

Poisson Geometry of the Grothendieck Resolution for a semisimple complex group

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Motivation

In our papers “On the variety of Lagrangian subalgebras, I, II”, Annales ENS, 2001 and 2006 we found a Poisson structure π on semisimple complex group G

(really due to Alekseev-Malkin, CMP, 1994)

Conjugacy classes in G are Poisson subvarieties, and each conjugacy class has an open symplectic leaf

Further, π is closely related via semi-classical limit to quantized enveloping algebra $U_q(\mathfrak{g})$

Purpose of this work: better understand π .

Further, there is a family of Poisson varieties (G_t, π_t) such that

for $t \neq 0$, $(G_t, \pi_t) \cong (G, \pi)$

for $t = 0$, $(G_0, \pi_0) \cong (\mathfrak{g}, \pi_{KK})$, where \mathfrak{g} is Lie algebra of G , π_{KK} is Kostant-Kirillov bracket.

Conclude: (G, π) gives deformation of Kostant-Kirillov structure

Principle: Our interesting applications of Poisson geometry to Lie theory come from deformation and degeneration

Examples: (1) Poisson proof of Kostant and Kumar results on \mathfrak{n} -cohomology and Schubert calculus uses deformations implicit in work of Etingof and Varchenko

(2) Proof of Thompson conjecture relating eigenvalues of sums and products of matrices

Idea: Information about (G, π) should help understand Kostant-Kirillov structure

Problem: Conjugacy classes are curved and have subtle geometry, especially in closure. Poisson structure is also complicated and degenerates in places

Proposed solution: Resolution of singularities

Find Poisson structure Π on Grothendieck resolution $G \times_B B$

so $\mu : (G \times_B B, \Pi) \rightarrow (G, \pi), (g, b) \mapsto gbg^{-1}$ is Poisson

Poisson geometry

Poisson algebraic variety $(X, \{\cdot, \cdot\})$ is:

Algebraic variety X with Lie algebra structure $\{\cdot, \cdot\}$ on sheaf of functions, so $\xi_f = \{f, \cdot\}$ is a vector field for $f \in O_X$

Symplectic leaf through $x \in X$ is maximal connected subset tangent to $\{\xi_f : f \in O_X\}$

For X smooth, $\pi \in \wedge^2(TX)$ satisfying Schouten identity $[\pi, \pi] = 0$ gives Poisson bracket:

$$\{f, g\} = \pi(df, dg), \quad f, g \in O_X$$

In such cases, denote induced Poisson variety (X, π)

If (X, ω) is smooth symplectic, induced identification $\wedge^2(T^*X) \cong \wedge^2(TX)$ gives Poisson bivector π

Kostant-Kirillov Poisson structure π_{KK}

$\mathfrak{g} = \mathfrak{g}^*$ is Poisson:

Let $\langle \cdot, \cdot \rangle$ be Killing form.

$X \in \mathfrak{g}$ gives $f_X \in O(\mathfrak{g})$, $f_X(Y) = \langle X, Y \rangle$. Let $\{f_X, f_Y\} = f_{[X, Y]}$ and extend using Poisson property.

Symplectic leaves are adjoint orbits, orbit closures are Poisson subvarieties

Grothendieck resolution for Lie algebra

G semisimple complex Lie group

$T \subset B$, T maximal torus in Borel B

G has Lie algebra \mathfrak{g} , B has Lie algebra \mathfrak{b} , etc.

B^- opposite T -stable Borel. N, N^- unipotent radicals of B, B^-

$G/B =$ flag variety of G

Identify $G/B =$ variety of Borel subalgebras of \mathfrak{g}
via $gB \mapsto Ad_g(\mathfrak{b})$

$\mathcal{N} =$ nilpotent cone

Grothendieck variety $\tilde{\mathfrak{g}} = G \times_B \mathfrak{b}$

$\mu : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$, $\mu(g, x) = Ad_g(x)$, proper, generically finite, X smooth.

Definition: A Poisson variety (X, π) is regular if $rk(\pi_x)$ is constant

Example: On \mathbb{C}^2 , $\partial_x \wedge \partial_y$ is regular

$x\partial_x \wedge \partial_y$ is singular, rank zero on $\{x = 0\}$

On smooth symplectic variety, induced Poisson structure is regular

(\mathcal{N}, π_{KK}) is singular: π_{KK} has constant rank on regular orbit O_{reg} , lower rank off O_{reg}

A morphism $f : X \rightarrow Y$ is Poisson if pullback $f^* : O_Y \rightarrow O_X$ is a Lie algebra homomorphism

Definition: Let (X, π_X) and (Y, π_Y) be Poisson varieties. A Poisson morphism $f : X \rightarrow Y$ is called a Poisson desingularization if (X, π_X) is regular and $f : X \rightarrow Y$ is proper, birational, and X smooth

$\mu : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is simultaneous resolution of Steinberg fibers

For $x \in \mathfrak{t}$, $F_x := Ad_G(x + \mathfrak{n})$, finite union of orbits under adjoint action

F_x called Steinberg fiber, it is a fiber of adjoint quotient $\mathfrak{g} \rightarrow \mathfrak{g} // G$, so (F_x, π_{KK}) is Poisson

$G \times_B (x + \mathfrak{n})$ twisted cotangent bundle, esp. symplectic, so Poisson

$G \times_B (x + \mathfrak{n})$ symplectic leaves of $\tilde{\mathfrak{g}}$

$\mu : G \times_B (x + \mathfrak{n}) \rightarrow F_x$ Poisson desingularization

Group analogue: Poisson structure π_0 on G

(Alekseev-Malkin, CMP, 1994)

G has a standard Poisson Lie group structure π_G

$$D = G \times G$$

$$\text{Double } \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}^*$$

$$\text{Diagonal } \mathfrak{g}_\Delta = \{(y, y) : y \in \mathfrak{g}\}$$

$$\mathfrak{g}^* = \{(x + z, y - z) : x \in \mathfrak{n}, y \in \mathfrak{n}^-, z \in \mathfrak{t}\}$$

$\langle \cdot, \cdot \rangle = K_1 - K_2$ on \mathfrak{d} , nondegenerate form,

K_1, K_2 Killing forms on first and second factors

$\mathfrak{g}_\Delta, \mathfrak{g}^-$ maximal isotropic in \mathfrak{d}
with respect to $\langle \cdot, \cdot \rangle$

Identifies \mathfrak{g}^* with dual of \mathfrak{g}_Δ

$\{x_i\}$ basis of \mathfrak{g}

$\{\xi_i\}$ dual basis of \mathfrak{g}^*

$$\Lambda = \frac{1}{2} \sum \xi_i \wedge x_i \in \wedge^2(\mathfrak{d})$$

Set $\Pi_D^\pm = \Lambda^R \pm \Lambda^L$,

Λ^R, Λ^L , right and left-invariant vector fields with value Λ at identity

(D, Π_D^-) Poisson Lie group

(D, Π_D^+) Poisson homogeneous for (D, Π_D^-) , i.e., multiplication map $(D \times D, \Pi_D^- \times \Pi_D^+) \rightarrow (D, \Pi_D^+)$ is Poisson

Identify $D/G_\Delta \cong G$, via $(x, y) \mapsto xy^{-1}$

Projection $p : D \rightarrow D/G_\Delta \cong G$

Definition: $\pi = p_*(\Pi_D^\dagger)$

Properties: (1) π Poisson

(2) (G, π) is (G, π_G) homogeneous under conjugation action, i.e.,

$(G \times G, \pi_G \times \pi) \rightarrow (G, \pi)$, $(g, x) \mapsto gxg^{-1}$ is Poisson

(3) Let $S \subset G$ be symplectic leaf for π . Then $T \cdot S := \{txt^{-1} : t \in T, x \in S\} = C_x \cap BwB^-$

where $C_x =$ conjugacy class of x , $w \in W$ Weyl group

(4) $C_x \cap BwB^-$ regular Poisson subvariety

Group analogue of Grothendieck resolution

$X = G \times_B B$, where B acts on B by conjugation

$$\mu : X \rightarrow G, \mu(g, b) = gbg^{-1}$$

Poisson structure Π on $G \times_B B$ (E-Lu):

Construct Π using coadjoint reduction:

π_D^\dagger is Poisson structure on D

Embedding $\psi : G \times B \rightarrow D, (g, x) \mapsto (gx, g)$

$Q := \psi(G \times B)$ coisotropic subvariety of D , i.e., consider $\tilde{\pi}_D^\dagger : T^*D \rightarrow TD$, anchor map $(d, \alpha) \mapsto (d, i_\alpha \pi_D^\dagger)$

$T_Q^*(D) =$ conormal bundle to Q in D

By definition, Q is coisotropic in D if $\tilde{\pi}_D^\dagger(T_Q^*(D)) \subset TQ$

Let B_Δ be diagonal in $B \times B$

Prove: $\pi_D^\dagger(T_Q^*(D)) =$ tangent space to right B_Δ -
action

π_D^\dagger descends to Q/B_Δ , giving Poisson structure Π
on $G \times_B B \cong Q/B_\Delta$

Group version of Steinberg fiber:

$$t \in T, X_t = G \times_B tN$$

$$\text{Steinberg fiber } F_t = \mu(X_t)$$

F_t = level set for invariant functions,
fiber of adjoint quotient $G \rightarrow G//G$

$\mu : X_t \rightarrow F_t$ generically finite

$\mu : X_t \rightarrow F_t$ resolution of singularities if G simply connected

For simplicity, assume G simply connected

F_t irreducible, finite union of conjugacy classes,
contains unique conjugacy class of
maximal dimension

For $w \in W$, $F_{t,w} = F_t \cap BwB^-$, $X_{t,w} = \mu^{-1}(F_{t,w})$

$F_{t,w}$ irreducible, normal, complete intersection in
 BwB^-

$F_{t,w}$ Poisson subvariety of (G, π)

$S \subset F_{t,w}$ symplectic leaf; $T \cdot S = C \cap BwB^-$ for unique conjugacy class C meeting S

$F_{t,w}$ is singular off $C \cap BwB_-$, Poisson structure also degenerates

$X_{t,w}$ smooth, irreducible

$\mu : X_{t,w} \rightarrow F_{t,w}$ resolution of singularities (requires G simply connected)

Poisson geometry of (X, Π)

(1) (X, Π) is (G, π_G) Poisson homogeneous space

(2) $\mu : (X, \Pi) \rightarrow (G, \pi)$ is Poisson

(3) $X_{t,w}$ Poisson subvariety of (X, Π)

(4) S symplectic leaf in $X_{t,w}$, $T \cdot S = \{t \cdot x : x \in S\}$
 $T \cdot S = X_{t,w}$

$(X_{t,w}, \Pi)$ smooth regular Poisson variety

$\mu : (X_{t,w}, \Pi) \rightarrow (F_{t,w}, \pi)$ is Poisson desingularization, i.e.,

Poisson structure on $F_{t,w}$ is singular, rank varies
Poisson structure on $X_{t,w}$ is regular.

Next paper: Use these results to find log-canonical coordinates on $X_{t,w}, F_{t,w}$

There is dual Poisson Lie group G^* . These results should lead to good understanding of G^*

Further: Cluster algebra structures, connections to combinatorics