

Math 60210, Basic Algebra, Problem Set 8, Fall 2009
due Tues, November 3
Do 7 of these problems

1. Let R be an integral domain and let $R[x]$ be the polynomial ring over R (see Ash, Example 2.1.3, # 5). For $f = \sum_{i=0}^n a_i x^i \in R[x]$ with $a_n \neq 0$, then by definition, $\deg(f) = n$. By convention, $\deg(0) = -\infty$. Prove that if $f, g \in R[x]$, then $\deg(f \cdot g) = \deg(f) + \deg(g)$ (use convention $-\infty + k = -\infty$ for $k \in \mathbf{Z}$). Prove that if $f, g \in R[x]$, then $R[x]$ is an integral domain and compute the set of units of $R[x]$.
2. Let R be an integral domain and let $R[[x]]$ be the ring of formal power series over R . Prove that $R[[x]]$ is an integral domain and compute the set of units of $R[[x]]$.
3. Let $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$. R is a subring of the integers. Consider the ideal $I = (2, 1 + \sqrt{-5})$. Prove that $R/I = \{0 + I, 1 + I\}$ has exactly two elements.
4. Let R be a ring and let I and J be two-sided ideals of R . Let $I + J = \{x + y : x \in I, y \in J\}$. Let $I \cdot J = \{\sum_{i=1}^n x_i y_i : x_i \in I, y_i \in J\}$. Prove that $I + J$, $I \cap J$, and $I \cdot J$ are two-sided ideals of R . If $R = \mathbf{Z}$ and $I = m\mathbf{Z}$ and $J = n\mathbf{Z}$ for integers m and n , compute $I + J$, $I \cap J$, and $I \cdot J$.
5. Let R be a ring with ideals I and J . Prove $I \cdot J \subset I \cap J$. If $R = I + J$, prove that $I \cap J = I \cdot J$. Give an example of a ring R with ideals I and J such that $I \cap J \neq I \cdot J$.
6. Let R be a Boolean ring (defined in problem set 7). Prove that the characteristic of R is 2.
7. Let F be a field. Prove that $M(n, F)$ is a simple ring.
8. Let F be a field and let $R = M(n, F)$. Let $x = E_{i,j}$ be the matrix in R which is 1 in the entry in the i th row and j th column, and is 0 in all the remaining $n^2 - 1$ entries. Show that the left ideal $Rx = \{A \in R : A(e_k) = 0 \forall k \neq j\}$. Show that the right ideal $xR = \{A \in R : A^{tr}(e_k) = 0 \forall k \neq i\}$.
9. Let F be a field. Determine the center of the ring $M(n, F)$ (Recall that the center $Z(R)$ of a ring R is $Z(R) := \{a \in R : ab = ba \forall b \in R\}$).
10. Dummit and Foote, Problem 12, 7.3, page 248.
11. Let R be a commutative ring and recall that $a \in R$ is called nilpotent if there is a positive integer n such that $a^n = 0$. Prove that the set of nilpotent elements of R forms an ideal $N(R)$ (note: $N(R)$ is called the nilradical of R). Compute the nilradical of $\mathbf{Z}/p^n\mathbf{Z}$.
12. Let $R = \mathbf{Z}[i]$, which is a subring of \mathbf{C} . Let $p \in \mathbf{Z}$ be a prime of \mathbf{Z} . Let $(p) = R \cdot p$ be the principal ideal generated by p in R . Prove that $|R/(p)| = p^2$. For $p = 3$, is $R/(p)$ a field? For $p = 5$, is $R/(p)$ a field?
13. Let R be a commutative ring and let $I \subset R$ be an ideal. Let $R[x]I$ be the ideal of $R[x]$ generated by I . Prove that $R[x]/R[x]I \cong (R/I)[x]$.