NOTES ON ALGEBRAIC HOMOGENEOUS SPACES

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Let G be a linear algebraic group and let H be a closed subgroup. The purpose of these notes is to explain the construction and properties of the homogeneous space G/H. Most of this material is from Humphreys book on Linear Algebraic Groups, chapters 4, 11, and 12, and Springer's book on Linear Algebraic Groups, 2nd edition, chapter 5.

Definition 0.1. *Let X be a topological space and let $A \subset X$. A is called locally closed if $A = F \cap U$, where F is closed in X and U is open in X.*

Lemma 0.2. *A subset A of a topological space X is locally closed if and only if A is open in $\overline{A}$. Further, A is locally closed if and only if $A = \overline{A} \cap U$ for some open set U.*

Proof: Suppose A is open in $\overline{A}$. Then by definition of the subspace topology, $A = \overline{A} \cap U$ for some open set U of X. Hence, A is locally closed. Conversely, suppose A is locally closed so $A = F \cap U$ for some F and U as above. Then $A \subset F$ so since F is closed, $\overline{A} \subset F$, so $A \subset \overline{A} \cap U \subset F \cap U \subset A$. It follows that $A = \overline{A} \cap U$, so A is open in $\overline{A}$ by properties of the subspace topology. This completes the proof of the first statement, and the second statement is clear from the proof.

Q.E.D.

Definition 0.3. *A subset Y of a topological space X is called constructible if $Y = A_1 \cup \cdots \cup A_k$ for locally closed subsets $A_1, \dots, A_k$.*

It follows easily from the definition that a finite union of constructible sets is constructible. Further, the complement of a constructible set is constructible.

Remark 0.4. *EXERCISE 1 A subset Y of X is constructible if and only if Y is obtained from a finite collection of open sets through a finite numbers of unions, intersections, and complements. Further, if Y is constructible and X is an algebraic variety, then Y has a dense open set U.*

Remark 0.5. *Let $X = \mathbb{C}^2$. The set $Y = X_{xy} \cup \{(0,0)\}$ is constructible but not locally closed.*

Lemma 0.6. *Let X be a topological space and let $A \subset X$. Then A is open if and only if for every $x \in A$, $x \notin \overline{X - A}$.*

We leave the proof as an exercise for the reader. It is useful to recall that if B is a subset of X, then $y \in \overline{B}$ if and only if for every open set U of X containing y, $B \cap U$ is nonempty.

Theorem 0.7. *Let $\phi : X \to Y$ be a morphism of algebraic sets. Then if $A \subset X$ is constructible, $\phi(A)$ is constructible.*

Proof : Since A is an algebraic set and the restriction of ϕ to A is a morphism, it suffices to show that $\phi(X)$ is constructible in Y. Since X is a finite union of its irreducible components and a finite union of constructible sets is constructible, we may further assume that X is irreducible. We can then replace Y by $\overline{\phi(X)}$, and hence assume ϕ is dominant and Y is irreducible.

We now proceed by induction on $\dim(Y)$. If $\dim(Y) = 0$, Y is a point and $\phi(X) = Y$ is constructible, so the result is clear. By a result proved in class, there exists a nonempty open set U of Y such that U is a subset of $\phi(X)$. Let $Y - U = W_1 \cup \cdots \cup W_t$ be the irreducible components of $Y - U$. Since W_i is closed in Y, $\dim(W_i) < \dim(Y)$. Let $\phi^{-1}(W_i)$ have irreducible components Z_{ij}. Then consider the restriction of ϕ to Z_{ij}, $\phi : Z_{ij} \to W_i$. Since $\dim(W_i) < \dim(Y)$, by induction $\phi(Z_{ij})$ is constructible. Further, $\phi(X) = U \cup \cup_{ij} \phi(Z_{ij})$, so $\phi(X)$ is constructible.

Q.E.D.

Theorem 0.8. *Let $\phi : X \to Y$ be a dominant morphism of irreducible varieties and let $r = \dim(X) - \dim(Y)$. There exists a nonempty open set U of Y such that if W is a closed subvariety of Y and $W \cap U$ is nonempty, then for every irreducible component Z of $\phi^{-1}(W \cap U)$, $\phi(Z)$ is dense in W and* $\dim(Z) = \dim(W) + r$.

Proof : This essentially follows from the proof of Theorem 0.22 from the notes on fiber dimension. The statement of the theorem we are proving adds to Theorem 0.22 by also asserting that $\phi(Z)$ is dominant. To prove this extra assertion, recall that we proved Theorem 0.22 by finding a normal open set U and factoring $\phi : \phi^{-1}(U) \to U$ as $\psi = (\phi, \eta) : \phi^{-1}(U) \to U \times \mathbb{C}^r$ for a map η, and that in this argument, ψ is finite. Then if Z is an irreducible component of $\phi^{-1}(W)$ meeting $\phi^{-1}(U)$, Z is an irreducible component of $\psi^{-1}(W \cap U \times \mathbb{C}^r)$, so since $U \cap \mathbb{C}^r$ is normal, $\psi(Z \cap \phi^{-1}(U)) = W \cap U \times \mathbb{C}^r$ by Proposition 0.15(d) from the notes on fiber dimension. Hence, $\phi(Z \cap \phi^{-1}(U)) = pr_Y(W \cap U \times \mathbb{C}^r) = W \cap U$ is dense in W, so $\phi(Z)$ is dense in W.

Q.E.D.

Definition 0.9. *Let $\phi : X \to Y$ be a continuous map of topological spaces. The map ϕ is called an open map if when $U \subset X$ is open, $\phi(U)$ is open in Y.*

Remark 0.10. *Let $\phi : X \to Y$ be a submersion of differentiable manifolds, i.e., the differential $d\phi_x : T_x(X) \to T_y(Y)$ is surjective for all $x \in X$. Then ϕ is an open map. This is a consequence of the inverse function theorem. In the situation where we apply the following open mapping theorem, it is easy to show that the map ϕ is submersive. This means that the classical counterpart of these results is easier.*

Theorem 0.11. *Let $\phi : X \to Y$ be a surjective morphism of irreducible varieties. Let $r = \dim(X) - \dim(Y)$. Suppose for every irreducible subvariety W of Y, each irreducible* *component Z of $\phi^{-1}(W)$ has the property that $\phi(Z) = W$ and $\dim(Z) = \dim(W) + r$. Then ϕ is an open map.*

Proof : Let U be open in X and let $V = \phi(U)$. To show V is open in Y, it suffices to show that for every $x \in U$, $y = \phi(x) \notin \overline{Y - V}$ using Lemma 0.6.

Suppose by contradiction that $y \in \overline{Y - V}$. Since U is constructible, V is constructible by Theorem 0.7, and it follows that $Y - V$ is constructible, so $Y - V = L_1 \cup \cdots \cup L_k$, where each L_i is locally closed. Hence, $y \in \overline{L}$ for some $L = L_i$. The point y is contained in the closure of some irreducible component of L, and we may replace L by that irreducible component, so L is irreducible. Since L is locally closed, by Lemma 0.2, $L = \overline{L} \cap W$ for an open set W.

Let $M = \phi^{-1}(\overline{L})$. Let $S = \phi^{-1}(W)$. We claim that S meets each irreducible component M_i of M. Indeed, by hypothesis, $\phi(M_i)$ is dense in $\overline{L}$, so since $W \cap \overline{L}$ is dense in $\overline{L}$ it follows that $\phi(M_i) \cap W$ is nonempty. Hence, $S \cap M_i$ is nonempty and hence dense in M_i since M_i is irreducible, so $S \cap M$ is dense in M.

But $S \cap M = \phi^{-1}(W \cap \overline{L}) \subset X - U$. Indeed, if $z \in \phi^{-1}(W \cap \overline{L}) \cap U$, then $\phi(z) \in V \cap (W \cap \overline{L}) = V \cap L$, which is empty. Hence, $M = \overline{S \cap M} \subset X - U$ since U is open. But $x \in M$ since $\phi(x) = y \in \overline{L}$, which is a contradiction since $x \in U$. It follows that $y \notin \overline{Y - V}$ so V is open.

Q.E.D.

Let Z be a an algebraic G-set and let $x \in Z$ and suppose that $G_x = H$. Consider the morphism $\pi : G \to G \cdot z =: Y$ given by $\pi(a) = a \cdot z$ for $a \in G$.

Remark 0.12. *We show that $\pi : G \to Y$ satisfies the hypotheses of Theorem 0.11 under the assumption that G is connected, so that G and its image Y are irreducible. Indeed, by Theorem 0.22 of the notes on fiber dimension (generalized to the setting of arbitrary morphisms of varieties), there exists an open set U of Y such that if W is a closed subvariety of Y, and $W \cap U$ is nonempty, and Z is an irreducible component of $\pi^{-1}(W)$ that meets $\pi^{-1}(U)$, then $\dim(Z) = \dim(W) + r$ and $\pi(Z)$ is dense in W. Let A be a closed subvariety of Y and suppose that B is an irreducible component of $\pi^{-1}(A)$. Then there exists $g \in G$ such that $gB \cap \pi^{-1}(U)$ is nonempty. Note that $l_g : A \to gA$ given by left multiplication lifts to an isomorphism $l_g : \pi^{-1}(A) \to \pi^{-1}(gA)$. It follows that the irreducible components of $\pi^{-1}(gA)$ are all of the form gC, where C is an irreducible component of $\pi^{-1}(A)$, so in particular gB is an irreducible component of $\pi^{-1}(gA)$. Since π is surjective and gB meets $\pi^{-1}(U)$, it follows that gA meets U, so $\pi(gB)$ is dense in gA and $\dim(gB) = \dim(gA) + r$. By multiplying by g^{-1} on the left, we obtain that $\pi(B)$ is dense in A and $\dim(B) = \dim(A) + r$.*

We now consider the morphism $\pi : G \to Y = G \cdot y$ from above, where $G_y = H$. We identify $G/H = Y$ via the map $gH \mapsto g \cdot y$.

Recall that an affine variety is normal if its ring of functions is integrally closed in its function field.

Definition 0.13. *Let Y be an algebraic variety. Y is called normal if there exists a finite open affine cover $Y = \cup U_i$ such that each U_i is normal.*

Note that a nonempty open subset of a normal variety is normal. This follows by Lemma 0.18 of the notes on fiber dimension.

Lemma 0.14. *Let Y be a G-variety and suppose the G-action on Y is transitive. Then Y is normal.*

Proof : Let $Y = \cup U_i$ be an open affine cover. Since each U_i is an affine variety, there exists an affine open set U of U_i such that U is normal (see notes on fiber dimension, Proposition 0.19). Let $x \in G$, and consider the isomorphism $l_x : Y \to Y$ given by $l_x(y) = xy$. Then $l_x(U)$ is an open affine variety, and it is normal since l_x is an isomorphism. Since the G-action on Y is transitive, $Y = \cup_{x \in G} l_x(U)$ is an open affine cover and each $l_x(U)$ is normal. Since Y is Noetherian, there is a finite collection $x_1, \ldots, x_k$ in G such that $Y \cup l_{x_i}(U)$.

Q.E.D.

Remark 0.15. *EXERCISE Let X be an algebraic variety with finite open affine cover $X = \cup U_i$. For $x \in X$, we define the tangent space $T_x(X) = T_x(U_i)$ if $x \in U_i$. The tangent space $T_x(X)$ is independent (up to isomorphism) of the choice of U_i containing x. We say $x \in X$ is a smooth point of X if $\dim(T_x(X)) = \dim(X)$. The algebraic variety X is called smooth if every point $x \in X$ is a smooth point. If X is an algebraic G-variety with transitive action by a linear algebraic group G, prove that X is smooth.*

Theorem 0.16. *ZARISKI'S MAIN THEOREM Let $\phi : X \to Y$ be a bijective morphism between complex algebraic varieties and assume Y is normal. Then ϕ is an isomorphism.*

This follows from results in chapter 5 of Springer, Linear Algebraic Groups, 2nd edition. Indeed since ϕ is surjective, ϕ is dominant, so $\phi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ is injective. By Theorem 5.1.6 of Springer, there exists a dense open set U of X such that if $x \in X$, the cardinality of $\phi^{-1}\phi(x)$ equals the degree of the field extension $[\mathbb{C}(X) : \phi^*(\mathbb{C}(Y))]$ (ϕ is separable because we are working in characteristic zero, so $[\mathbb{C}(X) : \phi^*(\mathbb{C}(Y))] = [\mathbb{C}(X) : \phi^*(\mathbb{C}(Y))]_s$, in Springer's notation. Since ϕ is bijective, the cardinality of $\phi^{-1}\phi(x)$ is one for every $x \in X$, hence for every $x \in U$. Theorem 5.2.8 of Springer asserts that a bijective birational morphism of algebraic varieties is an isomorphism. The morphism ϕ is assumed to be bijective, and we just showed that it follows that ϕ is birational. Hence, we can apply Theorem 5.2.8 to conclude that ϕ is an isomorphism.

Remark 0.17. *EXERCISE Give two examples of bijective morphisms between varieties that are not isomorphisms.*

Theorem 0.18. *Let $\chi : G \to Z$ be a morphism to an algebraic G-set Z, and suppose $\chi(gh) = \chi(g)$ for all $g \in G$ and $h \in H$. Then there exists a unique morphism $\tilde{\chi} : Y \to Z$ such that $\tilde{\chi} \circ \pi = \chi$.*

Proof : We define $\tilde{\chi} : Y \to Z$ by $\tilde{\chi}(g \cdot y) = \chi(g)$. $\tilde{\chi}$ is a well-defined map of sets since $\chi(gh) = \chi(g)$ for all $g \in G$, $h \in H$. It is easy to check that $\tilde{\chi} \circ \pi = \chi$, and further, since π is surjective, $\tilde{\chi}$ is the only map of sets with this property.

To show that $\tilde{\chi}$ is a morphism, we must show $\tilde{\chi}$ is continuous, and show that $\tilde{\chi}^*$ is a ring homomorphism $\mathcal{O}(U) \to \mathcal{O}(\tilde{\chi}^{-1}(U))$ for each open set U of Z. For simplicity, we assume that G is connected. The general case can be derived from this special case.

For continuity, let $U \subset Z$ be open. Then since $\tilde{\chi} \circ \pi = \chi$, $\chi^{-1}(U) = \pi^{-1}(\tilde{\chi}^{-1}(U))$. Since χ is continuous, $\chi^{-1}(U)$ is open, so $\pi(\chi^{-1}(U))$ is open. But $\pi(\chi^{-1}(U)) = \pi(\pi^{-1}(\tilde{\chi}^{-1}(U))) = \tilde{\chi}^{-1}(U)$. By Remark 0.12, we may apply Theorem 0.11 to conclude that π is an open map, so $\tilde{\chi}^{-1}(U)$ is open, and $\tilde{\chi}$ is continuous.

To complete the proof that $\tilde{\chi} : Y \to Z$ is a morphism, we take an open affine cover $Z = \cup V_i$, and let $U_i = \tilde{\chi}^{-1}(V_i)$ be the corresponding open sets in Y. From the definition of a morphism, We must show that if $f \in \mathcal{O}(V_i)$, then $f \circ \tilde{\chi} \in \mathcal{O}(U_i)$. For this, note that $f \circ \tilde{\chi} \circ \pi = f \circ \chi$. Since χ is right H-invariant, it follows easily that $f \circ \chi$ is right H-invariant. Suppose we can show that there exists $\lambda \in \mathcal{O}(U_i)$ such that $\lambda \circ \pi = f \circ \chi$. Then since π is surjective, $\pi^* : \mathcal{O}(U_i) \to \mathcal{O}(\chi^{-1}(U_i))$ is injective, so since

$\pi^*(\lambda) = \lambda \circ \pi = f \circ \chi = f \circ \tilde{\chi} \circ \pi = \pi^*(f \circ \tilde{\chi})$, it follows that $\lambda = f \circ \tilde{\chi}$. Hence, $f \circ \tilde{\chi} \in \mathcal{O}(U_i)$.

Thus, it suffices to show that for an open set U in Y, if $f \in \mathcal{O}_G(\pi^{-1}(U))$ is right H-invariant, there exists $\lambda \in \mathcal{O}_Y(U)$ such that $\pi^*(\lambda) = f$. For simplicity, we first do this when $U = Y$. The general case is similar.

Let $f \in \mathbb{C}[G]$ and suppose $f(xh) = f(x)$ for all $h \in H$ and $x \in G$. Consider the graph $\Gamma_G := \{(x, f(x)) : x \in G\}$. Since G and $\mathbb{C}$ are separated, it follows easily that Γ_G is closed in $G \times \mathbb{C}$. Consider also the set

$\Gamma := \{\pi(x), f(x)) : x \in G\} \subset Y \times \mathbb{C}$.

Note that $Y \times \mathbb{C}$ has a transitive action by the linear algebraic group $G \times \mathbb{C}$ with action $(g, z) \cdot (y, t) = (g \cdot y, z + t)$. Hence, $\psi := (\pi, id) : G \times \mathbb{C} \to Y \times \mathbb{C}$ is an open map by Remark 0.12. Hence, $\psi((G \times \mathbb{C}) - \Gamma_G) = (Y \times \mathbb{C}) - \Gamma$ is open, so Γ is closed in $Y \times \mathbb{C}$, and it follows that Γ is an algebraic variety.

Let $\eta : \Gamma \to Y$ be the restriction of projection on the first factor to Y. We claim that η is bijective. Indeed, if $\eta((\pi(x), f(x))) = \eta((\pi(x_1), f(x_1)))$, then $xH = \pi(x) = \pi(x_1) = x_1H$, so $x_1 = xh$, some $h \in H$. But then $f(x_1) = f(x)$ since f is right H-invariant, so $(\pi(x), f(x)) = (\pi(x_1), f(x_1))$, and η is injective. Further, η is surjective, since if $a \in Y$, $a = \pi(x)$ for some $x \in G$ so $a = \eta(\pi(x), f(x))$. Hence, η is a bijective morphism, so by Zariski's Main Theorem and the normality of Y, η is an isomorphism.

Now consider the morphism $g : \Gamma \to \mathbb{C}$ given by projection on the second factor. On a point $w = (\pi(x), f(x)) \in \Gamma$, $g(w) = f(x)$. Since $\eta : \Gamma \to Y$ is an isomorphism, it follows that there exists $\lambda \in \mathbb{C}[Y]$ such that $\eta^*(\lambda) = g$. Now consider the morphism $\mu = (\pi, f) : G \to Y$. It follows from definitions that $\eta \circ \mu = \pi$ and hence

$\pi^*(\lambda) = \mu^* \circ \eta^*(\lambda) = \mu^*(g)$. Since for $x \in G$, $\mu^*(g)(x) = g(\mu(x)) = g((\pi(x), f(x))) = f(x)$, it follows that $\mu^*(g) = f$, so $\pi^*(\lambda) = f$.

In the general case, replace Y by an open set U of Y and replace G by the open set $V = \pi^{-1}(U)$. Then the same argument we just gave shows that for any right H-invariant $f \in \mathcal{O}_G(V)$, there exists $\lambda \in \mathcal{O}_Y(U)$ such that $\pi^*(\lambda) = f$. By the preceding remarks, this completes the proof.

Q.E.D.

Theorem 0.19. *Let G be a linear algebraic group and let N be a closed normal subgroup. Then G/N is a linear algebraic group, and the morphism $\pi : G \to G/N$ is a homomorphism of algebraic groups.*

Proof : As before, construct a rational finite dimensional G-module V with a line $L = \mathbb{C}v$ such that $N = \{g \in G : g \cdot L \subset L\}$. It follows that L is a rational N-module, and we may consider the induced homomorphism of algebraic groups $N \to GL(L) = GL(1)$ given by the N-action on L. Since $v \in L$, it follows that $n \cdot v = \chi(n)v$ for some $\chi : N \to \mathbb{C}^*$. Since $N \to GL(1)$ is a group homomorphism, it follows that $\chi : N \to \mathbb{C}^*$ is an algebraic group homomorphism. Let $X(N)$ be the set of algebraic group homomorphisms from N to $\mathbb{C}^*$. For $\tau \in X(N)$, let $V_\tau = \{v \in V : h \cdot v = \tau(h)v, \forall h \in N\}$. Let $U = \sum_{\tau \in X(N)} V_\tau$. It is elementary to check that this sum is direct.

We claim that U is a G-stable submodule. Indeed, note that if $\tau \in X(N)$, and $g \in G$, then $\tau^g : N \to \mathbb{C}^*$ defined by $\tau^g(n) = \tau(g^{-1}ng)$ is a homomorphism of algebraic groups, so $\tau_g \in X(N)$. Let $v \in V_\tau$. Then

$ngv = gg^{-1}ngv = g\tau(g^{-1}ng)v = \tau^g(n)gv$ since $\tau^g(n)$ is a scalar. It follows that $gV_\tau \subset V_{\tau^g}$.

Let $W = \{A \in \mathrm{End}(U) : A \cdot V_\chi \subset V_\chi \forall \chi \in X(N)\}$. Define an algebraic group homomorphism $\eta : G \to GL(W)$ by the formula $\eta(g)(A) = gAg^{-1}$. Note that $A \in W$ implies $gAg^{-1} \in W$. Indeed, if $v \in V_\chi$, $g^{-1}v \in V_{\chi^{g^{-1}}}$, so $Ag^{-1}v \in V_{\chi^{g^{-1}}}$ by the definition of W, and then $gAg^{-1}v \in V_{\chi^{g^{-1}g}}$. It is elementary to check that $\chi = \chi^{g^{-1}g}$, so $gAg^{-1} \in W$. Further, W is a rational G-module since U is a rational G-module.

We claim that the kernel of η is N. Indeed, if $g \in N$ and $A \in W$, we show $gAg^{-1} = A$. For this, let $v \in V_\chi$, so

$gAg^{-1}v = gA\chi(g^{-1})v = \chi(g^{-1})gAv = \chi(g^{-1})\chi(g)Av$ since $Av \in V_\chi$ by the definition of W. Since $\chi(g^{-1})\chi(g) = 1$, it follows that $gAg^{-1} = A$, for all $A \in W$, so $\eta(g) = e$, and N is in the kernel of η.

Now suppose that $\eta(g) = e$, so $gAg^{-1} = A$ for all $A \in W$. It follows that $gA = Ag$ for all $A \in W$, so that the action of g on U commutes with the action of every A in W. It is easy to check that $W = \coprod_{\chi \in X(N)} \mathrm{End}(V_\chi)$. Thus, g commutes with the action of each $A \in \mathrm{End}(V_\chi)$ for each χ. Since the image of N is in W, and g commutes with the action of N, it follows easily that $gV_\chi \subset V_\chi$ for all $\chi \in X(N)$, so the image of g restricts to give an element of $\mathrm{End}(V_\chi)$, and this restriction is in the center of $\mathrm{End}(V_\chi)$. Since for a vector space M, the center of $\mathrm{End}(M)$ is scalar matrices, it follows that g acts as a scalar on each V_χ. In particular, g stabilizes each line in V_χ so $g \cdot L \subset L$, and hence $g \in N$. Thus, the kernel of η equals N.

To summarize, we have a homomorphism of algebraic groups $\eta : G \to GL(W)$ with kernel N. By Corollary 2, p. 10 of Brion, $\eta(G)$ is a closed subgroup of $GL(W)$, so η induces an isomorphism $G/N \to \eta(G)$, so G/N is isomorphic to a linear algebraic group.

Q.E.D.

Let $\pi : G \to G/N$ be the projection, $\pi(g) = gN$, where we are identifying $G/N = G \cdot y$, where Y is some algebraic G-set and $G_y = N$, and consider the morphism $\eta : G \to \eta(G)$. Since η is right N-invariant, by the universal property of the quotient from Theorem 0.18, there is an induced bijective morphism $\tilde{\eta} : G/N \to \eta(G)$. By Zariski's Main Theorem, $\tilde{\eta}$ is an isomorphism, so the construction of G/N given in the proof of the last theorem is the same as the homogeneous space construction.

Once we know that G/N exists, we can define the group operations on G/N just as in standard group theory. Indeed, the morphism $\pi \circ m : G \times G \to G \to G/N$ is right $N \times N$-invariant, so there is an induced morphism $\overline{m} : (G \times G)/(N \times N) \to G/N$. Since $(G \times G)/(N \times N)$ is identified with $G/N \times G/N$, we may regard $\overline{m}$ as the group multiplication on G/N. Similarly, if $i : G \to G$ is the inverse $x \mapsto x^{-1}$, then $\pi \circ i : G \to G/N$ is right N invariant, so descends to a morphism $\bar{i} : G/N \to G/N$, which is the inverse on G/N.