Microbundles

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Motivation

How do we define the notion of a tangent fibre bundle TM for a topological n-manifold M?

Attempt: For each $x \in M$, choose a neighborhood $U_x \cong \mathbb{R}^n$ as the fibre of TM over x.

Problems:

- It is difficult to choose the neighborhoods U_x so that they vary continuously with x.
- Even if such a choice were possible, it is not clear that the resulting object would give a topological invariant of M.

To get around the above problems, we instead consider a new type of bundle, called a microbundle, in which the fibre is only a 'germ' of a topological space. The theory of microbundles was originally developed by Milnor in [M1] and [M2].

Microbundles

Definition

A (topological) microbundle $\mathfrak{X}$ of fibre dimension n, with base space B and total space E or $E(\mathfrak{X})$, is a diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

with $ji = id_B$ satisfying the following local triviality condition:

- For each $b \in B$ there exist open neighborhoods U of b in B and V of $i(b)$ in E, with $i(U) \subset V$ and $j(V) \subset U$, as well as a homeomorphism $h : V \to U \times \mathbb{R}^n$, so that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "j_{|V}"] \arrow[dr, "h"'] & U \\
U \arrow[u, "i_{|U}"] \arrow[r, "\times 0"'] & U \times \mathbb{R}^n \arrow[u, "p_1"']
\end{tikzcd}$$

Equivalence of microbundles

- Note that what matters in a microbundle is its behavior near the zero section $i(B)$.
- If E_0 is any neighborhood of $i(B)$ in E, we identify the original microbundle $\mathfrak{X} : B \xrightarrow{i} E \xrightarrow{j} B$ with the new microbundle $\mathfrak{X}_0 : B \xrightarrow{i} E_0 \xrightarrow{j|E_0} B$.

Definition

More generally, two microbundles $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are isomorphic ($\mathfrak{X}_1 \cong \mathfrak{X}_2$) if there exist neighborhoods V_k of $i_k B$ in E_k ($k = 1, 2$) and a homeomorphism $h : V_1 \to V_2$ so that the following diagram commutes:

$$\begin{tikzcd}
V_1 \arrow[r, "j_1"] \arrow[dr, "h"'] & B \\
B \arrow[u, "i_1"] \arrow[r, "i_2"'] & V_2 \arrow[u, "j_2"']
\end{tikzcd}$$

Microbundle maps

Definition

Let $\mathfrak{X}_k : B_k \xrightarrow{i_k} E_k \xrightarrow{j_k} B_k$ $(k = 1, 2)$ be two microbundles. A microbundle map $\mathbf{f} : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a commutative diagram

$$\begin{tikzcd}
B_1 \arrow[r, "i_1"] \arrow[d, "f"] & V_1 \arrow[r, "j_1"] \arrow[d, "\mathbf{f}"] & B_1 \arrow[d, "f"] \\
B_2 \arrow[r, "i_2"'] & V_2 \arrow[r, "j_2"'] & B_2
\end{tikzcd}$$

where V_k is an open neighborhood of $i_k(B_k)$ in E_k $(k = 1, 2)$.

By definition, a microbundle isomorphism is a microbundle map with $f = id_B$ and $\mathbf{f}$ a homeomorphism.

Examples

- The standard trivial microbundle $\mathfrak{e}^n_B$: $B \xrightarrow{\times 0} B \times \mathbb{R}^n \xrightarrow{p_1} B$
  - A microbundle $\mathfrak{X}$ is called trivial if $\mathfrak{X} \cong \mathfrak{e}^n_B$.

Lemma

If $\mathfrak{X}$ is a trivial microbundle over a paracompact space B, then some open subset of $E(\mathfrak{X})$ is homeomorphic to all of $B \times \mathbb{R}^n$, rather than just to an open subset of $B \times \mathbb{R}^n$.

- For a vector bundle $\zeta : E \xrightarrow{j} B$ with zero section i, the diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

is called the underlying microbundle of ζ, denoted by $|\zeta|$.

Examples (continued)

For a topological manifold M, the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$$

is the tangent microbundle of M, denoted by $\mathfrak{t}_M$.

- Local triviality: Given $x \in M$, choose a neighborhood U_x of x that is homeomorphic to $\mathbb{R}^n$ under a homeomorphism f. Then $V = U \times U$ is homeomorphic to $U \times \mathbb{R}^n$ via $h(u_1, u_2) = (u_1, f(u_2) - f(u_1))$.
- A map $f : M \to N$ of topological manifolds induces a map $\mathbf{df} : \mathfrak{t}_M \to \mathfrak{t}_N$ of tangent microbundles, called the differential:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "f"] & M \times M \arrow[r] \arrow[d, "f \times f"] & M \arrow[d, "f"] \\
N \arrow[r] & N \times N \arrow[r] & N
\end{tikzcd}$$

The tangent microbundle

Theorem

If M is a smooth paracompact manifold with tangent vector bundle τ, then the underlying microbundle $|\tau|$ is isomorphic to the tangent microbundle $\mathfrak{t}_M$.

Proof.

Choose a Riemannian metric for M. For (p, v) in a neighborhood E' of $i(M) \subset E(\tau)$, let $exp(p, v) = g(1)$, where g is the unique geodesic $g : [0, 1] \rightarrow M$ satisfying $g(0) = p$ and $\frac{dg}{dt}(0) = v$.

Define $h : E' \rightarrow M \times M$ by $h(p, v) = (p, exp(p, v))$. h is the required homeomorphism between a neighborhood E'' of $i(M) \subset E'$ and a neighborhood V of the diagonal in $M \times M$. □

The tangent microbundle (continued)

Corollary

If a topological manifold M can be smoothed, then the tangent microbundle $\mathfrak{t}_M$ is isomorphic to $|\zeta|$ for some vector bundle ζ over M.

Is the converse true? That is, if the tangent microbundle $\mathfrak{t}_M$ is isomorphic to $|\zeta|$ for some vector bundle ζ, can M necessarily be given a smoothness structure?

Theorem

If the tangent microbundle $\mathfrak{t}_M$ is isomorphic to $|\zeta|$ for some vector bundle ζ, then $M \times \mathbb{R}^q$ can be given a smoothness structure for sufficiently large q.

PL microbundles

- PL microbundles are defined analogously to the topological case, using the category of polyhedra and PL maps instead of the category of topological spaces and continuous maps.
- If M is a PL manifold, the diagram

  $$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$$

  is the PL tangent microbundle of M.
- A PL microbundle η has a natural underlying topological microbundle, $|\eta|$.
- If $\mathfrak{X}$ is a topological microbundle over a polyhedron B, a PL structure on $\mathfrak{X}$ is a PL microbundle η over B so that $|\eta| \cong \mathfrak{X}$.
- Two PL structures η_0 and η_1 on $\mathfrak{X}$ are equivalent if they are concordant, that is, if there is a PL structure γ on $\mathfrak{X} \times I : B \times I \xrightarrow{i \times id} E \times I \xrightarrow{j \times id} B \times I$ so that $\gamma|(\mathfrak{X} \times k) = \eta_k \times k$ for $k = 0, 1$.

Microbundle constructions

Let $\mathfrak{X}$ denote the microbundle $B \xrightarrow{i} E \xrightarrow{j} B$.

- If $A \subset B$, the restricted microbundle $\mathfrak{X}|A$ is the diagram

  $$A \xrightarrow{i'} j^{-1}A \xrightarrow{j'} A$$

  where $i' = i|A$ and $j' = j|j^{-1}A$.

- Given a map $f : A \to B$, the induced microbundle $f^*\mathfrak{X}$ is the diagram

  $$A \xrightarrow{i'} E' \xrightarrow{p_1} A$$

  where $E' = \{(a, e) \in A \times E|\ f(a) = j(e)\}$ and $i'(a) = (a, i(f(a)))$.
  - If f is an inclusion map, $f^*\mathfrak{X} \cong \mathfrak{X}|A$.

Properties of induced microbundles

Theorem (Homotopy Theorem)

If A is paracompact and $g : A \longrightarrow B$ is homotopic to f, then $f^*\mathfrak{X} \cong g^*\mathfrak{X}$.

Lemma

Let $f : A \longrightarrow B$, where A is paracompact, and let CA be the cone on A. Then $\mathfrak{X}$ can be extended over $B \cup_f CA$ if and only if $f^*\mathfrak{X}$ is trivial.

Microbundle constructions (continued)

Let $\mathfrak{X}_k : B_k \xrightarrow{i_k} E_k \xrightarrow{j_k} B_k$ $(k = 1, 2)$ be two microbundles.

- The Cartesian product microbundle $\mathfrak{X}_1 \times \mathfrak{X}_2$ is the diagram

$$B_1 \times B_2 \xrightarrow{i_1 \times i_2} E_1 \times E_2 \xrightarrow{j_1 \times j_2} B_1 \times B_2.$$

- If $B_1 = B = B_2$, the Whitney sum microbundle $\mathfrak{X}_1 \oplus \mathfrak{X}_2$ is the diagram

$$B \xrightarrow{i'} E(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \xrightarrow{j'} B$$

  where $E(\mathfrak{X}_1 \oplus \mathfrak{X}_2) = \{(e_1, e_2) \in E_1 \times E_2 |\ j_1(e_1) = j_2(e_2)\}$, $i'(b) = (i_1(b), i_2(b))$, and $j'(e_1, e_2) = j_1(e_1)$.
  - Alternatively, $\mathfrak{X}_1 \oplus \mathfrak{X}_2 \equiv \Delta^*(\mathfrak{X}_1 \times \mathfrak{X}_2)$, where $\Delta : B \to B \times B$.

The group $k_{Top}B$

- Two microbundles $\mathfrak{X}$ and $\mathfrak{X}'$ over B are s-isomorphic if $\mathfrak{X} \oplus \mathfrak{e}_B^q \cong \mathfrak{X}' \oplus \mathfrak{e}_B^r$ for some integers q, r. The s-class of $\mathfrak{X}$ is denoted by $(\mathfrak{X})$. (Similarly for PL microbundles and vector bundles.)
- The s-classes of microbundles over B form an abelian group, denoted by $k_{Top}B$ under the operation of Whitney sum, $(\mathfrak{X}) + (\mathfrak{y}) = (\mathfrak{X} \oplus \mathfrak{y})$. (Similarly $k_{PL}B$ for PL microbundles and $k_O B$ for vector bundles.)

The group $k_{Top}B$ (continued)

Theorem

If $\mathfrak{X}$ is an n-microbundle over a finite dimensional simplicial complex B, then there exists a microbundle $\mathfrak{y}$ so that $\mathfrak{X} \oplus \mathfrak{y}$ is trivial.

- **Lemma.** If B is a bouquet of spheres meeting at a point and $r : B \to B$ maps each sphere into itself with degree -1, $\mathfrak{X} \oplus r^*\mathfrak{X}$ is trivial for any microbundle $\mathfrak{X}$ over B.
- **Proof of Theorem.** (by induction on the dimension d of B) Let B' be the $(d-1)$-skeleton of B. Let $\mathfrak{y}'$ be a microbundle over B' so that $(\mathfrak{X}|B') \oplus \mathfrak{y}'$ is trivial. $\mathfrak{y}' \oplus \mathfrak{e}^n_{B'}$ extends to a microbundle $\mathfrak{z}$ over B. Since $(\mathfrak{X} \oplus \mathfrak{z})|B' = \mathfrak{X}|B' \oplus \mathfrak{y}' \oplus \mathfrak{e}^n_{B'}$ is trivial, $\mathfrak{X} \oplus \mathfrak{z}$ extends to a microbundle $\mathfrak{w}$ over $B \cup Cone(B')$, which has the homotopy type of a bouquet of d-spheres. The lemma implies that $\mathfrak{w} \oplus r^*\mathfrak{w}$ is trivial, and hence $(\mathfrak{w} \oplus r^*\mathfrak{w})|B$ $= \mathfrak{X} \oplus (\mathfrak{z} \oplus (r^*\mathfrak{w}|B))$ is also trivial.

$k_{Top}B$ versus k_OB

- There is a canonical map $k_OB \to k_{Top}B$ that sends the s-class of ζ to the s-class of the underlying microbundle $|\zeta|$.
- Fact: If γ is a generator of $k_O(S^8) \cong \mathbb{Z}$, then $|\gamma| \in k_{Top}(S^8) \cong \mathbb{Z}$ is divisible by 7.
- Consequence 1: The map $k_OS^8 \to k_{Top}S^8$ is not surjective.

Corollary

There exists a topological manifold M_1 that cannot be smoothed.

Construction of M_1.

Choose an open set $U_1 \subset \mathbb{R}^q$ homotopy equivalent to S^8. Let $\mathfrak{X}$ be a microbundle over U_1 whose s-class does not belong to the image of the map $k_OU_1 \to k_{Top}U_1$. Then $\mathfrak{X}$ has a representative whose total space is a topological manifold M_1 that cannot be smoothed. □

$k_{Top}B$ versus $k_O B$ (continued)

- Consequence 2: If $X = S^7 \cup_7 e^8$, the map $k_O X \to k_{Top} X$ is not injective.

Corollary

The tangent vector bundle of a certain smooth manifold M_2 is not a topological invariant.

Construction of M_2.

Choose an open set $U_2 \subset \mathbb{R}^q$ homotopy equivalent to X. Then there exists a vector bundle ζ with fibre dimension k over U_2 that is not stably trivial but $|\zeta|$ is stably trivial. We may assume that $|\zeta|$ itself is trivial. Then some neighborhood M_2 of the zero section in $E(\zeta)$ and some neighborhood M_2' of the zero section in $E(\epsilon^k)$ are homeomorphic smooth manifolds but their tangent vector bundles are not isomorphic. $\square$

The Kister-Mazur Theorem

Let $\mathfrak{X}:\ B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle. We say that $\mathfrak{X}$ admits or contains a bundle if there exists an open neighborhood E' of $i(B)$ in E such that $j|E' : E' \to B$ is a topological bundle with fibre $\mathbb{R}^n$, structural group the origin-preserving self-homeomorphisms of $\mathbb{R}^n$, and zero section $i(B)$. Such a bundle is called admissible.

Theorem

(Kister, Mazur 1964) If a microbundle $\mathfrak{X}$ has base B which is an ENR, then $\mathfrak{X}$ admits a bundle, and this admissible bundle is unique up to isomorphism.

In principle, the Kister-Mazur theorem allows us to work with genuine $\mathbb{R}^n$-bundles. However, it is still convenient to work with microbundles, since, for example, the tangent microbundle of a topological manifold is canonical while the admissible tangent bundle is defined only up to isomorphism.

Normal microbundles

Let us assume that $M^m \subset N^n$ are topological manifolds with countable bases.

Definition

M has a microbundle neighborhood in N if there exists a neighborhood U of M in N and a retraction $j : U \to M$ such that the diagram

$$M \to U \xrightarrow{j} M$$

is a microbundle. We call it $\mathfrak{n}$, the normal microbundle of M in N.

If the neighborhood U and retraction j can be chosen such that $\mathfrak{n}$ is trivial, we say that M has a product neighborhood in N.

Existence of microbundle neighborhoods

- **Remark (1).** If M has a microbundle neighborhood in N, then M is locally flat in N.
- **Remark (2).** Having a locally flat embedding $i : M^m \to N^n$ does not guarantee the existence of a normal microbundle.

  Rourke and Sanderson [RS] gave an example of a locally flat PL embedding $g : S^{19} \times I \to S^{29}$ with no normal microbundle.
- **Lemma [M1].** If $i : M^m \to N^n$ is a closed embedding with retraction $N \xrightarrow{r} M$, then M has a microbundle neighborhood in $E(r^* \mathfrak{t}_M) \cong E(i^* \mathfrak{t}_N)$.
- **Consequence.** If $\mathfrak{t}_M$ is trivial, then $r^* \mathfrak{t}_M$ is trivial, and M has a microbundle neighborhood in $N \times \mathbb{R}^m$.

Microbundle neighborhoods in $N \times \mathbb{R}^q$

Again, let $M \xrightarrow{i} N$ be a closed imbedding. Since M is an ANR, we can assume there is a retraction $N \xrightarrow{r} M$.

Theorem

For $q >> 0$, $M \times 0$ has a microbundle neighborhood in $N \times \mathbb{R}^q$.

Proof.

1. Choose a microbundle $\mathfrak{y}$ over M such that $\mathfrak{t}_M \oplus \mathfrak{y} \cong \mathfrak{e}^q_M$.
2. M has a microbundle neighborhood in $E(r^*\mathfrak{t}_M)$, which has a microbundle neighborhood in $E(r^*\mathfrak{t}_M \oplus r^*\mathfrak{y})$, and thus M has a microbundle neighborhood in $E(r^*\mathfrak{t}_M \oplus r^*\mathfrak{y})$.
3. $r^*\mathfrak{t}_M \oplus r^*\mathfrak{y} \cong r^*(t_M \oplus \mathfrak{y}) \cong r^*\mathfrak{e}^q_M$. Therefore, $M \times 0$ has a microbundle neighborhood in $N \times \mathbb{R}^q$. ☐

Stable uniqueness of normal microbundles

Theorem

Given $M \subset N$ with normal microbundle $\mathfrak{n}$,

$$\mathfrak{t}_M \oplus \mathfrak{n} \cong \mathfrak{t}_N\big|_M.$$

If we pass to $k_{Top}M$, it follows that $(\mathfrak{t}_M) + (\mathfrak{n}) = i^*(\mathfrak{t}_N)$;
i.e.,$(\mathfrak{n}) = (i^*\mathfrak{t}_N) - (\mathfrak{t}_M)$.

Corollary

If the normal microbundle $\mathfrak{n}$ exists, then it is uniquely determined up to s-isomorphism.

The simplicial groups Top_m and PL_m

- Top_m is the simplicial group with typical k-simplex given by a microbundle isomorphism φ over Δ^k :

$$\varphi : \Delta^k \times \mathbb{R}^m \to \Delta^k \times \mathbb{R}^m$$

- Similarly define PL_m as the simplicial group with typical k-simplex an isomorphism $\Delta^k \times \mathbb{R}^m \to \Delta^k \times \mathbb{R}^m$ of m-dimensional PL microbundles over Δ^k.
- Let $BTop_m$, BPL_m denote the classifying spaces of the simplicial groups Top_m, PL_m.

$BTop_m, BPL_m$ classify m-dimensional microbundles

- Denote by $Micro_m(X)$ the set of isomorphism classes of microbundles of fibre dimension m.
- $PL - Micro_m(X) =$ the set of isomorphism classes of PL microbundles of fibre dimension m.

Theorem

BTop_m(resp. BPL_m) classifies topological (resp. PL) m microbundles with base X an ENR (resp. a polyhedron). that is, there is a one-to-one correspondence

$$Micro_m(X) \longleftrightarrow [X, BTop_m]$$

$$\text{(Resp) } PL - Micro_m(X) \longleftrightarrow [X, BPL_m]$$

The stable classifying spaces $BTop$ and BPL

- Using the natural inclusions $BTop_m \to BTop_{m+1}$, we see that any map $f : X \to BTop_m$ determines a map $X \to BTop := \operatorname{colim} BTop_m$ (resp. for PL).
- We get one to one correspondences

$$k_{Top}(X) \longleftrightarrow [X, BTop]$$

$$k_{PL}(X) \longleftrightarrow [X, BPL] \quad (\star)$$

$(\star)$ We must require X to be a polyhedron.

- Notice also that there is a forgetful map $BPL \to BTop$.

Classification of PL structures

Consider a polyhedron X and a topological m-microbundle $\mathfrak{z} : X \to E \to X$. Let $f : X \to BTop_m$ be the map that classifies $\mathfrak{z}$.

Theorem

There is a one-to-one correspondence between concordance classes of PL microbundle structures on $\mathfrak{z}$ and homotopy classes of lifts $\widetilde{f}$

$$\begin{tikzcd}
 & BPL_m \arrow[d] \\
X \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & BTop_m
\end{tikzcd}$$

Similarly, a lift $\tilde{f} : X \to BPL$ of the map $f : X \to BTop$ gives a PL microbundle structure on $\mathfrak{z} \oplus \mathfrak{e}^s_X$ for some $s \geq 0$ (and vice versa).

PL structures on topological manifolds

- By a PL structure on a topological manifold M^m, we mean a pair (K, f) where K is a PL manifold and $f : M \to K$ is a homeomorphism. Two structures $(K, f), (K', f')$ are equivalent if they are concordant.
- We would like to use microbundles to help us determine the existence of PL structures on $M^m \times \mathbb{R}^q, q >> 0$.
- **Method:** Get a PL microbundle structure on a bundle in the same s-class as $\mathfrak{t}_M$.
- **Problem:** PL microbundles are only defined over polyhedra.
- **Solution:** We can get a closed imbedding $i : M \to V$, where V is an open subset of $\mathbb{R}^n, n >> m$, and also a retraction $r : V \to M$ since M is an ENR. As V is an open subset of $\mathbb{R}^n$, it is a polyhedron. *We will put PL microbundle structure on a bundle in the same s-class as the induced bundle* $r^*\mathfrak{t}_M$ *over* V *and use it to get a PL structure on* $M \times \mathbb{R}^q, q >> 0$.

Lifts of $\widehat{\mathfrak{t}_M}$ determine PL structures on $r^*\mathfrak{t}_M \oplus \mathfrak{e}_V^s$

Recall $M^m \subset V^n$, with V an open subset of $\mathbb{R}^n$ and $r : V \to M$ a retraction.

Suppose we have a lift of $\widehat{\mathfrak{t}_M} : M \to BTop$, the map that classifies the stable tangent microbundle of M. This determines a lift $\widetilde{r^*\mathfrak{t}_M} : V \to BPL$ of the map $\widehat{r^*\mathfrak{t}_M}$ that classifies $r^*\mathfrak{t}_M$. Therefore, a lift of $\widehat{\mathfrak{t}_M} : M \to BTop$ determines a PL structure on $r^*\mathfrak{t}_M \oplus \mathfrak{e}_V^s$ for some $s \geq 0$ sufficiently large.

$$\begin{tikzcd}
 & & BPL \arrow[d] \\
V \arrow[r, "r"'] \arrow[urr, dashed, "\widetilde{r^*\mathfrak{t}_M}"] \arrow[rr, bend right, "\widehat{r^*\mathfrak{t}_M}"'] & M \arrow[r, "\mathfrak{t}_M"] \arrow[ur, dashed] & BTop
\end{tikzcd}$$

PL structure on $r^*\mathfrak{t}_M \oplus \mathfrak{e}^s_V \Rightarrow$ PL structure on $M \times \mathbb{R}^q$

- **Lemma.** There is an open embedding $\phi : E(r^*\mathfrak{t}_M) \to M \times \mathbb{R}^n$ so that we have commutativity below:

$$\begin{tikzcd}
M \arrow[r, "i"] \arrow[rrd, "()\times 0"'] & V \arrow[r] & E(r^*\mathfrak{t}_M) \arrow[d, "\phi"] \\
 & & M \times \mathbb{R}^n
\end{tikzcd}$$

  where $\phi((y, r(y), x)) = (x, x - y)$.

- Notice: A PL structure on $r^*\mathfrak{t}_M \oplus \mathfrak{e}^s_V$ determines a PL structure on a neighborhood of the zero section in $E(r^*\mathfrak{t}_M \oplus \mathfrak{e}^s_V) \cong E(r^*\mathfrak{t}_M) \times \mathbb{R}^s$.
- Then $\phi \times id_{\mathbb{R}^s}$ determines a PL structure on an open neighborhood W of the zero section $M \times 0$ in $\phi(E(r^*\mathfrak{t}_M)) \times \mathbb{R}^s \subset M \times \mathbb{R}^n \times \mathbb{R}^s \cong M \times \mathbb{R}^{n+s}$.

PL structure on $r^*\mathfrak{t}_M \oplus \mathfrak{e}^s_V \Rightarrow$ PL structure on $M \times \mathbb{R}^q$

- Have: PL structure on a neighborhood $W \subset M \times \mathbb{R}^{n+s}$ that contains $M \times 0$.
- Want: PL structure on $M \times \mathbb{R}^{n+s}$ itself.
- **Theorem.** If W is an open neighborhood of $M \times 0$ in $M \times \mathbb{R}^q$, then W contains an open neighborhood W' of $M \times 0$ which is homeomorphic to $M \times \mathbb{R}^q$.
- **Consequence:** $W' \cong M \times \mathbb{R}^{n+s}$ inherits PL structure from W, and therefore we have a PL structure on $M \times \mathbb{R}^{n+s}$ determined by the PL microbundle structure on $r^*\mathfrak{t}_M \oplus \mathfrak{e}^s_V$.

Theorem

If M^m is a topological manifold with tangent microbundle $\mathfrak{t}_M$, with classifying map $\widehat{\mathfrak{t}_M} : M \to BTop$, then a lifting of $\widehat{\mathfrak{t}_M}$ to BPL determines a PL structure on $M \times \mathbb{R}^q$, for some $q >> 0$.

A converse

Theorem

Let M be a topological manifold. A PL structure on $M \times \mathbb{R}^q$ determines a PL microbundle structure on $r^*(\mathfrak{t}_M) \oplus \mathfrak{e}_V^q$ and hence a lift of $\widehat{\mathfrak{t}}_M : M \to BTop$ to $\widetilde{\mathfrak{t}}_M : M \to BPL$.

Proof.

Let (K, f) be a PL structure on $M \times \mathbb{R}^q$. Consider the map $F = f \circ (r \times id_{\mathbb{R}^q}) : V \times \mathbb{R}^q \to K$. Approximate F by a PL map $\widehat{F}$, with F homotopic to $\widehat{F}$. Then $\widehat{F}^*(\mathfrak{t}_K)$ is a PL microbundle, and the Homotopy Theorem gives us a topological microbundle isomorphism from $F^*(\mathfrak{t}_K)$ to $|\widehat{F}^*(\mathfrak{t}_K)|$, i.e. $\widehat{F}^*(\mathfrak{t}_K)$ gives a PL microbundle structure on $F^*(\mathfrak{t}_K)$. But

$$F^*(\mathfrak{t}_K) \cong (r \times id_{\mathbb{R}^q})^*(f^*\mathfrak{t}_K) \cong (r \times id_{\mathbb{R}^q})^*\mathfrak{t}_{M\times\mathbb{R}^q} \cong r^*(\mathfrak{t}_M) \oplus \mathfrak{e}_V^q.$$

Thus $\widehat{F}^*(\mathfrak{t}_K)$ gives a PL microbundle structure on $r^*(\mathfrak{t}_M) \oplus \mathfrak{e}_V^q$.

A converse

Proof, contd.

PL microbundle structure on $r^*(\mathfrak{t}_M) \oplus \mathfrak{e}_V^q$ corresponds to a lift $\widetilde{r^*\mathfrak{t}_M}$ of the classifying map $\widehat{r^*\mathfrak{t}_M}$. Using i and the fact that $ri = id_M$, we see that $\widehat{r^*\mathfrak{t}_M}$ determines a lift of $\widehat{\mathfrak{t}_M}$.

$$\begin{tikzcd}
 & & & BPL \arrow[d] \\
M \arrow[r, "i"] \arrow[rrr, bend right, "\widehat{\mathfrak{t}_M}"'] & V \arrow[r, "r"'] \arrow[urr, dashed, "\widetilde{r^*\mathfrak{t}_M}"] & M \arrow[r] & BTop
\end{tikzcd}$$

$\square$

PL structures on topological manifolds

Theorem

Let M be an m-dimensional topological manifold. For large $q \geq 1$ the concordance classes of PL structures on $M \times \mathbb{R}^q$ are in one-one correspondence with the homotopy classes of lifts of the stable tangent bundle $\widehat{\mathfrak{t}_M} : M \to BTOP$ to $\widetilde{\mathfrak{t}_M} : M \to BPL$.

References

[KS] R. Kirby and L. Siebenmann: *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Math. Studies 88, Princeton University Press (1977).

[K] J. Kister: Microbundles are fibre bundles, *Ann. of Math. (2)* **80** (1964), 190–199.

[M1] J. Milnor: Microbundles: Part I, *Topology* **3**, Suppl. 1 (1964), 53–80.

[M2] J. Milnor: Topological manifolds and smooth manifolds, *Proceedings of the International Congress of Mathematicians*, Stockholm (1962), 132-138.

[RS] C. Rourke and B. Sanderson: An embedding without a normal microbundle, *Invent. Math.* **3** (1967), 293–299.