Math 60210, Basic Algebra, Problem Set 7, Fall 2009
due Tues, October 27
Do 7 of these problems

1. Let $F = \mathbf{Z}/p\mathbf{Z}$. Determine the number of p-Sylow subgroups of $SL(2, F)$.

2.

(A) Let R be a ring. An element $x \in R$ is called nilpotent if $x^n = 0$ for some $n > 0$. If $x \in R$ is nilpotent, prove that $1 + x \in R^*$.

(B) If F is a field and $A \in M(n, F)$ has the property that the entries $A_{i,j} = 0$ for $i \geq j$, prove that $(I + A)$ is invertible and give a formula for $(I + A)^{-1}$.

3-4. Let F be a field and let $V = F^n$, regarded as column vectors.

(A) Let $u_k(F) = \sum F \cdot E_{i,j}$, where the sum is over pairs (i, j) such that $j - i \geq k$. For $r = 1, \ldots, n$, let $V_r = \sum_{i=1}^{r} F \cdot e_i$. Prove that $u_k(F) = \{A \in M(n, F) : A \cdot V_r \subset V_{r-k}, r = 1, \ldots, n\}$ (by convention, $V_r = 0$ if $r \leq 0$). (Note: $u_k(F)$ is the set of matrices A such that the (i, j) entry $A_{i,j} = 0$ unless $j - i \geq k$)

(B) Let $U_k(F) = \{I + A : A \in u_k(F)\}$ for $k = 1, \ldots, n - 1$, where I is the $n \times n$ identity matrix. Prove that $U_k(F)$ is a normal subgroup of $T(n, F)$.

(C) Define a map $\psi : U_k(F)/U_{k+1} \to u_k(F)/u_{k+1}(F)$ by $\psi(I + A) = A$. Prove that ψ is a well-defined injective group homomorphism.

(D) Prove that $U_k(F)/U_{k+1}(F)$ is abelian and $T(n, F)$ is solvable.

5. Let notation be as in Problem 3-4 above.

(A) Let $g \in U_1(F)$ and let $B \in u_k(F)$. Prove that $gBg^{-1} - B \in u_{k+1}(F)$.

(B) For $g \in U_1(F)$ and $x \in U_k(F)$, prove that $gxg^{-1} - x \in u_{k+1}(F)$.

(C) Prove that if $g \in U_1(F)$ and $x \in U_k(F)$, then $[g, x] \in U_{k+1}(F)$ (hint: if $v \in V_r$, $v = x^{-1} \cdot u$ for some $u \in V_r$. Now apply Part (C)).

(D) Prove that $U(n, F) = U_1(F)$ is nilpotent.

6. Let G be a group of order p^r, $r > 0$. Show that G has a normal subgroup of order p^k for each $k \leq r$.

7. Let G be a group. Show that each of the subgroups $G_{(r)}$ defined in Ash, 5.7, is normal in G.

8. Let d be a square-free integer, i.e., m^2 does not divide d for every integer $m \geq 2$. Let $E = \mathbf{Q}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbf{Q}\}$. Prove that E is a subring of $\mathbf{C}$. Is E a field? Explain why or why not.

9. Let d be a square-free integer. Let $R = \mathbf{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbf{Z}\}$. Prove that R is a subring of $\mathbf{C}$. Compute the unit group R^* of R if $d < 0$.

10. Let $\zeta = e^{2\pi i/3}$. Let $R = \{a + b\zeta : a, b \in \mathbf{Z}\}$. Prove that R is a subring of $\mathbf{C}$ and compute the unit group R^*. Does R contain any of the rings $\mathbf{Z}[\sqrt{d}]$ from Problem 9. If so, which one?

11. Ash, Section 2.1, problem 11.

12. Let R be an integral domain and suppose $x \in R$ and $x^2 = 1$. Prove that $x = 1$ or -1.

13. Let R be a ring and let $a \in R$. Let $C(a) = \{x \in R : xa = ax\}$. Prove that $C(a)$ is a subring of R.

14. Let R be a ring and let $\{S_i\}_{i \in I}$ be a family of subrings of R. Prove that $\cap_{i \in I} S_i$ is a subring of R. Let $Z(R) = \{x \in R : xy = yx, \ \forall\ y \in R\}$. Show $Z(R) = \cap_{a \in R} C(a)$.

15. A ring R is called *boolean* if $a^2 = a$ for every $a \in R$. Prove that if R is a boolean ring, then R is commutative.