, by Theorem 3, there exists a smooth positive function ρ on a punctured neighbourhood of o such that

$$\iint_{\mathbb{R}^2} \rho \ \ \mathrm{Im}\ \left[(\chi_{R(z_0)} + i\ \chi_{S(z_0)})\frac{\bar{z}^4}{|z|^6}\ \eta\right] dxdy = 0$$

for all z_0 in a punctured disk about o; here η stands for the square of $\mathcal{F}$.

Now suppose that there exists a smooth vector field $v = v_1 + iv_2$ on a neighbourhood Ω of a o such that

(i) o is an isolated zero of v of index $n \in \mathbb{Z}$

and

(ii) v_1 and v_2 are non-negative on the sector Σ defined by $\dfrac{\pi}{4} \le \theta \le \pi$.

Then $\sqrt{z^4 v}$ defines a smooth singular foliation of index $j = (n+4)/2$ with square $\eta = \dfrac{z^4 v}{|z|^4|v|}$. If this foliation is Hessian then

$$\iint_{\mathbb{R}^2} \frac{\rho}{|z|^2|v|}\ \ \mathrm{Im}\ \left[(\chi_{R(z_0)} + i\ \chi_{S(z_0)})v\right] dxdy = 0$$

for all z_0 on a punctured disk about o, ρ being smooth and positive away from o.

Choosing $z_0 = \alpha i$ with $\alpha > 0$, the regions $R(z_0)$ and $S(_0)$ are pictured here:

Figure 3.

Notice that Im $\left[(\chi_{R(z_0)} + i\chi_{S(z_0)})v\right]$ takes the values $v_2, v_1 + v_2$ and v_1, on $R(z_0) - S(z_0), R(z_0) \cap S(z_0)$ and $S(z_0) - R(z_0)$ respectively, and is otherwise zero. As these regions lie in Σ it follows from condition (ii) on v and the above integral identity that $v \equiv 0$ on $R(z_0) \cap S(z_0)$. This contradicts the condition (i) on v. Hence the singular foliation $\sqrt{z^4 v}$ is not Hessian.

To complete the proof of Lemma 4, it remains only to construct fields v of every index $n \in \mathbb{Z}$ satisfying (i) and (ii). For the terminology and results used in the construction we refer the reader to [3], [14] or the work [2]. For each of the fields v constructed, Σ will be a hyperbolic sector and a neighbourhood of o will be a union of a finite number of hyperbolic and elliptic sectors. In particular, we note Bendixson's formula ([14], p. 222) for the index

$$n = 1 + \frac{e - h}{2}$$

where e and h denote the number of elliptic and hyperbolic sectors, respectively. We give the phase portrait of v for (i) $n > 0$, (ii) $n < 0$ and (iii) $n = 0$ in the figure below and this completes the proof.

Figure 4.

This last result and Lemma 2 now give Theorem 1 (announced in the introduction), the main result of this paper.

Proof of Theorem 1. If p is an isolated umbilic of positive curvature on a smooth surface in $\mathbb{R}^3$, we may assume its Gauss image is the south pole $s \in S^2$. Let $\mathcal{S}$ denote the Gauss image of one of the principal foliations around p. Then, by Lemma 2 of §2, the singular foliation $\mathcal{F} = \sigma(\mathcal{S})$ must be Hessian; here σ denotes stereographic projection from the north pole of S^2. Thus if we take $\mathcal{F}$ to be any of the singular foliations constructed in Lemma 4, it follows that the singular foliation $\mathcal{S} = \sigma^{-1}(\mathcal{F})$ around $s \in S^2$ is never the Gauss image of a principal foliation.

§4 Concluding remarks

There are new directions in which the investigation of the local Carathéodory conjecture might now proceed.

Theorem 2 and its consequence, Lemma 3, are in the spirit of the classical moment problems of Stieltjes and Hamburger [1], [12]. The following line of reasoning might be used to verify that a given singular foliation $\sqrt{g}$ (with $g(o) = 0$) is Hessian. Let us suppose that there exists a smooth positive kernel $p : \bar{D} \times \bar{D} \to \mathbb{R}$ on the unit disk such that the functions $\phi_z(w) = \operatorname{Im}\left[(\chi_{R(z)} + i\chi_{S(z)})\dfrac{\bar{w}^4}{|w|^6}g(w)\right]$ associated to each $z \in \bar{D}-\{o\}$ satisfy the following infimum condition

$$\inf_{z\in\bar{D}-\{o\}} \sum c_k \psi_{z_k}(z) \le 0$$

for any finite collection of $z_k \in \bar{D}-\{o\}$ and $c_k \in \mathbb{R}$; here

$$\psi_{z_k}(z) = \int_D p(z,w)\phi_{z_k}(w)dwd\bar{w}.$$

On the subspace $\mathcal{L}$ of $C(\bar{D}, R)$ generated by the constants and the functions ψ_z we define a linear functional λ by $\lambda(1) = 1$ and $\lambda(\psi_z) = 0$ for all $z \in \bar{D}-\{o\}$. By the infimum condition above, λ is a non-negative linear functional, i.e., $\lambda(f) \ge 0$ if $f \in \mathcal{L}$ with $f \ge 0$ on $\bar{D}$. By the Krein-Rutman theorem λ extends to a non-negative linear functional Λ on $C(\bar{D}, \mathbb{R})$ and, by Riesz's theorem, Λ is represented by integration against a probability measure μ. In particular

$$\int_D \left(\int_D p(z,w)\phi_{x_k}(w)dwd\bar{w}\right) d\mu(z) = 0$$

for each $z_k \in \bar{D}-\{o\}$. By Fubini's theorem

$$\int_D \rho(w)\phi_{z_k}(w)dwd\bar{w} = 0$$

for each $z_k \in \bar{D}-\{o\}$, where $\rho(w) = \int_D p(z,w)d\mu(z)$. These integral identities for the ϕ_{z_k} are, by Lemma 3, sufficient to guarantee that the foliation $\sqrt{g}$ is Hessian. If this procedure could be carried out for a foliation $\sqrt{g}$ of index ≥ 2 we would have a counterexample to the Loewner conjecture and therefore to the local Carathéodory conjecture.

When $g = (z^2 + b\bar{z}^2)^2, 0 \le b < 1$, the foliation $\sqrt{g}$ has index 2. It can be shown, from Lemma 3, that this foliation is not Hessian for b close to 0. The procedure outlined above might be used to decide this question when b is close to 1.

The foliations from hydrodynamics in §2 are singular Hessian foliations with integer index at o. The kind of argument given in [17], p. 178, shows that the index is ≤ 1, at least if the derivatives of the velocity components of all orders at o are not all zero. If on the other hand we are given a singular Hessian foliation $\mathcal{F}$ of *integer* index on a neighbourhood of o, one might attempt to impose the hydrodynamic formalism of §2 on this foliation. If this can be carried out then, by the remarks above, the Loewner and local Carathéodory conjectures would be shown to be essentially true in the case of integral index.

It is of interest to note also from §2 that a hydrodynamic foliation in a neighbourhood of an isolated stagnation point o naturally gives rise to a function ω whose Hessian gives this foliation. This function ω may then be used to construct a smooth surface M in $\mathbb{R}^3$ with an isolated umbilic and under this correspondence the streamlines correspond to principal curvature lines and the stagnation point to an umbilic. This is done via Lemma 2. It seems very remarkable that the hydrodynamic foliations have such canonical ties to surface theory, and the characterization of this class of surfaces deserves further study.

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