HOMEWORK 2, MATH 60210, BASIC ALGEBRA, DUE TUESDAY, SEPT. 8
INSTRUCTOR, SAM EVENS, FALL 2009

INSTRUCTIONS: Do 7 of these 12 problems.

1. Let X be a set and suppose $|X| \geq 3$. Show that A_X, the set of bijections of X, is a nonabelian group (cf. Ash, 1.2, problem 5).

2. Ash, 1.3, problem 4.

3. Ash, 1.3, problem 6.

4. Ash, 1.3, problem 8.

5. Ash, 1.3, problem 9.

6. Let G be a group and let H and K be two subgroups of G of finite index, i.e., $[G : H]$ and $[G : K]$ are finite. Prove that $H \cap K$ is a subgroup of finite index in G.

7. If x is an element of a group G, we say x is a torsion element if the order $|x|$ is finite.

(a) If G is abelian, show the set G_{tor} of all torsion elements of G is a subgroup.

(b) Give an example of a nonabelian group where G_{tor} is not a subgroup.

8. (cf. Dummit and Foote, problem 14 of 3.1) By definition, a group G is divisible if for each integer n and each $a \in G$, there exists $b \in G$ such that $b^n = a$.

(a) Show that every element of $G = \mathbf{Q}/\mathbf{Z}$ has finite order, but for each positive integer m, there exists an element of G of order m.

(b) Show that $\mathbf{Q}/\mathbf{Z}$ is the torsion subgroup of $\mathbf{R}/\mathbf{Z}$.

(c)(extra credit) Construct an isomorphism from $\mathbf{Q}/\mathbf{Z}$ to the subgroup of torsion elements of $\mathbf{C}^*$. Here $\mathbf{C}^*$ is the nonzero complex numbers, which is a group under multiplication.

9. (cf. D+F, problem 15, 3.1) By definition, a group G is divisible if for each integer n and each $a \in G$, there exists $b \in G$ such that $b^n = a$. Show that if G is divisible, and N is a normal subgroup of G, then G/N is divisible. Show that $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are divisible.

10. For a group G with subgroup H, let the normalizer $N_G(H) = \{g \in G : gHg^{-1} = H\}$. If H is finite, prove that $N_G(H) = \{g \in G : gHg^{-1} \subset H\}$.

11. (D+F, 3.1, problem 36) For a group G with center $Z(G)$, prove that if $G/Z(G)$ is cyclic, then G is abelian.

12. (D+F, 3.2, problem 4) Let G be a group of order pq, where p and q are prime. Prove that either G is abelian or $Z(G) = \{e\}$.