HOMEWORK 3, MATH 60210, BASIC ALGEBRA, DUE TUESDAY, SEPT. 15
INSTRUCTOR, SAM EVENS, FALL 2009

INSTRUCTIONS: Do 7 of these 13 problems.

1. Let X be a G-set, where G is a group. Define a map $\phi : G \to A_X$ by $\phi(g) = l_g$, where $l_g(x) = g \cdot x$ for $x \in X$. Prove that ϕ is a group homomorphism, and $Ker(\phi) = \cap_{x \in X} G_x$.

2. For a group G, let $Int(G) = \{c_x : x \in G\}$, where $c_x(g) = xgx^{-1}$. Let $Aut(G)$ is the set of all isomorphisms $\phi : G \to G$.

(a) Prove that $Aut(G)$ is a subgroup of A_G.

(b) Prove that $Int(G)$ is a normal subgroup of $Aut(G)$, and $Int(G) \cong G/Z(G)$.

3. Ash, 1.4, problem 6.

4. (see D+F, problem 18 of 3.2): Let G be a finite group and let H and N be subgroups of G with N normal in G. Prove that if $|H|$ and $[G : N]$ are relatively prime, then $H \subset N$.

5. Prove that $\mathbf{R}/\mathbf{Z} \cong S^1$, where $S^1 = \{z \in \mathbf{C}^* : |z| = 1\}$.

6. Prove that $\mathbf{C}/\mathbf{Z} \cong \mathbf{C}^*$.

7. Let F be a field. Prove that $T(n, F)/U(n, F) \cong D(n, F)$.

8.Let k and n be positive integers, and consider the group homomorphism $\pi : \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$. Show $\pi(k\mathbf{Z}) = \pi(d_k\mathbf{Z})$ for some d_k dividing n, and compute d_k in terms of n and k.

9. Recall the dihedral group $G = D_{2n}$ introduced in lecture, or as presented in Ash, 1.2.3, so $|D_{2n}| = 2n$. Compute $Z(G)$ (it depends on n).

10. Compute the group $Int(D_{2n})$.

11. Let $G = D_{2n}$, so G is the group generated by a rotation $r = r_{2\pi/n}$ and a reflection s, where $s(x, y) = (x, -y)$. Let k be a divisor of n, and let $H_k := \{r^{jk} : j = 0, \ldots, \frac{n}{k}\}$. Prove that H_k is a normal subgroup of G.

12. (notation from previous problem) Is G/H_k a dihedral group D_{2m}? If so, which one? Prove that your answer is correct.

13. (cf. D+F, problem 6) Show that $\mathbf{R}^*/\{\pm 1\} \cong \mathbf{R}^{>0}$, the group of positive real numbers under multiplication.