**Math 60210, Basic Algebra, Problem Set 12, Fall 2009**
**due Thurs., Dec 10**
Do 6 of these problems

**1.** Let $R$ be an integral domain. Show that if $a \in R$ is nonzero, then $\{a\}$ is a linearly independent subset of the $R$-module $R$. Show that $R \cdot a$ is a free $R$-module. Which elements of $R \cdot a$ give a basis of $R \cdot a$?

**2.** Let $R$ be an integral domain. Show that if $a, b \in R$, then the set $\{a, b\}$ is linearly dependent. Show that a nonzero ideal $I$ of $R$ is free if and only if $I$ is principal. Give an example of an integral domain $R$ with a free module $M$ and a submodule $N$ that is not free.

**3.** Let $R$ be a ring with left ideal $I$. Show that $R/I$ is a cyclic $R$-module.

**4-5.** Let $F$ be a field and let $R = M(n, F)$. Regard $F^n$ as column vectors, and show that $F^n$ is a $R$-module using the action $(A, v) \mapsto A \cdot v$ of matrix multiplication. Show that $F^n$ is a cyclic $R$-module. For which $v \in F^n$ is $F^n = R \cdot v$? Determine $Ann_R(F^n)$ and $Ann_R(e_1)$, where $e_1$ is the first standard basis vector.

**6.** Let $R$ be a ring and let $M, N$, and $P$ be $R$-modules.

(a) If $f : M \to N$ and $g : N \to P$ are $R$-module homomorphisms, show that $g \circ f : M \to P$ is a $R$-module homomorphism.

(b) Let $f : M \to N$ be a $R$-module isomorphism. Show that the inverse $f^{-1} : N \to M$ is a $R$-module isomorphism.

**7.** Let $R = \mathbf{Z}$ and consider the $R$-module $\mathbf{Z}_m \times \mathbf{Z}_n$ with $a \cdot (x, y) = (ax, ay)$ for $x \in \mathbf{Z}_m$ and $y \in \mathbf{Z}_n$. Compute $Ann_{\mathbf{Z}}(1, 1)$ and show that $\mathbf{Z} \cdot (1, 1) \cong \mathbf{Z}_s$ for some integer $s$. Determine $s$.

**8.** Let $R$ be a ring and let $M$ be a module. If $N_1 \subset N_2 \subset \cdots \subset N_k \subset ...$ is an ascending chain of submodules of $M$, prove that $\cup_{i \geq 1} N_i$ is a submodule of $M$ and $\cap_{i \geq 1} N_i$ is a submodule of $M$.

**9.** Let $M = \mathbf{Z}^2$ and let $v = 2e_1$, where $e_1$ and $e_2$ is the standard basis of $\mathbf{Z}^2$. Is there a basis $v, w$ of $\mathbf{Z}^2$ for some $w \in M$? Explain your answer.

**10.** Let $F$ be a field and let $R = F[x]$. Let $M = F[x]/(x^2)$ and let $N = F[x]/(x) \oplus F[x]/(x)$. Is $M \cong N$ as a $F$-vector space? Is $M \cong N$ as a $R$-module?

**11.** Let $R = \mathbf{Z}$ and let $M = \mathbf{Z}^2$ with standard basis vectors $e_1, e_2$. Let $N = R \cdot e_1$. Prove that $M/N \cong R$ as $R$-modules.

**12.** Let $F$ be a field and let $R = F[x]$. Prove that there is a bijective correspondence between the collection of $R$-modules and the collection of $F$-vector spaces with a linear operator $L$ (you do not need to repeat assertions explained in class).