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EIGENVALUE ESTIMATES AND SINGULARITIES OF HESSIAN FIELDS

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§0 Introduction.

The Hessian operator of a C^3-smooth real-valued function f defined on an open set in $\mathbb{R}^2$ is

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

The singularities are those points where H_f is a multiple of the identity. Away from the singularities the eigendirections of H_f give two orthogonal line fields, one corresponding to the larger eigenvalue λ and the other corresponding to the smaller eigenvalue μ. If f has an isolated Hessian singularity at $o \in \mathbb{R}^2$ then the index j of this singularity is defined as the index of either line field in a neighbourhood of o: as line fields may not be orientable $j \in \frac{1}{2}\mathbb{Z}$.

A general line field may, of course, have any index j between $-\infty$ and $+\infty$. Although no topological restriction is known for the phase portrait of a Hessian line field, there is a longstanding belief that it must satisfy $j \leq 1$. Indeed this is a conjecture of Loewner [ ] and it is known to be equivalent to the local Caratheodory conjecture in differential geometry [ ]; the latter conjecture states that any isolated umbilic on a smooth surface in $\mathbb{R}^3$ must have index ≤ 1 [ ]. In complex notation, Loewner's conjecture was stated as follows:

Let g be a C^1-smooth complex-valued function on a neighbourhood of o with an isolated zero at o. If g, considered as a real vector field, has index > 2 then the system of wave equations

$$f_{\bar{z}\bar{z}} = g$$

has no real solution f[ ].

Under certain hypotheses on the eigenvalues of H_f in a neighbourhood of o we prove Loewner's conjecture. As these hypothesis are automatically satisfied when f is radially symmetric – and H_f then has index 1 – our result is sharp under the hypotheses.

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Theorem1. *Let f be a C^3-smooth real-valued function on a neighbourhood Ω of $o \in \mathbb{R}^2$ for which the Hessian H_f has an isolated singularity at o. Let λ and μ, $\lambda > \mu$, be the eigenvalues of H_f on $\Omega - \{o\}$. If*

$$\min_{C(r)} \lambda > \max_{C(r)} \mu$$

for each circle $C(r)$ of radius r about o, $0 < r < \epsilon$, for some $\epsilon > 0$, then the singularity has index $j \leq 1$.

Theorem2. *Let f be a C^3-smooth real-valued function on a neighbourhood Ω of $o \in \mathbb{R}^2$ for which the Hessian H_f has an isolated singularity at o. Let λ and μ, $\lambda > \mu$, be the eigenvalues of H_f on $\Omega - \{o\}$. If $\lambda > \lambda o > \mu$ on $\Omega - \{o\}$ then the singularity has index $j \leq \frac{1}{2}$.*

§1 Proof of Theorem 1.

We first want to establish a formula for the index $j(A)$ of the fields of eigendirections associated with a symmetric tensor A in the plane for which the eigenvalues are different, except at the isolated point $p = 0$. Let ξ be any continuous vector field with an isolated singularity at 0. The formula reads

$$2j(A) = j((A - \frac{trA}{2} I)\xi) + j(\xi). \tag{1}$$

In order to apply the above formula to the hessian operator H_f of f we bring in the identity $\nabla\langle\nabla f, \nabla g\rangle = H_f \nabla g + H_g \nabla f$, where g is any C^1 function. In particular, if $g(x, y) = x^2 + y^2$, we have

$$H_f \nabla g = \nabla(\langle\nabla f, \nabla g\rangle - 2f). \tag{2}$$

We now apply (1) to $A = H_f$, $\xi = \nabla g$. By the invariance of the degree, we can also replace the quantity $\frac{1}{2}\Delta f$ by any continuos function h which, outside the origin, lies *strictly* between the two (distinct) eigenvalues $\mu < \lambda$ of H_f. It follows from (1) and (2) that

$$2j(H_f) = j(H_f \nabla g - h\nabla g) + 1 = 1 + j(\nabla\langle\nabla f, \nabla g\rangle - 2\nabla f - h\nabla g). \tag{3}$$

By the main hypothesis of the theorem, there is a continuous function $k = k(p)$ defined in a neighborhood of $\mu(0) = \lambda(0)$ in $\mathbb{R}$, which is C^1 for $p \neq \lambda(0)$, such that $\mu < k \circ g < \lambda$ on a punctured neighborhood of 0 in $\mathbb{R}^2$. Setting $h = k \circ g$ one has

$$h\nabla g = \nabla(K \circ g), \tag{4}$$

where $K' = k$ and the formula is valid in a punctured neighborhood of the origin. Setting $F := \langle\nabla f, \nabla g\rangle - 2f - K \circ g$, (3) can be rewritten as

$$2j(H_f) = 1 + j(\nabla F). \tag{5}$$

In order to finish the proof of Theorem 1 we must argue that $j(\nabla F). \leq 1$ This is a standard result but, for completeness, we provide the details. Notice that the vector field ∇F extends to be continuous at the origin (with value zero) and, outside the origin, $\nabla F = (H_f - hI)\nabla g$. In particular, ∇f is of class C^1 outside the origin. Multiplying ∇f by a positive smooth function ϕ that decays to zero sufficiently fast at the the origin we obtain the vector field $\phi\nabla f$ which is of class C^1 everywhere. Of course ∇f and $\phi\nabla f$ have the same trajectories, in particular the same index. If $\phi\nabla f$ were to have index bigger than one then, by theorem 9.1, p. 166 of Hartman's book, ∇f would have a trajectory $\alpha(t)$ tending to 0 as t tends to ∞ and $-\infty$. Since F increases along the trajectories of ∇f, it follows that $\alpha(\mathbb{R})$ would consist entirely of critical pints of F, a contradiction to $\nabla F = (H_f - hI)\nabla g$ since 0 is an isolated hessian umbilic. This concludes the proof of Theorem 1.

§2 Proof of Theorem 2.

We apply the same reasoning of the proof of Theorem 1 given in §1, this time with the choices $h = \lambda(0) = \mu(0)$, $g(x, y) = x$. Observing $j(\nabla g) = 0$), equation (3) then becomes

$$2j(H_f) = j(H_f\nabla g - h\nabla g) = j(\nabla(\langle\nabla f, \nabla g\rangle - 2f - \lambda(0)g)).$$

Hence $2j(H_f)$ is the index of a gradient field and as such is at most one. This concludes the proof of Theorem 2.

References

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