HANDOUT ON INJECTIVE MODULES

MATH 60220, Prof. Sam Evens

These notes are supplementary to class discussion, and much of them is in Ash, 10.6 and 10.7. In the sequel, R is a ring.

A R-module E is called injective if for each injective homomorphism $f : A \to B$ of R-modules, the associated homomorphism of abelian groups $f^* : Hom_R(B, E) \to Hom_R(A, E)$, defined by $\phi \mapsto \phi \circ f$, is surjective. Otherwise said, the functor $Hom_R(-, E)$ is exact.

In the remainder of these notes, an injective R-module homomorphism is called *an embedding.*

The goal is to prove that for each R-module M, there is an embedding $f : M \to E$, where E is an injective R-module. The strategy is to first solve the problem in the special case where $R = \mathbf{Z}$, the ring of integers. Then we use a "change of rings" functor to prove the assertion in general.

Definition 1 (10.6.5 in Ash). *Let R be an integral domain. A R-module M is called divisible if for each $y \in M$ and each nonzero $r \in R$, there is $x \in M$ so $y = r \cdot x$.*

If M has no R-torsion, i.e., $r \cdot m = 0$ implies $r = 0$ or $m = 0$, then x in the previous definition is unique and we may think of x as $\frac{y}{r}$. Thus, the definition of divisible means in some rough sense that any element of the module may be divided by an arbitrary nonzero element of the ring.

Lemma 2. *If M is a divisible R-module and $N \subset M$ is a submodule, then M/N is divisible.*

One proves the following results.

Ash, 10.6.4. *An R-module E is injective if and only if for every left ideal I of R, the induced homomorphism of abelian groups $f^* : Hom_R(R, M) \to Hom_R(I, M)$ is surjective. Here we denote $f : I \to R$ for the embedding of the ideal I in R.*

10.6.4 means that we may test whether E is injective by checking that the functor $Hom_R(-, E)$ is exact on short exact sequences

$$0 \mapsto I \mapsto R \mapsto R/I \to 0,$$

rather than having to check exactness on all short exact sequences.

Ash, 10.6.6. *If R is a principal ideal domain (PID), then a R-module M is injective if and only if it is divisible.*

10.6.6 is a fairly easy consequence of 10.6.4. 10.6.4 has a somewhat nonintuitive proof, but it is not long. Both are reasonably explained in Ash.

Ash, 10.7.1. *Every* $\mathbf{Z}$*-module* M *may be embedded in an injective* $\mathbf{Z}$*-module* E*.*

Sketch proof: $\mathbf{Z}$ may be embedded in $\mathbf{Q}$, which is injective since it is trivially seen to be divisible. By 4.3.6, there is a free $\mathbf{Z}$-module F and submodule B so $F/B \cong M$. Let $F = \oplus_{i \in I}\mathbf{Z} \cdot x_i \cong \oplus_{i \in I}\mathbf{Z}$, where $\{x_i\}_{i \in I}$ is a basis of F. Write the isomorphism induced by composition as $\chi : F \mapsto^{\cong} \oplus_{i \in I}\mathbf{Z}$. Let $F_{\mathbf{Q}} = \oplus_{i \in I}\mathbf{Q}$, and write $i : \oplus_{i \in I}\mathbf{Z} \to F_{\mathbf{Q}}$ for the embedding given by $\oplus(a_i) \mapsto \oplus(a_i)$, i.e., each copy of $\mathbf{Z}$ is embedded in the corresponding copy of $\mathbf{Q}$ in the obvious way. Then $i \circ \chi : F \to F_{\mathbf{Q}}$ is an embedding. Further, $F_{\mathbf{Q}}$ is evidentally divisible. It follows from Lemma 2 above that $F_{\mathbf{Q}}/i \circ \chi(B)$ is divisible, so by 10.6.6, $F_{\mathbf{Q}}/i \circ \chi(B)$ is injective. It is easy to check that the natural map $F/B \to F_{\mathbf{Q}}/i \circ \chi(B)$ is an embedding.

10.7.1 solves our main problem when $R = \mathbf{Z}$. We use this case to solve the problem for general R.

First, let R be a ring and denote by f the canonical homomorphism $f : \mathbf{Z} \to R$ such that $f(n) = n \cdot 1_R$. By forgetting the ring structure on R, we may regard R as a $\mathbf{Z}$-module.

Let M be a $\mathbf{Z}$-module. For a $\mathbf{Z}$-module M, denote by $f_*(M)$ the abelian group $Hom_{\mathbf{Z}}(R, M)$. Then $f_*(M)$ is naturally a R-module. For $\phi : R \to M$ in $f_*(M)$ and $r \in R$, define $r \cdot \phi$ by the formula, $(r \cdot \phi)(x) = \phi(xr)$.

Exercise 1. *Verify that* $f_*(M)$ *is a left* R*-module using this definition of* $f \cdot \phi$*.*

$f_*(M)$ has the following additional structure. Let A be a R-module, so A is also a $\mathbf{Z}$-module by forgetting the R-module structure. For a $\mathbf{Z}$-module K, define
$\psi_A : Hom_{\mathbf{Z}}(A, K) \to Hom_R(A, Hom_{\mathbf{Z}}(R, K))$ by
$\psi_A(h) \in Hom_R(A, Hom_{\mathbf{Z}}(R, K))$ is defined by $\psi_A(h)(a)(x) = h(x \cdot a)$, where $a \in A$ and $x \in R$. Here $f_*(K) = Hom_{\mathbf{Z}}(R, K))$ is regarded as a R-module as in Exercise 1.

Exercise 2. *Show that for* $h \in Hom_{\mathbf{Z}}(A, K)$*,* $\psi_A(h) \in Hom_R(A, Hom_{\mathbf{Z}}(R, K))$*.*

For A and K as above, define
$\tau_A : Hom_R(A, Hom_{\mathbf{Z}}(R, K)) \to Hom_{\mathbf{Z}}(A, K)$ by $\tau_A(g)(a) = g(a)(1)$, for $a \in A$.
It is very easy to check that $\tau_A(g) \in Hom_{\mathbf{Z}}(A, K)$.

Exercise 3. *Show that* ψ_A *and* τ_A *are inverse isomorphisms.*

Let now B also be a R-module, with R-module homomorphism $\phi : A \to B$.
Define $\phi^*_{\mathbf{Z}} : Hom_{\mathbf{Z}}(B, K) \to Hom_{\mathbf{Z}}(A, K)$ by $\phi^*_{\mathbf{Z}}(f) = f \circ \phi$.
Define $\phi^*_R : Hom_R(B, Hom_{\mathbf{Z}}(R, K)) \to Hom_R(A, Hom_{\mathbf{Z}}(R, K))$ by $\phi^*_R(g) = g \circ \phi$.
We have the following diagram:

Exercise 4. *Show the diagram is commutative, i.e.,* $\phi_R^* \circ \psi_B = \psi_A \circ \phi_{\mathbf{Z}}^*$.

We may now explain how to construct injective R-modules.

Proposition 1. *Let* E *be an injective* $\mathbf{Z}$*-module. Then* $f_*(E) = Hom_{\mathbf{Z}}(R, E)$ *is an injective* R*-module.*

Proof: Let $i : M \to N$ be an embedding of R-modules. We must show $i_R^* : Hom_R(N, f_*(E)) \to Hom_R(M, f_*(E))$ is surjective. By Exercise 4 applied with $i : M \to N$ in place of $\phi : A \to B$ and with E in place of K, we have $i_R^* \circ \psi_N = \psi_M \circ i_{\mathbf{Z}}^*$. Since E is an injective $\mathbf{Z}$-module, $i_{\mathbf{Z}}^* : Hom_{\mathbf{Z}}(N, E) \to Hom_{\mathbf{Z}}(M, E)$ is surjective by definition of injective module. By Exercise 3, it follows that $\psi_M \circ i_{\mathbf{Z}}^*$ is surjective, so $i_R^* \circ \psi_N$ is surjective. In particular, i_R^* is surjective, which completes the proof.

Theorem 10.7.4. *Let* M *be a left* R*-module. Then there is an embedding* $M \to G$*, where* G *is an injective* R*-module.*

Proof : Regard M as a $\mathbf{Z}$-module. By 10.7.1, there is an embedding of $\mathbf{Z}$-modules $h : M \to E$, where E is an injective $\mathbf{Z}$-module. Define $\eta : M \to f_*(E) = Hom_{\mathbf{Z}}(R, E)$ by the formula $\eta(m)(r) = h(r \cdot m)$.

Exercise 5. η *is a* R*-module homomorpism.*

By Proposition 1, $G = f_*(E)$ is an injective R-module. It remains to check that η is injective. For this, note that for $m \in M$, $\eta(m) = 0$ implies that $0 = \eta(m)(1) = h(m)$. Since h is an embedding, $m = 0$, so the proof of 10.7.4 is complete.

Hopefully, you can see from the argument that the proof is essentially a triviality once the relevant constructions are made and understood. It is at some level more an issue of familiarizing oneself with the relevant terminology than making an argument.

As if you aren't sick enough of this already, let's explain Exercise 4 in the language of natural transformations (cf. Ash, 10.3.4, except Ash discusses covariant functors and our functors are contravariant, so presentation is somewhat different). Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two contravariant functors.

Definition. *A natural transformation of functors* $T : F \to G$ *assigns to each object* X *of* $\mathcal{C}$ *a morphism* $T_X : F(X) \to G(X)$ *with the property that for any two objects* X, Y *of* $\mathcal{C}$ *and each morphism* $f : X \to Y$ *in* $Mor_{\mathcal{C}}(X, Y)$*, then* $G(f) \circ T_Y = T_X \circ F(f)$*, i.e., the following diagram commutes:*

Let $\mathcal{C} = (R - mod)$ be the category of R-modules and let $\mathcal{D} = (\mathbf{Z} - mod)$ be the category of $\mathbf{Z}$-modules. Fix a $\mathbf{Z}$-module K. Define two functors

$$F, G : (R - mod) \to (\mathbf{Z} - mod)$$

by the rules, $F(B) = Hom_{\mathbf{Z}}(B, K)$ for a R-module B, and given $\phi : A \to B$, define $F(\phi) = \phi^*_{\mathbf{Z}}$,

and for A, B, ϕ as above, define

$G(B) = Hom_R(B, f_*K)$ and $G(\phi) = \phi^*_R$.

Define $T_B : F(B) \to G(B)$ by $T_B = \psi_B$ as defined before.

Exercise 6. *: Prove that this T is a natural transformation between the contravariant functors F and G.*

Further, we remark that f_* as defined before is a covariant functor from $(\mathbf{Z}-mod)$ to $(R - mod)$. We may define a covariant functor from $(R - mod)$ to $(\mathbf{Z} - mod)$ by forgetting the R-module structure. This is an example of a forgetful functor (see 10.3, problem 7(a)), and may be denoted by $f^*(M) = M$, where on the left M is regarded as a R-module, and on the right M is regarded as a $\mathbf{Z}$-module by forgetting the R-module structure. In this language, we may restate the conclusion of Exercise 3 as stating that for R-module B and $\mathbf{Z}$-module K, $Hom_R(B, f_*(K)) \cong Hom_{\mathbf{Z}}(f^*B, K)$. Then f_* and f^* are called adjoint functors, and f_* is called right adjoint of f^* and f^* is called left adjoint of f_*. There are many examples of this in mathematics.