Math 60210, Basic Algebra, Problem Set 4, Fall 2009
due Tues, September 22
Do 7 of these problems

1. Let $\sigma \in S_n$.

(A) If $\tau = (i_1, \ldots, i_k)$ is k-cycle in S_n, prove that $\sigma\tau\sigma^{-1} = (\sigma(i_1), \ldots, \sigma(i_k))$.

(B) If τ and η are in S_n prove that there is σ in S_n such that $\sigma\tau\sigma^{-1} = \eta$ if and only τ and η have the same cycle decomposition, i.e., when they are written as products of disjoint cycles, the cycles that appear in τ have the same lengths as they cycles that appear in η.

2. Let $G = S_4$. Prove that there exists a normal subgroup V of S_4 such that $S_4/V \cong S_3$ (hint on course website).

3. Prove that S_n is generated by its simple transpositions $\tau_1, \tau_2, \ldots, \tau_{n-1}$, where $\tau_i = (i, i+1)$ is the transposition interchanging i and $i+1$. Show that $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ (these are called braid relations; there is a picture relating them to diagrams of braided strings).

4. Prove that $S_n =< \sigma, \tau >$, where $\sigma = (1, 2, 3, \ldots, n)$ is the n-cycle and $\tau = (1, 2)$.

5. Let $Q_8 = \{1, -1, I, -I, J, -J, K, -K\}$, the quaternion group (cf. class discussion or Dummet and Foote, page 36). Prove that Q_8 is a nonabelian group. Is $Q_8 \cong D_8$? Explain why or why not.

6. Read Exercise 5.8.8 in Ash. Show that the multiplication defined in that problem on $N \times H$ makes $N \times H$ into a group, denoted $N \rtimes_f H$. If A is a subgroup of H, show that $N \rtimes_f A$ is a subgroup of $N \rtimes_f H$. Show that if $f(x)$ is the identity operator on N for all $x \in H$, then $N \rtimes_f H \cong N \times H$, the direct product group.

7. Let G be a group with subgroups H and N, and suppose N is normal, $G = NH$, and $H \cap N = \{e\}$. Let $\phi : H \to Aut(N)$ denote the map $\phi(x) = c_x$, where $c_x(y) = xyx^{-1}$. Prove that $N \rtimes_\phi H \cong G$, using the definition of $N \rtimes_\phi H$ given in class.

8. Let

$$1 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 1$$

be a short exact sequence of groups, where f_1 and f_2 are group homomorphisms. Prove that f_1 is injective, f_2 is surjective, and $G_3 \cong G_2/f_1(G_1)$.

9. Let H_1, H_2, N_1, N_2 be groups and suppose that $\phi_1 : H_1 \to Aut(N_1)$ and $\phi_2 : H_2 \to Aut(N_2)$ are group homomorphisms. Suppose $\chi : H_1 \to H_2$ and $\psi : N_1 \to N_2$ are group isomorphisms. Define $c_\psi : Aut(N_1) \to Aut(N_2)$ by the formula $c_\psi(f) = \psi \circ f \circ \psi^{-1}$ for $f \in Aut(N_1)$. Suppose that $c_\psi \circ \phi_1 = \phi_2 \circ \chi$. Prove that $N_1 \rtimes_{\phi_1} H_1$ is isomorphic to $N_2 \rtimes_{\phi_2} H_2$.

10. Show that if F is a field, then $T(n, F)$ is the semidirect product of its subgroups $U(n, F)$ and $D(n, F)$. Show that the dihedral group D_{2n} is the semidirect product of its subgroups $< r >$ and $< s >$, where r is rotation by an angle of $2\pi/n$ and $s(x, y) = (x, -y)$ is reflection through the x-axis.

Problems from Ash:

1.5: 4, 5.

5.8: 7, 11.