Math 60210, Basic Algebra, Problem Set 6, Fall 2009
due Tues, October 13

Do 6 of these problems; note that some of the problems count as more than one problem. For example, if you do 10-11-12, you only need to do three other problems.

1. Let G be a finite group with normal subgroup N. Prove that there exists a composition series $G_0 = \{e\} \subset G_1 \subset \cdots \subset G_r = G$ such that $G_i = N$ for some i.

2. Let $G = \mathbf{Z}/28\mathbf{Z}$. Find two different composition series for G and verify that the composition factors are the same up to reordering.

3. (A) Let $G = D_{56}$. Find a composition series for G.

(B) Explain how to find a composition series for D_{2n} for any integer n (this can be done in rough terms, i.e., just give the idea).

4. Let $F = \mathbf{Z}/3\mathbf{Z}$. Give a composition series for $GL(2, F)$.

5. Let G be a nonabelian group of order p^3 where p is prime. Let $Z = Z(G)$, and let $C = \mathbf{Z}/p\mathbf{Z}$.

(A) Prove that $Z \cong C$ and $G/Z \cong C \times C$.

(B) Let H be a subgroup of G of order p^2. Prove that H is normal and $Z \subset H$.

6-7. Let G be a nonabelian group of order 8. Prove that either $G \cong D_8$ or $G \cong Q_8$ (there are hints on the course website).

8. Let F be a field and assume $|F| > 2$. Show that $T(n, F)$ is solvable but not nilpotent, and show that $U(n, F)$ is nilpotent.

9. Let G be a finite group with normal subgroup H. Let p be a prime dividing $|H|$ and suppose that p and $[G : H]$ are relatively prime. Prove that every p-Sylow subgroup of G is contained in H.

10-11-12. Let G be a group of order 60 and let n_5 be the number of 5-Sylow subgroups. Prove that if $n_5 > 1$, then G is a simple group. Prove that A_5 is simple (you can use the following series of hints):

(A) Show that $n_5 = 6$.

(B) Let P be a 5-Sylow subgroup. Show that $|N_G(P)| = 10$ (use proof of Sylow (2) and (3)).

(C) Let $H \subset G$ be a normal subgroup and suppose 5 divides $|H|$. Prove that all six 5-Sylow subgroups of G are contained in H.

(D) Let H be as in (C). Prove that if H is a proper normal subgroup, then $|H| = 30$.

(E) Let H be as in (D). Prove that the number of 3-Sylow subgroups or the number of 5-Sylow subgroups of G is one, and conclude that $|H|$ cannot be a multiple of 5.

(F) Suppose there exists a proper normal subgroup H and $|H| = 6$. Prove that G has a normal 3-Sylow subgroup, which contradicts the assumption that G is simple.

(G) Suppose there exists a proper normal subgroup of order 12. Prove that G has a normal 2-Sylow subgroup or a normal 3-Sylow subgroup.

(H) Suppose there exists a proper normal subgroup H of order 2, 3 or 4. Prove that G/H has a normal 5-Sylow subgroup, and use this to prove that G has a proper normal subgroup of order divisible by 5, contradicting (E). Use the correspondence theorem for this, and for one of these cases, you may want to recall that a group of order 30 has a unique 5-Sylow subgroup.

13.

(A) Show that every conjugacy class in $SL(2, \mathbf{C})$ meets $T(2, \mathbf{C})$.

(B) Does every conjugacy class in $SL(2, F)$ meet $T(2, F)$, where $F = \mathbf{Z}/3\mathbf{Z}$ (hint: use eigenvalues of matrices as a way to organize conjugacy classes)? If not, find a conjugacy class in $SL(2, F)$ that does not meet $T(2, F)$.