A NEW ELEMENT IN CLASSICAL SURFACE THEORY AND THE CARATHÉODORY-LOEWNER CONJECTURES

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Abstract. The character of the principal foliations on a smooth surface in $\mathbb{R}^3$ is not well understood, in particular whether there are geometric or topological restrictions on their singularities (umbilics) distinguishing them from the singularities of arbitrary smooth foliations on an abstract smooth surface. The non-existence of elliptic sectors in the principal foliations would give the Carathéodory conjecture that the index of an isolated umbilic is $j \leq 1$. We show that if an elliptic sector exists in the principal foliations on a smooth surface in $\mathbb{R}^3$ it would have to be highly complicated.

1. Introduction

There are few results in classical surface theory that do not proceed from some curvature assumption and, of these, fewer still that are local. Thinking about the conjectures of Carathéodory and Loewner, long unsolved, led us to examine an area of this theory which has not received the attention it deserves and where there lies a result of such a local nature. This work identifies this new phenomenon which at once explains the Carathéodory-Loewner conjectures and, in establishing the phenomenon over a broad range, solves the conjectures over a broad range in the smooth category. The ultimate idea is captured in the following conjecture [11].

> **The elliptic sector conjecture:** *At an isolated umbilic in a smooth surface in $\mathbb{R}^3$ the principal foliations cannot have an elliptic sector.*

This would give Carathéodory's conjecture on any smooth surface, via Bendixson's theorem on the index of an isolated singularity of a smooth foliation on a smooth surface (see below); the notions of elliptic, hyperbolic and parabolic sectors are central and [1] and [6] will be general references for this subject. In this paper we show that any elliptic sector occurring at a singularity of the principal foliations of surface theory must be pathological, in the sense we now explain.

Let E be an elliptic sector of *any* smooth foliation $\mathcal{F}$ on a smooth oriented Riemannian 2-manifold $(M, <, >)$; without loss of generality we may assume M oriented. Write $\mathcal{F} = \mathcal{F}^+$ and denote the orthogonal foliation by $\mathcal{F}^-$. Let e_1 be the unit field on E tangent to the leaves of $\mathcal{F}^+$ and directed so that it gives the standard orientation to that part of E enclosed by each leaf of $\mathcal{F}^+$. Let $\{e_1, e_2\}$ be the positively oriented orthonormal frame field on E so determined. Choosing any $p \in E$, the arc length parameter along the e_2-leaf through p gives a variation of the e_1-leaves on E and its variation vector field on the elliptic sector is denoted V. If V is bounded along each e_1-leaf in E we say the sector is a *normal* elliptic

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sector; if V approaches zero at the ends of each e_1-leaf in E we say the sector is 0-*normal*. While the field V depends on the choice of basepoint used to construct it, these definitions are independent of the choice of $p \in E$.

The 0-normal elliptic sectors of a foliation on a Riemannian 2-manifold are at the limit of what is "drawable", so that sectors which are not even normal might be considered pathological. It remains to be checked whether non-normal sectors can occur when the leaves are assumed rectifiable or when the 2-manifold and the foliations are analytic.

In treating the non-existence of elliptic sectors in the principal foliations on a surface in $\mathbb{R}^3$ we find we only use the Codazzi equation of the second fundamental form of the surface in the induced metric $<,>$. Thus we might as well consider the eigenfoliations of any smooth symmetric operator A satisfying Codazzi's equation on any smooth Riemannian 2-manifold $(M,<,>)$. Codazzi's equation for A is

$$(\nabla_X A)Y - (\nabla_Y A)X = 0,$$

where ∇ is the metric connexion on M. The relevance of the eigenfoliations of a Codazzi operator to optics was noted by Gullstrand [4] in 1905, to hydrodynamics by Loewner [9] in 1953 and, in more recent applications, to relativity by Bessières-Lafontaine-Rozoy [2] and to elasticity by Smyth [10].

Theorem 1. *Let A be any smooth symmetric operator satisfying Codazzi's equation on a smooth Riemannian 2-manifold. Then (i) the eigenfoliations of A have no 0-normal elliptic sectors at any singularity and (ii) the eigenfoliations have no normal elliptic sectors at any isolated singularity of index $j > 1$.*

Remark 1. *The singularity in (i) need not be isolated. If a singularity of the type occurring in (ii) exists, then there must exist elliptic sectors there, by Bendixson's theorem. Following the proof in §5 we will see that the normality conditions can be weakened further.*

The Carathéodory conjecture is first mentioned in Cohn-Vossen's report [3] to the ICM at Bologna in 1928 and its formulation for smooth surfaces appears in Hamburger [5]. The conjecture says that any smooth immersion of the sphere in $\mathbb{R}^3$ must have at least two umbilic points; the stronger local form of the conjecture claims that the index j of the principal foliations at an isolated umbilic in a smooth surface smoothly immersed in $\mathbb{R}^3$ satisfies $j \leq 1$. The conjecture attracted many attempts for the case of analytic surfaces analytically immersed in $\mathbb{R}^3$, beginning with Hamburger [5]. Each is a lengthy power series argument, with a solution tree having many branches, in which gaps have been detected where particular branches might not have been followed to their ends. There is a very readable recent account by Ivanov [7] of a resolution of the gaps and difficulties attending the web of special cases arising in this analytic endeavour. At the end of the day the method does not enlighten our geometric understanding and the conjecture is certainly still open for smooth surfaces.

The relevance of the study to hydrodynamics was noticed by Loewner [9], who observed that the streamlines of a plane compressible fluid flow near a stagnation point is an eigenfoliation of a certain Codazzi operator — the Hessian operator of a certain function — and independently conjectured the same index result as the Carathéodory conjecture (the index of the streamline foliation is ≤ 1). The Hessian of a smooth function on $\mathbb{R}^2$ satisfies Codazzi's equation, by flatness of $\mathbb{R}^2$. The Carathéodory and Loewner conjectures are known to be equivalent [13].

The conjecture clearly posits a topological distinction between singularities of principal foliations and singularities of arbitrary smooth foliations. The source of this distinction is the real question here. However, seventy-five years after the first of many announcements of the solution in the analytic case [3], there was still no hint of what might be the geometric source of such an expectation. We supply that here.

Bendixson's formula (see [1], [6])for the index of an isolated singularity of a smooth foliation says that

$$j = 1 + \frac{e-h}{2},$$

where e and h are the number of elliptic and hyperbolic sectors of the foliation at the singularity, and these are assumed finite; the remaining possibility, when $j > 1$, is that there are infinitely many elliptic sectors at the singularity [1].

The simplest explanation of the Carathéodory conjecture would be the non-existence of elliptic sectors in the principal foliations of any smooth surface in $\mathbb{R}^3$ — that is, our conjecture above.

Obviously the Carathéodory-Loewner conjectures pale in comparison to the conjectured non-existence of elliptic sectors but the approach identifies the playing field where these conjectures will be decided — elliptic sectors — no examples of which are known to the author. We can be surprised that the importance of such qualitative information on the principal foliations could have been missed in classical surface theory.

Theorem 1 is a consequence of a series of lemmas and is proved in §5. In a precursor of this result, Lazarovici [8] used hyperbolic p.d.e. to show the non-existence of elliptic sectors diffeomorphic to those in the dipole foliation.

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2. Normal elliptic sectors

In this section we will be concerned with an elliptic sector E at $p_0 \in M$ of *any* smooth foliation $\mathcal{F}$ on a smooth oriented Riemannian 2-manifold $(M, <,>)$. Write $\mathcal{F} = \mathcal{F}^+$ and denote the orthogonal foliation by $\mathcal{F}^-$.

Let e_1 be the unit field on E tangent to the leaves of $\mathcal{F}^+$ and directed so that it gives the standard orientation to the part of E enclosed by each leaf of $\mathcal{F}^+$. Let $\{e_1, e_2\}$ be the positively oriented orthonormal frame field on E so determined. The geodesic curvatures associated with the frame field $\{e_1, e_2\}$ are given by $< \nabla_{e_1} e_1, Je_1 >= k_1$ and $< \nabla_{e_2} e_2, Je_2 >= k_2$, where ∇ is the Riemannian connexion and J is the complex structure determined by the orientation of M. The functions k_1 and k_2 are smooth on E.

We now show that each $p \in E$ determines a vector field V on E normal to $\mathcal{F}^+$ whose essential properties at the ends of each leaf of that foliation are independent of the choice of p. Fix $p \in E$ and let $\gamma : (a, b) \longrightarrow M$ be an arc length parametrization of the e_1-leaf through p so that $\lim_{s\to a} \gamma(s) = p_0$ and $\lim_{s\to b} \gamma(s) = p_0$; this leaf may not be rectifiable so that $a = -\infty$ and $b = \infty$ are allowed. Let $\gamma(L) = p$. The oriented e_2-leaf through p in E may be considered as a curve $\tau : (c, d) \longrightarrow M$ parametrized by arc length ϵ with $\lim_{s\to c} \tau(\epsilon) \in \partial E$ and $\lim_{\epsilon\to d} \tau(\epsilon) = p_0$. We allow the possibility that τ is non rectifiable, that is $d = \infty$. We may assume $\tau(0) = p$. Similarly, the e_2-leaf through any point $\gamma(s)$ has forward arc length

$d(s)$ measured from the point $\gamma(s)$ and backward arc length $c(s) < 0$ to ∂E. Let $\triangle = \{(s,u) | a \le s \le b, c(s) \le u \le d(s)\}$.

For any $(s,u) \in \triangle$ define $\Gamma(s,u) \in M$ to be that point of E at an oriented distance u from $\gamma(s)$ along the e_2-leaf through $\gamma(s)$. The map Γ is defined, continuous on $\triangle$ and smooth on its interior. On this latter set $\Gamma_s = \gamma_*(\frac{\partial}{\partial s})$ and $\Gamma_u = \gamma_*(\frac{\partial}{\partial u})$ define smooth vector fields along the map Γ. Note that $\Gamma_u(s,u) = e_2(\Gamma(s,u))$ from the definition of Γ. Furthermore, $\Gamma_{uu}(s_0,u_0)$ will denote the covariant derivative in M of the vector field Γ_u along the curve $\Gamma(s_0,u)$ evaluated at $u = u_0$; the quantities $\Gamma_{su} = \Gamma_{us}$ and Γ_{ss} are defined similarly.

Since E is an elliptic sector, each $\epsilon \in (c,d)$ uniquely determines a smooth function $\alpha(s,\epsilon)$ defined on (a,b) with $\alpha(L,\epsilon) = \epsilon$, such that $\gamma_\epsilon(s) = \Gamma(s,\alpha(s,\epsilon))$ is an e_1-curve from $(p_0)_\epsilon = p_0$ to $p_\epsilon = \gamma_\epsilon(L)$. Notice $\alpha : (a,b) \times (c,d) \longrightarrow \mathbb{R}$ is smooth and increasing in the second variable with $\alpha(s,0) \equiv 0$ for all s. Since $u = \alpha(s,\epsilon)$ along $\gamma_\epsilon(s)$ we have

$$\frac{d}{ds}(\gamma_\epsilon(s)) = \Gamma_s + \frac{\partial\alpha}{\partial s}(s,\epsilon)\Gamma_u. \tag{1}$$

Now $\gamma_\epsilon(s)$ being an e_1-curve and $\Gamma_u = e_2$, this gives

$$\frac{\partial\alpha}{\partial s}(s,\epsilon) = - < \Gamma_s, \Gamma_u >_{\gamma_\epsilon(s)} . \tag{2}$$

To avoid notational overload we will generally suppress the point of evaluation, vector quantities being evaluated at $\gamma_\epsilon(s)$ and scalars at (s,ϵ). From (1) and (2)

$$\frac{d}{ds}(\gamma_\epsilon(s)) = \Gamma_s - < \Gamma_s, \Gamma_u > \Gamma_u \tag{3}$$

and the monotonically parametrized curve $\gamma_\epsilon(s)$ has arc length parameter satisfying

$$ds_\epsilon = \sqrt{< \Gamma_s, \Gamma_s > - < \Gamma_s, \Gamma_u >^2}\ ds.$$

From (2)

$$\frac{\partial^2\alpha}{\partial s \partial\epsilon} = -(< \Gamma_{su}, \Gamma_u > + < \Gamma_s, \Gamma_{uu} >)\frac{\partial\alpha}{\partial\epsilon} = k_2 < \Gamma_s, e_1 > \frac{\partial\alpha}{\partial\epsilon} \tag{4}$$

since $\Gamma_u = e_2$ is a unit vector field and $\Gamma_{uu} = \nabla_{e_2} e_2 = -k_2 e_1$.

Now fix any $q = \gamma(h)$ in the e_1-leaf through p and let $q_\epsilon = \gamma_\epsilon(h)$ be the corresponding point on the e_1-leaf through $p_\epsilon = \gamma_\epsilon(L)$. Integrating the previous equation with respect to s we have

$$\frac{\partial\alpha}{\partial\epsilon}(h,\epsilon) = e^{-\int_h^L (k_2 < \Gamma_s, e_1 >)_{\gamma_\epsilon(s)} ds} \tag{5}$$

since $\alpha(L,\epsilon) = \epsilon$, for all ϵ. Since $\gamma_\epsilon(s)$ is monotonically parametrized by s, (3) can be rewritten $\sqrt{< \Gamma_s, \Gamma_s > - < \Gamma_s, \Gamma_u >^2} e_1 = \Gamma_s - < \Gamma_s, \Gamma_u > \Gamma_u$ so that the expression in (5) is $e^{-\int_h^L k_2 \sqrt{< \Gamma_s, \Gamma_s > - < \Gamma_s, \Gamma_u >^2}\ ds}$ and (5) becomes

$$\frac{\partial\alpha}{\partial\epsilon}(h,\epsilon) = e^{-\int_{C^+(\gamma_\epsilon(h),\gamma_\epsilon(L))} k_2 ds_\epsilon} \tag{6}$$

where s_ϵ is the arc length element along the oriented e_1-leaf segment $C^+(\gamma_\epsilon(h), \gamma_\epsilon(L))$ from $\gamma_\epsilon(h)$ to $\gamma_\epsilon(L)$. Here we have used the fact that $\frac{ds_\epsilon}{ds} = \sqrt{< \Gamma_s, \Gamma_s > - < \Gamma_s, \Gamma_u >^2}$.

Now $V = \frac{\partial\gamma_\epsilon(s)}{\partial\epsilon} = \Gamma_u(\gamma_\epsilon(s))\frac{\partial\alpha}{\partial\epsilon}(s,\epsilon)$ is *the variation vector field* of the foliation $\mathcal{F}^+$ determined by the choice of $p \in E$.

We call E a 0-*normal* elliptic sector of $\mathcal{F}^+$ if V approaches zero at the ends of each e_1-leaf, so that, $\lim_{s\to a}\frac{\partial\alpha}{\partial\epsilon}(s,\epsilon)=0$ for each relevant value of ϵ. We call E a *normal* sector of $\mathcal{F}^+$ if V is bounded along each leaf, that is, if $\frac{\partial\alpha}{\partial\epsilon}(s,\epsilon)$ is bounded in s for each relevant value of ϵ.

Each $p\in E$ determines a unique oriented leaf segment of $\mathcal{F}^+$ which is born at p_0 and ends at p; this is denoted $C^+(p)$. For any $q\in C^+(p)$ the oriented segment of $C^+(p)$ from q to p is denoted $C^+(q,p)$. Similarly p determines a unique oriented leaf segment of $\mathcal{F}^-$ which begins at p and dies at p_0; this is denoted $C^-(p)$.

From (6) and from the definition of 0-normal and normal elliptic sectors we have the following lemma.

Lemma 1. *Let E be an elliptic sector of a foliation $\mathcal{F}$ on a Riemannian 2-manifold $(M,<,>)$. (i) If E is 0-normal then for each $p\in E$ the improper integral $\int_{C^+(p)}k_2ds$ diverges to $+\infty$. (ii) If the elliptic sector E is normal then for each $p\in E$ the integrals $\int_{C^+(q,p)}k_2ds$ are bounded away from $-\infty$ as q recedes to p_0 from p along $C^+(p)$.*

3. Codazzi operators

Let A be a smooth symmetric tensor field of type $(1,1)$ on a smooth oriented Riemannian 2-manifold $(M,<,>)$ and assume A satisfies Codazzi's equation

$$(\nabla_XA)Y-(\nabla_YA)X=0.$$

The trace function $2H=trA$ and $detA$ are the elementary symmetric functions of A and $\sigma=\sqrt{(trA)^2-4detA}$ is the square root of the discriminant of A. The zeroes of σ are the points where A is a multiple of I. Away from the zero set U of σ this function is smooth; σ^2 is smooth everywhere on M.

On the open set $M-U$ the eigenspaces of A give two smooth orthogonal foliations, the *eigenfoliations* of A; U is the singularity set of these foliations on M. Let $\{e_1,e_2\}$ be a positively oriented orthonormal frame field locally on $M-U$, *with respect to which A diagonalizes*

$$A=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}.$$

The geodesic curvatures k_1 and k_2 associated with the frame field $\{e_1,e_2\}$ were defined in §2. We can check that $Z=k_1e_1+k_2e_2$ is independent of the choice of local positive orthonormal frame diagonalizing A and gives a well-defined vector field on all of $M-U$, and its dual 1-form is a global connexion form on $M-U$. The 1-form given on $M-U$ by $\omega(X)=-2<X,(A-HI)JZ>$ is expressed in terms of the above local frame field by $\omega(e_1)=\sigma k_2$ and $\omega(e_2)=\sigma k_1$, where $\sigma=\lambda-\mu\neq 0$; we call ω the *1-form of A*. Note that as we move from one diagonalizing oriented frame field to another on $M-U$, the function σ may reverse sign on overlaps but σ^2 is a smooth function on M.

Now suppose E is an elliptic sector of one of the eigenfoliations $\mathcal{F}$ of A at a singularity $p_0\in U$. We can choose a positive orthonormal frame field $\{e_1,e_2\}$ on all of E diagonalizing A so that e_1 positively orients the boundary of the region bounded by each leaf of that foliation. Then the e_2-leaves run orthogonally inward to die at p_0. The e_1-foliation $\mathcal{F}$ on E is written $\mathcal{F}=\mathcal{F}^+$ and the e_2-foliation $\mathcal{F}^-$; these foliations are oriented on E by the fields e_1 and e_2, respectively. The function $\sigma=\lambda-\mu$ is smooth non-vanishing on E and approaches zero at p_0. The geodesic curvature of the oriented leaves of $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) on E is given by k_1 (resp. k_2) and k_1 and k_2 are smooth functions on E.

If in the Codazzi equation above we replace X by e_1 and Y by e_2, we obtain from the definition of the curvatures k_1 and k_2 above:

$$e_1(\mu) = \sigma k_2, \; e_2(\lambda) = \sigma k_1. \tag{7}$$

In terms of the trace function $2H = \lambda + \mu$, Codazzi's equation may be rewritten

$$e_1(\lambda) = e_1(2H) - \sigma k_2, \; e_2(\mu) = e_2(2H) - \sigma k_1, \tag{8}$$

or as

$$e_1(\sigma) = 2\{e_1(H) - \sigma k_2\}, \; e_2(\sigma) = 2\{\sigma k_1 - e_2(H)\}. \tag{9}$$

For each $p \in E$ the lengths $l_1(p)$ and $l_2(p)$ of the leaf segments $C^+(p)$ and $C^-(p)$, respectively, are positive extended real-valued functions on E. The region $L(p)$ with positively oriented boundary $C^+(p) + C^-(p)$ is called *the lens determined by p.*

Let p be any point in the elliptic sector E. Integrating the first two of these equations along the oriented leaf $C^+(p)$ of the $\mathcal{F}^+$-foliation through p which is born at p_0 we have

$$\mu(p) - \mu(p^+) = \int_{C^+(p^+,p)} \sigma k_2 ds,$$

where p^+ is any point on $C^+(p)$ between p_0 and p and the integration on the right takes place along the subarc $C^+(p^+, p)$ of $C^+(p)$ from p^+ to p. By continuity of μ on M

$$\mu(p) - \mu(p_0) = \int_{C^+(p)} \sigma k_2 ds,$$

and the improper integral on the right converges. Similarly, integrating the second of the Codazzi equations, we arrive at another convergent improper integral

$$\lambda(p_0) - \lambda(p) = \int_{C^-(p)} \sigma k_1 ds,$$

where $C^-(p)$ is the oriented leaf of the $\mathcal{F}^-$ -foliation through p which dies at p_0. Since p_0 is an umbilic, the sum of these equations is

$$-\sigma(p) = \int_{C^+(p)} \sigma k_2 ds + \int_{C^-(p)} \sigma k_1 ds,$$

with all improper integrals on the right converging whether or not the arcs $C^+(p)$ and $C^-(p)$ are rectifiable. Thus in the presence of an elliptic sector E in the eigenfoliations of A, Codazzi's equation gives rise to a linear integral equation (with no rectifiability assumptions) for the nowhere vanishing function $\sigma = \lambda - \mu$ which must hold identically on E.

Lemma 2. *If there exists an elliptic sector E in an eigenfoliation of a Codazzi operator A at p_0, then for every lens $L(p)$ in E the integral equation*

$$-\sigma(p) = \int_{C^+(p)} \sigma k_2 ds + \int_{C^-(p)} \sigma k_1 ds = \int_{\partial L(p)} \omega, \tag{10}$$

holds, where $\sigma = \lambda - \mu$, ω is the 1-form of A and all improper integrals in the equation converge.

In particular, if C^+ is a full leaf of the $\mathcal{F}^+$-foliation in an elliptic sector of that foliation, then $\int_{C^+} \sigma k_2 ds = 0$. While this integral equation is not used in the proof of Theorem 1, it motivated our approach to the conjecture.

Lemma 3. *The 1-form ω of A is bounded on any relatively compact subset of the non-umbilic set $M-U$. In particular, $\sigma\sqrt{k_1^2+k_2^2}$ is bounded on any elliptic sector E.*

Proof. If ϕ is a smooth function on M, its gradient is denoted $\nabla\phi$ and its Hessian operator is given by $h_\phi(X) = \nabla_X \nabla\phi$. The matrix of h_ϕ with respect to any local orthonormal frame field $\{e_1, e_2\}$ on M is

$$\begin{pmatrix} e_1e_1(\phi) - k_1e_2(\phi) & e_2e_1(\phi) - k_2e_2(\phi) \\ e_1e_2(\phi) + k_1e_1(\phi) & e_2e_2(\phi) + k_2e_1(\phi) \end{pmatrix}$$

and its trace is the laplacian

$$\triangle\phi = e_1e_1(\phi) + e_2e_2(\phi) - k_1e_2(\phi) + k_2e_1(\phi).$$

The function σ^2 is smooth on M and the identity

$$\frac{1}{2}\triangle\sigma^2 = ||\nabla\sigma||^2 + \sigma\triangle\sigma \tag{11}$$

holds on $M-U$. If around any point of $M-U$ we now take a positive local orthonormal frame field $\{e_1, e_2\}$ diagonalizing A, then from the earlier expressions (9) for $e_1(\sigma)$ and $e_2(\sigma)$, we get

$$\begin{aligned} ||\nabla\sigma||^2 &= 4(||\nabla H||^2 - 2\sigma k_1 e_2(H) - 2\sigma k_2 e_1(H) + \sigma^2(k_1^2+k_2^2)), \\ &= 4(||\nabla H||^2 - 2s\sigma||\nabla H||\sqrt{k_1^2+k_2^2} + \sigma^2(k_1^2+k_2^2)), \end{aligned} \tag{12}$$

where $-1 \le s \le 1$, by Cauchy-Schwarz. A somewhat longer calculation, using the definition of the laplacian above and the expression for the Gauss curvature $K = -e_1(k_2) + e_2(k_1) - k_1^2 - k_2^2$ in terms of k_1 and k_2, gives

$$\triangle\sigma = 2\{P + \sigma[K + 2(k_1^2+k_2^2)]\}, \tag{13}$$

where

$$P = e_1e_1(H) - e_2e_2(H) - k_1e_2(H) - k_2e_1(H).$$

Looking at $B = h_H$, the Hessian of H, and more particularly at its traceless part the smooth operator $B_0 = B - \frac{1}{2}trBI$ on M we see that $|P| \le 2\sqrt{-detB_0}$, on $M-U$ and P is therefore bounded on any relatively compact subset of $M-U$.

Thus substituting (12) and (13) in (11) we obtain

$$\frac{1}{4}\triangle\sigma^2 = \{2(||\nabla H|| - s\sigma\sqrt{k_1^2+k_2^2}\ )^2 + 2\sigma^2(2-s^2)(k_1^2+k_2^2) + \sigma P + \sigma^2 K\}.$$

Since P, K and $\triangle\sigma^2$ are bounded on any relatively compact subset of $M-U$, it follows that $\sigma^2(k_1^2+k_2^2)$ and ω are similarly bounded. □

Lemma 4. *Let p_0 be an isolated singular point of an eigenfoliation of a Codazzi operator A with index $j > 1$. Then this foliation has an elliptic sector E at p_0 and $(\nabla A)(p_0) = 0$; in particular, as p approaches p_0 in E, $\lim_{p\to p_0} \sigma\sqrt{k_1^2+k_2^2} = 0$ and $(\nabla H)(p_0) = 0$.*

This is the substance of the main result in our paper [12].

4. Geometric information on Codazzi operators on normal elliptic sectors

In this section we continue the study of an elliptic sector E in the eigenfoliations of A and establish an identity linking H, σ, k_1 and k_2 under the normality conditions on this sector appearing in Theorem 1.

On E the eigenvalue μ of A is a smooth function so that $e_1e_2(\mu) - e_2e_1(\mu) - [e_1, e_2](\mu) = 0$. From the first Codazzi equation (7), $e_1(\mu) = \sigma k_2$, this takes the form

$$e_1e_2(\mu) + k_2e_2(\mu) = e_2(\sigma k_2) - \sigma k_1k_2,$$

and the right-hand side $\chi = e_2(\sigma k_2) - \sigma k_1k_2$ is a smooth function on E. With the notation of the previous section, we fix ϵ and write

$$\phi(s) = e_2(\mu)_{\gamma_\epsilon(s)}e^{-\int_{C^+(\gamma_\epsilon(s),\gamma_\epsilon(L))} k_2ds_\epsilon}$$

for each $s \in (a, b)$. Then

$$\frac{d\phi}{ds}(s) = \{(e_1e_2(\mu) + k_2e_2(\mu))_{\gamma_\epsilon(s)}e^{-\int_{C^+(\gamma_\epsilon(s),\gamma_\epsilon(L))} k_2ds_\epsilon}\}\frac{ds_\epsilon}{ds}$$

or

$$\frac{d\phi}{ds}(s) = \{\chi(\gamma_\epsilon(s))e^{-\int_{C^+(\gamma_\epsilon(s),\gamma_\epsilon(L))} k_2ds_\epsilon}\}\frac{ds_\epsilon}{ds}.$$

Hence for any $a < h < v < b$ we have

$$\begin{aligned}\int_{C^+(\gamma_\epsilon(h),\gamma_\epsilon(v))} & \{\chi(\gamma_\epsilon(s))e^{-\int_{C^+(\gamma_\epsilon(s),\gamma_\epsilon(L))} k_2ds_\epsilon}\}ds_\epsilon \\ = & \ \phi(v) - \phi(h) \\ = & \ e_2(\mu)_{\gamma_\epsilon(v)}e^{-\int_{C^+(\gamma_\epsilon(v),\gamma_\epsilon(L))} k_2ds_\epsilon} - e_2(\mu)_{\gamma_\epsilon(h)}e^{-\int_{C^+(\gamma_\epsilon(h),\gamma_\epsilon(L))} k_2ds_\epsilon}.\end{aligned}$$

At this point we must distinguish two cases.

(i) Assume that E is a 0-normal elliptic sector. Now since, by the second form of Codazzi's equation (8), $e_2(\mu) = e_2(2H) - \sigma k_1$ and since σk_1 is bounded on E, by Lemma 3, it follows that $e_2(\mu)$ is bounded on E. Since E is 0-normal it follows, from Lemma 1, that $\int_{C^+(p)} k_2ds$ diverges to $+\infty$ so that the limit of the last term in the previous equation, as h approaches a, is zero. Thus, taking $\epsilon = 0$, we have

$$e_2(\mu)(q) = e^{\int_{C^+(q,p)} k_2}\int_{r\in C^+(q)} \chi(r)e^{-\int_{C^+(r,p)} k_2}$$

where q is any point of $C^+(p)$ and the improper integral on the right converges. This is best written

$$e_2(\mu)(q) = \int_{r\in C^+(q)} \chi(r)e^{-\int_{C^+(r,q)} k_2}, \tag{14}$$

and this equation is the turning point of the proof. Since all integrals are along oriented curves and are with respect to the arc-length parameter induced from the metric M, we simplify the integral notation by not registering the arc length of $C^+(r, q)$ in the notation.

Noting that $e_2(\mu) = e_2(2H) - \sigma k_1$ from the second form of the Codazzi equation (8), and differentiating this identity in the direction e_1 at q, we obtain

$$e_1e_2(2H) - e_1(\sigma k_1) = \chi = e_2(\sigma k_2) - \sigma k_1k_2$$

or

$$e_1(\sigma k_1) + e_2(\sigma k_2) = e_1e_2(2H) + \sigma k_1k_2.$$

Since σ is smooth on $E, e_1e_2(\sigma) - e_2e_1(\sigma) - [e_1, e_2](\sigma) = 0$ holds on E and by Codazzi's equation (9) this can be rewritten

$$e_1(\sigma k_1) + e_2(\sigma k_2) = e_1e_2(H) + e_2e_1(H) - k_1e_1(H) + k_2e_2(H).$$

Eliminating between the last two equations gives

$$e_1e_2(H) - e_2e_1(H) + k_1e_1(H) - k_2e_2(H) + \sigma k_1k_2 = 0,$$

and since

$$e_1e_2(H) - e_2e_1(H) = [e_1, e_2](H) = -k_1e_1(H) - k_2e_2(H),$$

on E we obtain the equation

$$k_2(e_2(2H) - \sigma k_1) \equiv 0. \tag{15}$$

(ii) Assume that E is a normal elliptic sector and $j(p_0) > 1$. In this case we find, by Lemma 4, that $j(p_0) > 1$ implies that $\nabla H(p_0) = 0$, and as p approaches p_0 through E that $\lim_{p\to p_0} \sigma k_1 = 0$. By Codazzi's equation (8), we therefore have $\lim_{p\to p_0} e_2(\mu) = 0$. If E is now assumed normal then, by Lemma 1, the integrals $\int_{C^+(q,p)} k_2 ds$ are bounded away from $-\infty$ as q recedes to p_0 from p along $C^+(p)$; thus the expressions $e^{-\int_{C^+(q,p)} k_2 ds}$ are bounded as q recedes to p_0. These last two facts mean that $e_2(\mu)_q e^{-\int_{C^+(q,p)} k_2 ds}$ approaches zero as q recedes to p_0 along $C^+(p)$. Hence, the earlier equation (14)

$$e_2(\mu)(q) = \int_{r\in C^+(q)} \chi(r) e^{-\int_{C^+(r,q)} k_2}$$

holds in case (ii) also. From this point on the argument is the same as in case (i).

With the notation above we therefore have the following result.

Lemma 5. *Let E be an elliptic sector at p_0 of an eigenfoliation of a Codazzi operator A. Then*

$$k_2(e_2(2H) - \sigma k_1) \equiv 0$$

if either (i) E is a 0-normal elliptic sector

or (ii) E is a normal elliptic sector and p_0 is an isolated singularity of index $j(p_0) > 1$.

Remark 2. *We can define an individual leaf of $\mathcal{F}$ in E to be 0-normal (resp. normal) if the field V (see §2) approaches zero (resp. is bounded) at the ends of that leaf. Part (i) (resp. part (ii)) of the key result, Lemma 5, remains true if the set of 0-normal (resp. normal) leaves is dense in the leaf space of E.*

5. Proof of Theorem 1

Proof. By Codazzi's equation (8), Lemma 5 is equivalent to $k_2e_2(\mu) \equiv 0$ on E.

If k_2 is not identically zero on E, then on the non-empty open set $I = \{p \in E | k_2(p) \neq 0\}$ we have $e_2(\mu) \equiv 0$, by Lemma 5. Hence $e_1e_2(\mu) \equiv 0$ on I. If I is dense in E then $e_1e_2(\mu) \equiv 0$ on E. Otherwise, the interior J of the complement of I in E is a non-trivial open set on which $k_2 \equiv 0$ and therefore, by Codazzi's equation (7), $e_1(\mu) \equiv 0$; thus on J the equation

$$e_1e_2(\mu) - e_2e_1(\mu) + k_1e_1(\mu) + k_2e_2(\mu) \equiv 0$$

reduces to $e_1e_2(\mu) \equiv 0$. Thus $e_1e_2(\mu) \equiv 0$ on $I \cup J$ and therefore on all of E.

With this last conclusion and Lemma 5 above, only the two middle terms of the equation

$$e_1e_2(\mu) - e_2e_1(\mu) + k_1e_1(\mu) + k_2e_2(\mu) \equiv 0$$

survive, and with Codazzi's equation (7) this reduces to

$$+e_2e_1(\mu) - k_1e_1(\mu) = e_2(\sigma k_2) - \sigma k_1k_2 \equiv 0.$$

Using Codazzi's equation (9) and Lemma 5 once again, this gives $\sigma e_2(k_2) \equiv 0$, or $k_2 \equiv const.$ along each e_2-curve. Thus each e_2-curve is either (a) a geodesic ($k_2 \equiv 0$) or (b) $k_2 \neq 0$ and, by Lemma 5, μ is constant along the curve and so $\mu \equiv \mu(p_0)$.

Now take any e_1-curve C^+; at points where $k_2 \neq 0$ we have $\mu = \mu(p_0)$ and on the complement of this set in C^+ we have $e_1(\mu) = \sigma k_2 = 0$, by Codazzi; since μ is locally constant on C^+ we must have $\mu = \mu(p_0)$ along C^+. Hence $\mu \equiv \mu(p_0)$ on E and so $e_1(\mu) = \sigma k_2 \equiv 0$ on E. Thus $k_2 \equiv 0$ on E.

But if $k_2 \equiv 0$ on E, then all e_2-leaves are geodesics ending at p_0 and, by Gauss' Lemma, their orthogonal trajectories (the e_1-leaves) cannot close at p_0, a contradiction. $\square$

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