Math 60210, Basic Algebra, Problem Set 10, Fall 2009
due Tues, November 17

Do at least 6 of these problems

Comment: there will be no homework due the Tuesday after Thanksgiving

1. Let $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$. Prove that the ideal $(2, 1 + \sqrt{-5})$ is not a principal ideal.

2. Suppose d is an odd integer, $d \geq 3$, and d is not a square. Let $R = \{a + b\sqrt{-d} : a, b \in \mathbf{Z}\}$. Prove that 2 is irreducible in R, but 2 is not prime.

3. Let F be a field and let

$$R = F[x^2, xy, y^2] = \{p(x) = \sum_{i,j \in \mathbf{Z}_{\geq 0}, i+j \in 2\mathbf{Z}} a_{i,j} x^i y^j : a_{i,j} \in F\}.$$

Prove that x^2 is irreducible in R but is not prime in R.

4. Let F be a field and let $\alpha, \beta \in F$. Let $R = F[x, y]$. Prove that $(x - \alpha)$ and $(y - \beta)$ are prime ideals of R but are not maximal ideals of R. Prove that $(x - \alpha, y - \beta)$ is a maximal ideal of R.

5. Let R be an integral domain. Prove that $R[x]$ is a principal ideal domain if and only if R is a field. (Extra Credit: is this true if we only require R to be a commutative ring).

6. Let F be a field and let $p \in F[x]$, and suppose the degree of p is 2 or 3. Prove that p is irreducible in $F[x]$ if and only if $p(\alpha) \neq 0$ for all $\alpha \in F$.

7-8. Ash, 2.6, problems 1-6.

9. Let R be a commutative ring and suppose that R is a prinicipal ideal ring, i.e., if $I \subset R$ is an ideal, then $I = (a)$ for some $a \in R$. Prove that if $f : R \to S$ is a surjective homomorphism of commutative rings, than S is a principal ideal ring. If R is a principal ideal domain and $P \subset R$ is a prime ideal, prove that R/P is a principal ideal domain.

10. Let F be a field. Show that $F[x, y]$ is not a principal ideal domain.

11-12. Let $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$. Let $I_1 = (2, 1 + \sqrt{-5})$, $I_2 = (2, 1 - \sqrt{-5})$, $I_3 = (3, 1 + \sqrt{-5})$, $I_4 = (3, 1 - \sqrt{-5})$.

(a) Explain why $2, 3, 1 + \sqrt{-5}$ and $2 + \sqrt{-5}$ are irreducible in R. Deduce that R is not a unique factorization domain.

(b) Prove that $I_1 \cdot I_2 = (2)$ and $I_3 \cdot I_4 = (3)$.

Note: Similar reasoning shows that $I_1 I_3 = (1 + \sqrt{-5})$, and $I_2 I_4 = (1 - \sqrt{-5})$. The factorization

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

shows that R is not a unique factorization domain. However, we do have a factorization of the principal ideal (6) of R into a product of maximal ideals;

$$(6) = (2) \cdot (3) = (1 + \sqrt{-5})(1 - \sqrt{-5}) = (I_1 \cdot I_2) \cdot (I_3 \cdot I_4) = (I_1 \cdot I_3) \cdot (I_2 \cdot I_4).$$

This should be interpreted as saying $(6) = I_1 \cdot I_2 \cdot I_3 \cdot I_4$ is a unique factorization of the principal ideal (6) into the prime ideals I_j, $j = 1, \cdots, 4$ (it follows from previous homework that I_1 is prime, since R/I_1 is a field. Elementary arguments show $I_2 = I_1$. To show I_3 is prime, prove $R/I_3 \cong \mathbf{Z}_3$, and similarly $R/I_4 \cong \mathbf{Z}_3$).

Some nineteenth century number theorists thought about this issue in the following way. They argued that although $2, 3$ and $1 \pm \sqrt{-5}$ are irreducible elements of R, they should not be regarded as irreducible, but instead as products of certain "ideal numbers". An "ideal number" is really an ideal in the modern sense, so in their language the factorization $(6) = I_1 \cdot I_2 \cdot I_3 \cdot I_4$ is a unique factorization of 6 into prime “ideal numbers”. More precisely, in the ring of integers of an algebraic number field, every nonzero ideal may be factored uniquely as a product of prime ideals.