HOMEWORK 1, MATH 60210, BASIC ALGEBRA, DUE TUESDAY, SEPT. 1 INSTRUCTOR, SAM EVENS, FALL 2009

INSTRUCTIONS: Do 7 of these 10 problems.

1. Ash, 1.1, problem 8.

2. Ash, 1.1, problem 9.

3. Ash, 1.1, problem 10.

4. Let G be a group and let $x \in G$. Prove that x and x^{-1} have the same order.

5. Let G be a group and let $x \in G$ and suppose x has odd order. Prove that $x = x^{2k}$ for some integer k.

6. Let G be a group and let $x, g \in G$. Show that x and gxg^{-1} have the same order. Prove that for all $a, b \in G$, ab and ba have the same order.

7. Let $G = \mathbf{Z}_n$, the integers mod n. Let S be the set of positive integers k such that there exists $x \in G$ such that the order of x is k. Determine S in terms of n, and for each $k \in S$, find $x \in G$ such that the order of $x = k$.

8. Let F be a field with identity 0 under addition and multiplicative identity 1. For matrices $A, B \in M(n, F)$, let AB denote their matrix product. Let $\mathfrak{t}(n, F) = \{A \in M(n, F) : (A)_{ij} = 0 \ \forall \ i > j\}$. Let $V = F^n$ be the set of all column vectors with n entries in F, so an element of V is of the form

$$v = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_n \end{pmatrix} \tag{0.1}$$

with $\lambda_1, \dots, \lambda_n \in F$. If $v \in V$, let $A \cdot v$ be the usual action of a $n \times n$ matrix on a column vector v. Consider the standard basis vectors $e_1, \dots, e_n \in F^n$, where the ith row of the vector e_i is 1 and all other entries of e_i are 0. The above vector

$$v = \lambda_1 e_1 + \dots + \lambda_n e_n$$

For $i = 1, \dots, n$, let

$$V_i = \{\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_i e_i : \lambda_1, \dots, \lambda_i \in F\}$$

so a vector v as above is in V_i if and only if the last $n - i$ entries are 0.

(a) Prove that

$$\mathfrak{t}(n, F) = \{A \in M(n, F) : A \cdot v \in V_i \ \forall \ v \in V_i\}$$

(b) Let $T(n, F) = GL(n, F) \cap \mathfrak{t}(n, F)$. Prove that $T(n, F)$ is a group under matrix multiplication. If F is finite and $|F| = q$, compute the number of elements of $T(n, F)$.

9. Let $U(n, F) = \{A \in T(n, F) : (A)_{ii} = 1 \forall i = 1, \dots, n\}$. Prove that $U(n, F)$ is a subgroup of $GL(n, F)$ under matrix multiplication and if F is a finite field with q elements, compute the cardinality of $U(n, F)$. Note: the following idea may be useful. Recall the subspaces V_i of F^n from the previous problem. If $v \in V_i$, let $\overline{v} = v + V_{i-1} = \{v + u : u \in V_{i-1}\}$, and if $g \in T(n, F)$, let $g \cdot (v + V_{i-1}) = g \cdot v + V_{i-1}$. Show that $T(n, F)$ acts on $v + V_{i-1}$, and hence $T(n, F)$ acts on the set

$$W := \{\overline{v_1} \times \cdots \times \overline{v_k} \times \cdots \times \overline{v_n} : v_j \in V_j\}$$

Show that $U(n, F)$ is the stabilizer in $T(n, F)$ of a particular point in W.

10. Let $D(n, F) = \{A \in GL(n, F) : (A)_{ij} = 0, \ \forall\, i \neq j\}$. $D(n, F)$ is the set of invertible diagonal matrices.

(a) Prove that $D(n, F) = \{A \in GL(n, F) : A \cdot e_i \in Fe_i\}$.

(b) Prove that $D(n, F)$ is a group under matrix multiplication. Suppose that F is finite and $|F| = q$. Compute the number of elements in $D(n, F)$. How is the cardinality of $T(n, F)$ related to the cardinalities of $U(n, F)$ and $D(n, F)$?