# CORRECTED PROOF OF RESULT PROPOSITION 2.26 FROM CLASS

## MATH 80210, Prof. Sam Evens

I gave an incorrect proof of the following result in class. These notes give a correct proof. The end of the proof follows the idea of the proof of Lemma 2.22 in Mukai instead of quoting Lemma 2.22 in Mukai. Also, this statement is not really the same as Proposition 2.26 in Mukai. It plays the same role in the proof, so I gave it the same name.

**Proposition 2.26.** *Let $B$ be an integral domain. Suppose that $A \subset B$ is a subring and $B$ is a finitely generated $A$-algebra. Then there exists $\gamma \in A^* := A - \{0\}$ and $b_1, \dots, b_m \in B$ such that $B[\gamma^{-1}]$ is integral over $A[\gamma^{-1}][b_1, \dots, b_m]$ and $b_1, \dots, b_m$ is algebraically independent over $A[\gamma^{-1}]$.*

*Proof:.* Let $S = A^*$ and let $K = S^{-1}A$, the fraction field of the integral domain $A$. Since $B$ is a finitely generated $A$-algebra, it follows easily that $S^{-1}B$ is a finitely generated $K$-algebra. Since $B$ is an integral domain, the canonical homomorphism $B \to S^{-1}B$ is injective, and we will regard $B$ as a subring of $S^{-1}B$. Therefore, by the Noether Normalization Lemma, there are $\alpha_1, \dots, \alpha_m \in S^{-1}B$ such that $S^{-1}B$ is integral over $K[\alpha_1, \dots, \alpha_m]$ and $\alpha_1, \dots, \alpha_m$ are algebraically independent over $K$. Since each $\alpha_i \in S^{-1}B$, $\alpha_i = \frac{b_i}{s_i}$ for some $b_i \in B$, $s_i \in S$.

Since $\alpha_1, \dots, \alpha_m$ are algebraically independent over $K$, the rings $K[\alpha_1, \dots, \alpha_m]$ and $K[t_1, \dots, t_m]$ are isomorphic. It is easy to show that $K[b_1, \dots, b_m] = K[\alpha_1, \dots, \alpha_m]$ since each $\frac{1}{s_i} \in K$. It follows that $b_1, \dots, b_m$ are algebraically independent over $K$, since if they were algebraically dependent, the ring they generate would have fraction field with transcendence degree over $K$ strictly less than $m$, which contradicts the fact that the fraction field of $K[t_1, \dots, t_m]$ has fraction field of dimension $m$ over $K$.

Let $y \in B \subset S^{-1}B$. Then since $S^{-1}B$ is integral over $S^{-1}A[b_1, \dots, b_m]$, there is a monic polynomial $f(t) = t^n + \beta_{n-1}t^{n-1} + \dots + \beta_0$ with $\beta_i \in K[b_1, \dots, b_m]$. It is easy to check that $K[b_1, \dots, b_m] = S^{-1}(A[b_1, \dots, b_m])$, so each $\beta_i = \frac{c_i}{d_i}$ for some $c_i \in A[b_1, \dots, b_m]$ and some $d_i \in S$. Let $d_y = \Pi_{i=0}^{n-1} d_i \in S$. Let $g(t) = d_y \cdot f(t) \in A[b_1, \dots, b_m][t]$. Then $f(t) \in A[{d_y}^{-1}][b_1, \dots, b_m]$ is monic and $f(y) = 0$, so $y$ is integral over $A[{d_y}^{-1}][b_1, \dots, b_m]$.

Since $B$ is a finitely generated $A$-algebra, $B = A[y_1, \dots, y_r]$ for some $y_1, \dots, y_r \in B$. Let $e_i = d_{y_i}$, so that $y_i$ is a root of a monic polynomial $f_{y_i}(t)$ in the ring $A[{e_i}^{-1}][b_1, \dots, b_m][t]$, and hence $y_i$ is integral over $A[{e_i}^{-1}][b_1, \dots, b_m]$. Let $\gamma = \Pi_{i=1}^r e_i$. Then since $\frac{1}{e_i} = \frac{e_1 \dots e_{i-1} e_{i+1} \dots e_r}{\gamma}$, it follows that $A[{e_i}^{-1}] \subset A[\gamma^{-1}]$. It follows that each $y_i$ is integral over $A[\gamma^{-1}][b_1, \dots, b_m]$. Since $A[\gamma^{-1}][b_1, \dots, b_m]$ and $y_1, \dots, y_r$ are all integral over $A[\gamma^{-1}][b_1, \dots, b_m]$, it follows from Corollary 20, page 667 in Dummit and Foote that $A[\gamma^{-1}][b_1, \dots, b_m][y_1, \dots, y_r]$ is integral over $A[\gamma^{-1}][b_1, \dots, b_m]$. But $A[\gamma^{-1}][b_1, \dots, b_m][y_1, \dots, y_r] = B[\gamma^{-1}]$ since $B = A[y_1, \dots, y_r]$. Hence $B[\gamma^{-1}]$ is integral over $A[\gamma^{-1}][b_1, \dots, b_m]$. Finally, since the set $b_1, \dots, b_m$ is algebraically independent over $K$, it is algebraically independent over the subring $A[\gamma^{-1}]$ of $K$.

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