Math 60210, Basic Algebra, Problem Set 5, Fall 2009
due Tues, September 29
Do 7 of these 12 problems

1. Let $\sigma = (1\ 2) \in S_n$, and let C_σ be the conjugacy class of σ.

(A) Compute $|C_\sigma|$.

(B) Compute the subgroup $C_{S_n}(\sigma)$, i.e., determine the elements of S_n which centralize σ.

2. Determine the conjugacy classes in the dihedral group $G = D_{2p}$, where p is an odd prime. For each conjugacy class C, compute $C_G(x)$ for some $x \in C$.

3. Determine the conjugacy classes in $G = D_{24}$. For each conjugacy class C in G, compute $C_G(x)$ for some $x \in C$.

4. Let F be a field and let $SL(n, F) := \{A \in GL(n, F) : \det(A) = 1\}$. Show $SL(n, F)$ is a group and determine $|SL(n, F)|$ if F is a finite field with q elements.

5. Let F be a field. Let $FP(n) = \{v \in F^{n+1} - \{0\}\}/\sim$, where $v \sim w$ if $v = \lambda w$ for some $\lambda \in F - \{0\}$.

(A) Construct a bijection between $FP(n)$ and the set of all one-dimensional linear subspaces of F^{n+1}.

(B) Show that $GL(n+1, F)$ acts on $FP(n)$ by the formula $(g, [v]) \mapsto [g \cdot v]$, where $[u]$ is the equivalence class in $FP(n)$ defined by a nonzero vector $u \in F^{n+1}$, and $(g, v) \mapsto g \cdot v$ is the usual action of matrices on column vectors. Show the $GL(n+1, F)$ action on $FP(n)$ is transitive and compute the stabilizer of the point $[e_1]$.

(C) Let $e_1, \ldots, e_{n+1}$ be the standard basis of F^{n+1}, and let $FP(n)_0 = \{[v] \in FP(n) : v = \sum_{i=1}^{n+1} a_i e_i, a_{n+1} \neq 0\}$. Construct a bijection $FP(n)_0 \to F^n$.

6. Let $n = 1$ in the previous problem. The subgroup $SL(2, \mathbf{R})$ of $GL(2, \mathbf{C})$ acts on $\mathbf{C}P(1)$ by restricting the $GL(2, \mathbf{C})$ acton. Prove that $SL(2, \mathbf{R})$ has precisely three orbits on $\mathbf{C}P(1)$, given by

$$H_+ = \{[(z, 1)] : z = x + yi, y > 0\},$$
$$H_- = \{[(z, 1)] : z = x - yi, y > 0\},$$
$$\mathbf{R}P(1) = \{[(u, v)] : u, v \in \mathbf{R}\}.$$

Identify the stabilizers of the points $(i, 1) \in H_+$, $(-i, 1) \in H_-$, and $(1, 0) \in \mathbf{R}P(1)$ (note: H_+ and H_- are called the upper and lower half planes. If we use part (C) of the previous problem, we may regard H_+ and H_- as subsets of the complex plane $\mathbf{C}$, and then H_+ consists of complex numbers with imaginary part greater than zero, etc. It may be useful to compute the action of $SL(2, \mathbf{R})$ on $H_\pm$ using the identification of $H_\pm$ as a subset of the complex plane $\mathbf{C}$. It may be useful to understand the action of $D(2, \mathbf{R}) \cap SL(2, \mathbf{R})$ and $T(2, \mathbf{R})$ on $H_\pm$ viewed as a subset of $\mathbf{C}$).

7. Let $V = \mathbf{R}^n$, and let $G = O(n, \mathbf{R}) = \{g \in GL(n, \mathbf{R}) : g^{tr} = g^{-1}\}$. Determine the G-orbits on V. Show $O(n-1, \mathbf{R}) \cong H := \{g \in G : g \cdot e_n = e_n\}$. Construct a bijection $O(n, \mathbf{R})/H \to S^{n-1}$, where $S^{n-1} = \{v \in V : v \cdot v = 1\}$ is the usual $n-1$-dimensional sphere.

8. Let F be a field with 3 elements. Show that $|SL(2, F)| = |S_4|$. Is $SL(2, F) \cong S_4$? Explain your answer.

9. Let G be a nonabelian group of order 6 and let H be a subgroup of order 2. Let $X = G/H$. Construct a group isomorphism $\phi : G \to A_X$. Prove that $G \cong S_3$.

10. (see D+F, 4.2, problem 8)

(A) Let H be a subgroup of a group G and suppose $[G : H] = n$ is finite. Then there exists a normal subgroup K of G such that $K \subset H$ and $|G/K| \leq n!$.

(B) Suppose in part (A), G is finite and $|G|$ does not divide $n!$. Prove that G is not simple.

11. Let X be a G-set and let $x \in X$. For $g \in G$, let $y = g \cdot x$. Prove that $gG_xg^{-1} = G_y$.

12. Find 2-Sylow subgroups of S_4, S_5, and S_6. For extra credit, describe how you would find a 2-Sylow subgroup of S_n for any n (hint: D_n acts by permutations on the vertices of a regular n-gon, and hence may be viewed as a subgroup of S_n).