Math 60210, Basic Algebra, Problem Set 9, Fall 2009
due Tues, November 10
Do 7 of these problems

1. Let F be a field and let $R = F[x]$, the polynomial ring in one variable over F. Let $\alpha, \beta \in F$ and assume that $\alpha \neq \beta$. Let $I_\alpha = (x-\alpha)$ and $I_\beta = (x-\beta)$ be the principal ideals in R generated by $x - \alpha$ and $x - \beta$ respectively. Prove that I_α and I_β are relatively prime.

2. Let $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$. If $\alpha \in R$ is nonzero, prove that $R/(\alpha)$ is finite. Conclude that if $I \subset R$ is a nonzero ideal, then R/I is finite.

3. Let R be a commutative ring. Prove that the set of prime ideals of R has a minimal element, i.e., there is a prime ideal I of R such that if $J \subset R$ is a prime ideal, then if $J \subset I$, then $J = I$ (hint: use Zorn's Lemma, and order prime ideals so $J \leq I$ if $I \subset J$. Then a minimal prime ideal is a maximal element of the set of prime ideals according to this ordering).

4-5. Let R_1 and R_2 be rings. Prove that $(R_1 \times R_2)^* = R_1^* \times R_2^*$. Prove that if m and n are relatively prime integers, then $\phi(mn) = \phi(m)\phi(n)$, where $\phi(k) = |\mathbf{Z}_k^*|$ (hint: use the Chinese Remainder Theorem to relate $\mathbf{Z}_{mn}$ to $\mathbf{Z}_m$ and $\mathbf{Z}_n$.) Compute $\phi(p^n)$ for p prime, and give a formula for $\phi(n)$ in terms of the prime factorization of n.

6. Let F be a field and let $R = F[[x]]$ be the ring of formal power series over F. Show that every nonzero ideal of R is a principal ideal (x^k) for some k. What are the prime ideals of R?

7. Let $\phi : R \to S$ be a ring homomorphism of commutative rings. If $P \subset S$ is a prime ideal, show that $\phi^{-1}(P)$ is a prime ideal of R. If $M \subset S$ is a maximal ideal, is $\phi^{-1}(M)$ necessarily a maximal ideal of R? Eplxain why or why not.

8. Ash, 2.5, problem 7.

9. Ash, 2.5, problem 9.

10. Prove that $\mathbf{Z}[x]$ is not isomorphic to $\mathbf{Q}[x]$.

11. (D+F, problem 8, p. 256) Let R be an integral domain and let $a, b \in R$. Prove that $(a) = (b)$ if and only if $b = ua$ for some $u \in R^*$.

12. (D + F, problem 7, p. 256) Let R be a commutative ring and let $A = R[x]$. Prove that (x) is a prime ideal of A if and only if R is an integral domain. Prove that (x) is a maximal ideal of A if and only if R is a field.