THE NATURE OF ELLIPTIC SECTORS IN THE PRINCIPAL FOLIATIONS OF SURFACE THEORY

Brian Smyth

University of Notre Dame

Abstract. Little is known of the geometry of the principal foliations on a smooth surface in $\mathbb{R}^3$. The unsolved Carathéodory conjecture for such a surface would follow from Bendixson's formula for the index of an isolated singularity of a smooth foliation on a smooth 2-manifold, were it known that on a smooth surface the principal foliations never admit an elliptic sector at an isolated umbilic. Evidence in favour of the nonexistence of such elliptic sectors in principal foliations is the main result here and offers a first geometric explanation of why the conjecture might be true for smooth surfaces.

§1. Introduction

The classical Carathéodory conjecture [2] (for which a proof has recently been given for analytic surfaces analytically immersed in $\mathbb{R}^3$ [4], see also [6]) posits a special topological behaviour near an isolated umbilic for the principal foliations of a smooth surface, a behaviour not enjoyed by arbitrary smooth foliations near an isolated singularity. The local form of this conjecture states that the index j of the principal foliations at an umbilic is ≤ 1. After many years it is still not understood what the geometric source for such an expectation might be.

Bendixson's formula for the index of an isolated singularity of a smooth foliation says that

$$j = 1 + \frac{e-h}{2},$$

where e and h are the number of elliptic and hyperbolic sectors of the foliation at the singularity ([1] or [3]). Obviously the Carathéodory conjecture would follow from a proof of the non-existence of elliptic sectors in the principal foliations at an isolated umbilic and such a conjecture is not only stronger but of genuine geometric interest.

In the course of this work the author was a visitor at Max-Planck-Institut für Mathematik in Bonn and gratefully acknowledges its hospitality.

Partially supported by NSF Grant DMS00 71729
Subject classification numbers: 53, 53A05

§2. The conjecture and the theorem

The theme of this work is then the following:

The elliptic sector conjecture.

At an isolated umbilic in a smooth surface in $\mathbb{R}^3$ the principal foliations cannot have an elliptic sector.

The hypothetical occurrence of such sectors in the principal foliations of surfaces is our main interest here. The essence of the theorem is that if such sectors occur at all, they must be pathological. The work depends critically on Codazzi's equation and is carried out on the surface itself. The first hint of the existence of such a result came in the work of Laurentiu Lazarovici [4], the basic assumption there being a convexity condition on the sector and the rather more intricate proof used hyperbolic p.d.e. applied to Hessian foliations — the planar normalization of principal foliations via Gauss-stereographic projection.

Let $f : M \longrightarrow \mathbb{R}^3$ be a smooth immersion of a smooth oriented surface M in $\mathbb{R}^3$. Let g denote the induced metric, ξ the oriented unit normal field along the immersion f, and A the second fundamental form of ξ; A is defined by $X\xi = -f_{*_p}(A_pX)$ for all vectors X tangent to M at p, where f_{*_p} is the differential of f at p. The inner product space (M_p, g_p) has orientation coming from M and counterclockwise rotation through $\frac{\pi}{2}$ defines the complex structure J_p. The metric connexion of g is denoted ∇ and for our purposes here the important equation is Codazzi's equation

$$(\nabla_X A)Y - (\nabla_Y A)X = 0.$$

On the complement of the umbilic set the eigenvalues of A are denoted by λ and μ and we may assume $\lambda > \mu$. The eigenspaces of A give two smooth, possibly non-orientable, foliations on the complement of the umbilic set. These are called the principal foliations of the immersed surface. The foliation determined by λ will be denoted $\mathcal{F}^+$ and that determined by μ will be denoted $\mathcal{F}^-$.

Let p_0 be an isolated umbilic of the immersion f and let E be an elliptic sector of the foliation $\mathcal{F}^+$ at p_0. Then E is bounded by a leaf of $\mathcal{F}^+$ that is born and dies at p_0, with no other umbilic occurring within E. Hence $\mathcal{F}^+$ is orientable on E. We orient $\mathcal{F}^+$ so that it gives the standard orientation to the boundary ∂E. The orientation of $\mathcal{F}^+$ is given by a unit vector field e_1 on E and $e_2 = Je_1$ orients the foliation $\mathcal{F}^-$ on E. The curvatures of the oriented leaves of $\mathcal{F}^+$ are given by $g(\nabla_{e_1}e_1, Je_1) = k_1$ and those of $\mathcal{F}^-$ by $g(\nabla_{e_2}e_2, Je_2) = k_2$. The functions k_1 and k_2 are real-valued on E.

Each $p \in E$ determines a unique oriented leaf segment of $\mathcal{F}^+$ which is born at p_0 and ends at p; this is denoted $C^+(p)$. Its length $l_1(p)$ is a positive *extended* real-valued function on E. Similarly $C^-(p)$ stands for the unique oriented leaf of $\mathcal{F}^-$ which begins at p and dies at p_0 and its length is denoted $l_2(p)$. The functions l_1 and l_2 are extended real-valued functions on E. The region $L(p)$ with positively oriented boundary $C^+(p) + C^-(p)$ is called *the lens determined by p*.

The Codazzi equation on E may be used to deduce the following integral formula.

Lemma.

$$-(\lambda-\mu)(p)=\int_{C^+(p)}(\lambda-\mu)k_2ds+\int_{C^-(p)}(\lambda-\mu)k_1ds.$$

All of the integrals occurring in this integral formula are proper whether or not the leaves are orientable.

Since $\lambda-\mu>0$ on the elliptic sector E there is an immediate contradiction if k_1 and k_2 are non-negative on *some* lens $L(p)$ in this elliptic sector. This contradiction gives a simple direct proof of Lazarovici's result (Main Theorem, [4]).

A closer analysis of the integral formula gives us the following result:

Theorem. *Let p_0 be an isolated umbilic of a smooth surface smoothly immersed in $\mathbb{R}^3$. If there exists an elliptic sector E in one of the principal foliations at p_0, then for every lens $L(p)$ in E either*

(a) $\sup_{L(p)}\{l_1,l_2\}=\infty$

or

(b) $\inf_{L(p)}\{k_1,k_2\}=-\infty$.

Of course the ultimate objective should be to use the integral formula in a more unified way to obtain the full conjecture. It should be noted that the integrals in the Lemma both converge and together give the integral over the boundary of the lens of a canonical 1-form determined by A. The sign of its differential on the interior of the lens is important. For an analytically immersed surface one might try to improve this result, showing that (a) and (b) do not occur; in this case the principal foliations are given by a $\mathbb{C}$-valued analytic quadratic differential (Hopf). At an isolated zero of this differential one needs to prove rectifiability of all characteristic curves to eliminate (a).

References

1. V. I. Arnol'd, Yu S. Il'yashenko, *Ordinary differential equations, Vol. 1, Dynamical systems*, Encyclopaedia of Mathematical Sciences, Springer Verlag, Berlin-Heidelberg-New York, 1988.
2. H. L. Hamburger, *Beweis einer Carathéodoryschen Vermütung. I*, Ann. of Math. **41** (1940), 63-86.
3. P. Hartman, *Ordinary differential equations*, John Wiley, New York, 1964.
4. V. V. Ivanov, *The Analytic Carathéodory Conjecture*, Siberian Math. J. **43** (2002), 251-322.
5. L. Lazarovici, *Elliptic sectors in surface theory and the Carathéodory-Loewner conjectures*, J. Differential Geom. **55** (2000), 453-473.
6. H. Scherbel, *A new proof of Hamburger's index theorem on umbilical points*, Dissertation No. 10281, ETH-Zürich.

Brian Smyth (smyth.1@nd.edu)
Mathematics Department, Notre Dame, Indiana 46556, USA