MATH 13150: Freshman Seminar
Unit 9

1. More on prime numbers

1.1. **Another interpretation of prime numbers.** In this section, we want to focus on another property of primes numbers.

PRIME PROPERTY: A prime number is a number that only divides a product ab of two numbers a and b if it divides either a or b.

Let's start with the example of the prime number 2. The claim is that if 2 divides a product $a \cdot b$ of two numbers a and b, then 2 divides either a or b. Since a number is divisible by 2 exactly when it is even, this says the following:

(1): If a or b is even, then $a \cdot b$ is even.
(2): If on the other hand, a and b are both odd, then $a \cdot b$ is odd.

This corresponds to your experience. Certainly any time you try to multiply an even number with any other number, you get an even number. For example, $4 \cdot 1375$ will be even because 4 is even. On the other hand, $7 \cdot 133$ will be odd, because 7 and 133 are odd. This is a special case of the principle.
Here are some other examples:

For the prime 5:
(1) If a or b is divisible by 5, then $a \cdot b$ is divisible by 5
(2) If neither a nor b is divisible by 5, then $a \cdot b$ is not divisible by 5.

Again this corresponds to your experience. While $15 \cdot 1237$ will be a multiple of 5 since 15 is a multiple of 5, $13 \cdot 37$ is not going to be a multiple of 5. Indeed, the 1's digit of $13 \cdot 37$ is going to be 1, so $13 \cdot 37$ is not divisible by 5 because it does not end with 0 or 5.

For another example, let's look at a multiplication table:

*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	4	6	8	10	12	14	16
3	3	6	9	12	15	18	21	24
4	4	8	12	16	20	24	28	32
5	5	10	15	20	25	30	35	40
6	6	12	18	24	30	36	42	48
7	7	14	21	28	35	42	49	56
8	8	16	24	32	40	48	56	64

Let's consider the prime 3. Notice that the only times a product $a \times b$ of two numbers is a multiple of 3 is when either a or b is a multiple of 3. For example, $5 \times 2 = 10$ is not a multiple of 3 and 3 does not divide 5 or 2. Another way of sayiing this is that

the only entries in the multiplication table that are multiples of 3 are in the 3 or 6 row, or in the 3 or 6 column.
Similarly, if you consider the prime 7, you will see that while every entry in the 7 row and every entry in the 7 column are multiples of 7, no other entries are.
On the other hand, the basic principle fails for composite numbers.

Let n be a composite number. Then we can find numbers a and b so that n divides $a \cdot b$, but n does not divide either a or b.

In fact, a and b are easy to find in examples for any given n. Let's take $n = 6$. Let $a = 4$ and let $b = 15$. Then $a \cdot b = 60$, and 6 divides 60, but 6 does not divide either 4 or 15.
We can find an even simpler example. Again take $n = 6$. Let $a = 2$ and $b = 3$. Then 6 divides $a \cdot b = 6$, but 6 does not divide a or b.
This example explains why the prime property fails for a composite number. If the composite number factors as $n = a \cdot b$, with $a, b < n$, then n divides $a \cdot b = n$, but n does not divide a or b. For example, $21 = 3 \cdot 7$ is composite, and certainly 21 divides itself, but 21 does not divide 3 or 7. My point is that there is nothing special about the number 21. For any composite number you can do the same thing.
Let's now reinterpret the prime property in terms of division.

PRIME PROPERTY FOR DIVISION: Suppose a prime number p divides a number n and does not divide a factor a of n. Then p divides $\frac{n}{a}$.

Let's illustrate this with an example involving the prime number 7. You can check 7 divides 84. Also, note that 4 is a factor of 84 and 7 does not divide 4. The principle says that 7 should divide $\frac{84}{4} = 21$, and of course you can see it does.
Let's see why it does. The point is that we can factor
$84 = 2 \times 2 \times 3 \times 7$, so we can write
$\frac{84}{4} = \frac{2 \times 2 \times 3 \times 7}{2 \times 2} = 3 \times 7.$
The idea is that we cannot use the 2's in the denominator to cancel off the 7 in the numerator. I'm saying that this works in general.
On the other hand, the same principle fails miserably for composite numbers. In other words, if we take a composite number (e.g. 6), then 6 divides 24 and does not divide 8, and 6 does not divide $\frac{24}{8} = 3$. The reason should be pretty clear in terms of the last calculation. If we look at:
$\frac{24}{8} = \frac{2 \times 2 \times 2 \times 3}{2 \times 2 \times 2} = 3,$
we have cancelled out all the 2's in the numerator (fancy math term for top) of the fraction $\frac{24}{8}$. Then 6 does not divide $\frac{24}{8}$ since 6 cannot divide an odd number. 6 only divides a number if it has a factor of 2 and a factor of 3 in it.
In fact, the PRIME PROPERTY at the beginning of this section and the PRIME PROPERTY FOR DIVISION are just different ways of saying the same thing. Indeed,

if we look at the PRIME PROPERTY FOR DIVISION and let $n = a \times b$, then the statement that

A prime number p divides $n = a \times b$ and p does not divide a implies that p divides $\frac{n}{a} = \frac{a \cdot b}{b}$

is identical to the statement:

A prime number p divides $n = a \times b$ and p does not divide a implies that p divides b

This last statement is just a way of restating the PRIME PROPERTY.

Now let's see a reason why the PRIME PROPERTY is true. We'll do it in the special case of the prime number 3. To be more concrete, let's take $a = 7$, and certainly 3 does not divide 7. We want to show that if 3 divides the number $7 \cdot b$, then 3 divides the number b.
To do this, note that since 3 does not divide 7, $\gcd(3, 7) = 1$. By the discussion of Unit 7, we can write 1 as a combination of 3 and 7, or to be more explicit, we write

$1 = 7 - 2 \cdot 3$.

Now multiply this equation by b. This gives,

$b = 7 \cdot b - 2 \cdot 3 \cdot b$.

Since 3 divides $7 \cdot b$ by assumption, and 3 obviously divides $2 \cdot 3 \cdot b$, and 3 must divide the difference of two multiples of 3, it follows that 3 divides b.
To do this argument for a general prime p and a general number a that is not divisible by p, note that since p does not divide a, $\gcd(a, p) = 1$. Indeed, if a number divides a and p, it would have to divide p, so the number would have to be 1 or p. But p does not divide a by assumption, so the number must be 1, and $\gcd(a, p) = 1$. Now use the results of Unit 7 to conclude that we can write 1 as a combination of a and 7, that is,

$1 = x \cdot a + y \cdot p$ for some numbers x, y, which can be positive or negative.

Now multiply each side by b to get:

$b = x \cdot a \cdot b + y \cdot p \cdot b$.

Then both terms on the right side of the equation are divisible by p, since p divides $a \cdot b$, and p obviously divides $y \cdot p \cdot b$. It follows that p also divides b.

1.2. **Pascal's triangle revisited.** Let's go back to Pascal's triangle. Look at the 7-row of Pascal's triangle. It's entries are:

1 7 21 35 35 21 7 1

Notice that all of the entries besides the $1's$ on the end are divisible by 7, i.e., $7, 21$, and 35 are all multiples of 7.
Let's look at the 11-row. It's entries are:

1 11 55 165 330 462 462 330 165 55 11 1

Notice that all the entries besides the $1's$ on the end are divisible by 11, i.e., $11, 55, 165, 330$, and 462 are multiples of 11. You can check this using the alternating digits rule for divisibility by 11. This observation we have seen for the 7-row and the 11-row are special cases of the following general fact.

FACT: For a prime number p, every entry in the p-row of Pascal's triangle besides the 1's on the end are divisible by p.

So for example, this means that since 13 is prime, all entries of the 13-row of Pascal's triangle besides the $1's$ on the ends are multiples of 13. Let's see why this works. The entries of the 13-row of Pascal's triangle are:

$\binom{13}{0} = 1,\ \binom{13}{1},\ \binom{13}{2},\ \binom{13}{3},\ \binom{13}{4},\ \ldots\ \binom{13}{12},\ \binom{13}{13} = 1.$

Let's try to see why $\binom{13}{4}$ should be a multiple of 13 without actually computing the number. We can write $\binom{13}{4}$ as:

$$\binom{13}{4} = \frac{13!}{4! \cdot 9!} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2 \cdot 1}.$$

Certainly 13 divides the numerator $13 \cdot 12 \cdot 11 \cdot 10$ of $\binom{13}{4}$ and 13 does not divide the denominator $4 \cdot 3 \cdot 2 \cdot 1$ since there is no factor of 13 in the denominator. So by the PRIME PROPERTY FOR DIVISION, 13 must divide $\binom{13}{4}$. It would provide us less insight to compute $\binom{13}{4}$ and then check on a calculator that 13 divides $\binom{13}{4}$. The same argument shows that 13 divides any interior entry $\binom{13}{k}$ if $k = 1, \ldots, 12$ of the 13-row of Pascal's triangle. Similarly, for any prime number p, we can see that p divides any entry $\binom{p}{k}$ with $k = 1, \ldots, p-1$. We simply write:

$$\binom{p}{k} = \frac{p!}{k! \cdot (p-k)!},$$

and note that p divides the numerator $p!$ of $\binom{p}{k}$, while p does not divide the denominator $k! \cdot (p-k)!$ since p does not appear as a factor in either $k!$ or $(p-k)!$.
On the other hand, for every composite number n, we can find an interior entry of the n-row of Pascal's triangle so that n does not divide $\binom{n}{k}$. For example, take $n = 4$ and consider the 4-row of Pascal's triangle, which has entries

1 4 6 4 1.

Clearly, 4 does not divide 6. It is easy to explain this. The 6 that appears in the 4-row is $\binom{4}{2} = \frac{4!}{2! \cdot 2!}$, which we can write as:

$$\frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1}.$$

We can use the two 2's in the denominator to cancel off the 4 in the numerator. We only have $2 \cdot 3$ left in the numerator, which is not divisible by $4 = 2 \times 2$, since there are factors of 2 in 4 are only one in 6.

EXAMPLE: Find an entry in the 35-row of Pascal's triangle that is not divisible by 5.

To solve this, look at $\binom{35}{5}$, and write it as a fraction.

$$\binom{35}{5} = \frac{35 \cdot 34 \cdot 33 \cdot 32 \cdot 31}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}.$$

The only term in the numerator divisible by 5 is 35, and $35 = 5 \cdot 7$. This means if we factor the numerator of this fraction as a product of primes, it will look like:

5 times a product of some other primes.

When we cancel this 5 in the numerator with the 5 in the denominator, there are no factors of 5 left. This means the fraction is not a multiple of 5, so 5 does not divide $\binom{35}{5}$.

1.3. **Multiplying and Dividing using prime factorizations.**

It is useful to recall a few facts about manipulating exponents. Let a and n be numbers. We write a^n for a multiplied with itself n times. For example, $3^4 = 3 \cdot 3 \cdot 3 \cdot 3$.

If m and n are two numbers, then $a^m \cdot a^n = a^{m+n}$.

For example, $3^3 \cdot 3^2 = 3^{3+2} = 3^5$. It is easy to see why this is true:
$3^3 \cdot 3^2 = (3 \cdot 3 \cdot 3) \cdot (3 \cdot 3) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5$.

If m and n are two numbers, then $\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$.

For example, $\frac{3^7}{3^2} = 3^{7-2} = 3^5$, and $\frac{3^2}{3^7} = \frac{1}{3^5}$.

If m and n are numbers, then $(a^m)^n = a^{m \cdot n}$.

For example, $(3^5)^4 = 3^{5 \cdot 4} = 3^20$.

Now, we'll use these properties to see how to multiply and divide numbers using their prime factorizations. The idea is that we work one prime at a time.

PROBLEM: Let $m = 2^3 \cdot 3^5 \cdot 7^2 \cdot 11 \cdot 17^4$ and let $n = 2^2 \cdot 3^3 \cdot 5^2 \cdot 11^3 \cdot 13$. Find $m \cdot n$ and $\frac{m}{n}$.

To do this, just rearrange the factors so that factors with the same prime can easily be combined.

$m \cdot n = 2^3 \cdot 3^5 \cdot 7^2 \cdot 11 \cdot 17^4 \cdot 2^2 \cdot 3^3 \cdot 5^2 \cdot 11^3 \cdot 13 = 2^3 \cdot 2^2 \cdot 3^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 11^3 \cdot 13 \cdot 17^4$,
which equals

$$2^5 \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11^4 \cdot 13 \cdot 17^4.$$

When we do this, we multiply $11 \cdot 11^3$ using $11 = 11^1$, so $11 \cdot 11^3 = 11^1 \cdot 11^3 = 11^4$.
To compute $\frac{m}{n}$, we reason similarly.

$$\frac{m}{n} = \frac{2^3 \cdot 3^5 \cdot 7^2 \cdot 11 \cdot 17^4}{2^2 \cdot 3^3 \cdot 5^2 \cdot 11^3 \cdot 13} = \frac{2^3}{2^2}\frac{3^5}{3^3}\frac{1}{5^2}\frac{7^2}{1}\frac{11}{11^3}\frac{1}{13}\frac{17^4}{1},$$

which equals

$$\frac{2 \cdot 3^2 \cdot 7^2 \cdot 17^4}{5^2 \cdot 11^2 \cdot 13}.$$

You could multiply these numbers out to rewrite the answer as $\frac{73665522}{39325}$, but this provides little insight.

PROBLEM : Find the prime factorization of $84 \cdot 120$.

To do this, first find the prime factorizations of 84 and 120:
$84 = 2^2 \cdot 3 \cdot 7$ and $120 = 2^3 \cdot 3 \cdot 5$.
Then we multiply:

$$84 \cdot 120 = 2^2 \cdot 3 \cdot 7 \cdot 2^3 \cdot 3 \cdot 5 = 2^2 \cdot 2^3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 2^5 \cdot 3^2 \cdot 5 \cdot 7.$$

If we also wanted to compute $\frac{84}{120}$ using prime factorizations, we could do this:

$$\frac{84}{120} = \frac{2^2 \cdot 3 \cdot 7}{2^3 \cdot 3 \cdot 5} = \frac{2^2}{2^3}\frac{3}{3}\frac{1}{5}\frac{7}{1} = \frac{7}{2 \cdot 5}.$$

1.4. **When does one number divide another?**

PROBLEM: Does $m = 2^3 \cdot 3^2 \cdot 11 \cdot 13^3$ divide $n = 2^2 \cdot 5 \cdot 11^2 \cdot 13 \cdot 17$.
To do this, we compute $\frac{n}{m}$ as in the last section. If we can write $\frac{n}{m}$ as a number with no denominator, then m divides n. If we can't, then m does not divide n. We compute

$$\frac{n}{m} = \frac{5 \cdot 11 \cdot 17}{2 \cdot 3^2 \cdot 13^2}.$$

There is nothing in the numerator we can use to cancel the 2 term in the denominator, so m does not divide n.
The same idea gies the following principle:

DIVISIBILITY PRINCIPLE: A number m divides a number n if every prime dividing m divides m at least as many times as it divides n.

PROBLEM: Does m divide n if $m = 2^3 \cdot 3^2 \cdot 7$ and $n = 2^4 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11^{21}$.

To solve this problem, use the principle. The primes dividing m are $2, 3$ and 7. Since 2^3 is the power of 2 in m, 2 divides m 3 times. Since 2 divides n 4 times, the principle if OK for 2. For 3, we see that 3 divides m 2 times, and also 3 divides n 2 times, so the principle if OK for 3. For 7, we say that 7 divides m 1 time, but divides n 2 times, so the principle is OK for 7. Since this is OK for each of the primes dividing m, we conclude that m divides n.
Alternatively, we can compute

$$\frac{n}{m} = \frac{2 \cdot 5 \cdot 7 \cdot 11^{21}}{1},$$

is a whole number, so m divides n.

PROBLEM: Does m divide n if $m = 2^3 \cdot 3^4 \cdot 5^2$ and $n = 2^{146} \cdot 3^3 \cdot 5^{21}$.

The answer is NO, because the primes 3 divides m 4 times, but only divides n 3 times.
Alternatively, compute

$$\frac{n}{m} = \frac{2^{143} 5^{19}}{3},$$

which is not a whole number. There is no way to cancel the 3 left in the denominator with anything in the numerator. This means $\frac{n}{m}$ is not a whole number, so m does not divide n.

NEW PROBLEM: How many numbers divide 720?

It would be a pain to try to list them all, but fortunately we don't have to give a complete list. The prime factorization of 720 is $720 = 2^4 \cdot 3^2 \cdot 5^1$ (note that we wrote 5^1 in place of 5). By the divisibility criterion we just discussed, for a number to divide 720, its only prime factors are $2, 3$, and 5. Further, it must be of the form:
$m = 2^a \cdot 3^b \cdot 5^c$, where $a = 0, 1, 2, 3$, or 4, $b = 0, 1$, or 2, and $c = 0$ or 1. This means we have 5 different choices for a, 3 different choices for b, and 2 different choices for c. These numbers can be chosen independently, so there are $5 \cdot 3 \cdot 2 = 30$ different numbers dividing 720.

PROBLEM: How many numbers divide $2^3 \cdot 5^7 \cdot 11^4$?

Application of the same idea shows that the answer is:

$$(3 + 1) \cdot (7 + 1) \cdot (4 + 1) = 160.$$

PROBLEM: How many numbers divide 75? List them.

Using the prime factorization $75 = 3^1 \cdot 5^2$, the answer is $(1 + 1) \cdot (2 + 1) = 6$. To list them, make a table with the different possible divisors $3^a \cdot 5^b$ with $a = 0$ or 1, and $b = 0, 1$ or 2. We get:

$3^0 5^0$	$3^0 5^1$	$3^0 5^2$
$3^1 5^0$	$3^1 5^1$	$3^1 5^2$

or listing these as ordinary numbers,

1	5	25
3	15	75

so $1, 5, 25, 3, 15, 75$ are the six numbers dividing 75.

1.5. **Greatest common divisors and least common multiples.** It is very easy to compute the greatest common divisor and least common multiple of two numbers from their prime factorizations. Let's see how to do this in an example.

PROBLEM : Find $\gcd(m, n)$ and $\text{lcm}(m, n)$ where $m = 2^3 3^4 5^7 11^3 13^3$ and $n = 2^5 3^4 5^3 11^2 17^2$.

To compute $\gcd(m, n)$, find the primes dividing both m and n. They are

$2, 3, 5, 11$.

For each of these primes, take the least power of the prime you see in m or n. For the prime 2, the powers are 2^3 in m and 2^5 in n, so 2^3 is the least power. For the prime 3, the powers are 3^4 in m and 3^4 in n, so 3^4 is the least power (if you get a tie, you just call either the least power). For the prime 5, the powers are 5^7 in m and 5^3 in n, so 5^3 is the least power. For the prime 11, 11^2 is the least power. Then $\gcd(m, n)$ is just the product of the least powers we found. This means:

$\gcd(m, n) = 2^3 3^4 5^3 11^2$.

It is very easy to see that $2^3 3^4 5^3 11^2$ divides both m and n using the results about divisibility from the last section. Similarly, it is very easy to see that if a number divides both m and n, it must divide $2^3 3^4 5^3 11^2$ using the criterion from the last section. This is an argument explaining why $\gcd(m, n) = 2^3 3^4 5^3 11^2$.

To compute $\text{lcm}(m, n)$, find the primes dividing either of m and n or both. They are:

$2, 3, 5, 11, 13, 17$.

For each of these primes, take the biggest power of the prime that you see. For the prime 2, the biggest power is 2^5, for the prime 3, the biggest power is 3^4, for the prime 5, the biggest power is 5^7, for the prime 11, the biggest power is 11^3, for the prime 13, the biggest power is 13^3 (that is the only power of 13 in m or n), and for 17, the biggest power is 17^2.

To find $\text{lcm}(m, n)$, just take the product of the biggest powers that we saw. This means:

$\text{lcm}(m, n) = 2^5 3^4 5^7 11^3 13^3 17^2$.

Again, using the divisibility criterion from the last section, it is very easy to see that both m and n divide $2^5 3^4 5^7 11^3 13^3 17^2$. Furthermore, there is no smaller number divisible by both m and n, so $2^5 3^4 5^7 11^3 13^3 17^2$ is really $\text{lcm}(m, n)$.

The same basic idea can be used to compute $\gcd(m, n)$ and $\text{lcm}(m, n)$ for any two numbers m and n. That is,

TO COMPUTE $\gcd(m, n)$

1. Find the primes dividing both m and n.
2. For each of these primes, find the least power of the prime appearing in m or n.

3. Then $\gcd(m, n)$ is the product of these primes to the least power.

TO COMPUTE $\operatorname{lcm}(m, n)$

1. Find the primes dividing either m or n, or both.
2. For each of these primes, find the biggest power of the prime appearing in m or n.
3. Then $\operatorname{lcm}(m, n)$ is the product of these primes to the biggest power.

PROBLEM: Compute $\gcd(120, 260)$ and $\operatorname{lcm}(120, 260)$.

First, let's solve this using the technique just discussed. We factor 120 and 260 as a product of primes:

$120 = 2^3 \cdot 3 \cdot 5$, $260 = 2^2 \cdot 5 \cdot 13$. The primes dividing both 120 and 260 are 2 and 5. Taking least powers gives

$\gcd(120, 260) = 2^2 \cdot 5 = 20$. The primes dividing either 120 or 260 or both are $2, 3, 5, 13$. Taking greatest powers gives

$\operatorname{lcm}(120, 260) = 2^3 \cdot 3 \cdot 5 \cdot 13 = 1560$. Now, let's try doing it by the technique from Unit 6 using the Euclidean algorithm. The steps for computing $\gcd(120, 260)$ are:

$260 = 2 \cdot 120 + 20$

$120 = 6 \cdot 20 + 0$ so $\gcd(120, 260) = 20$. Then we get $\operatorname{lcm}(120, 260) = \dfrac{120 \cdot 260}{20} = 1560$.

In other words, both procedures for computing gcd and lcm give the same answer. Let's also note that we can now explain why $\gcd(m, n) \cdot \operatorname{lcm}(m, n) = m \cdot n$. Let's try this with the m and n from our first example:

$m = 2^3 3^4 5^7 11^3 13^3$ and $n = 2^5 3^4 5^3 11^2 17^2$. We saw that $\gcd(m, n) = 2^3 3^4 5^3 11^2$ and $\operatorname{lcm}(m, n) = 2^5 3^4 5^7 11^3 13^3 17^2$. Now compute $\gcd(m, n) \cdot \operatorname{lcm}(m, n)$ and compare it to $m \cdot n$:

$$\gcd(m, n) \cdot \operatorname{lcm}(m, n) = 2^3 3^4 5^3 11^2 \cdot 2^5 3^4 5^7 11^3 13^3 17^2,$$

while

$$m \cdot n = 2^3 3^4 5^7 11^3 13^3 \cdot 2^5 3^4 5^3 11^2 17^2,$$

and it is pretty easy to see that these two products are the same after we rearrange the factors. Indeed, if we collect like factors

$$\gcd(m, n) \cdot \operatorname{lcm}(m, n) = 2^3 2^5 3^4 3^4 5^3 5^7 11^2 11^3 13^3 17^2,$$

while

$$m \cdot n = 2^3 2^5 3^4 3^4 5^7 5^3 11^3 11^2 13^3 17^2,$$

so $\gcd(m, n) \cdot \operatorname{lcm}(m, n) = m \cdot n$, since we can easily see these last two expressions are the same. The same argument works in general to justify the fact we discovered in Unit 6 by looking at examples. There is nothing special about the particular m and n we used in these examples.

Theorem 1.1. *For any two numbers m and n,*
$\gcd(m,n)\cdot \operatorname{lcm}(m,n)=m\cdot n$.

Let's do one more example to point out how we deal with powers like $3=3^1$.

PROBLEM: Compute $\gcd(m,n)$ and $\operatorname{lcm}(m,n)$ if $m=2^5\cdot 3\cdot 19^2\cdot 23^3$ and $n=2\cdot 3\cdot 17^3\cdot 23^2$.
The primes dividing both m and n are:
$2,3,23$. The least power of 2 is $2=2^1$. The least power of 3 is $3=3^1$. The least power of 23 is 23^2. This gives:
$\gcd(m,n)=2\cdot 3\cdot 23^2$. To compute $\operatorname{lcm}(m,n)$, the primes dividing either or both are:

$2,3,17,19,23$ Taking greatest powers (note that the greatest power of 3 is 3^1), we get:
$\operatorname{lcm}(m,n)=2^5 3 17^3 19^2 23^2$.

1.6. Which primes divide $\binom{n}{k}$?

PROBLEM: Does 3 divide $\binom{15}{10}$? Does 2 divide $\binom{15}{10}$?
To answer this, write

$$\binom{15}{10}=\frac{15\cdot 14\cdot 13\cdot 12\cdot 11}{5\cdot 4\cdot 3\cdot 2\cdot 1},$$

and use the prime factorizations of each of the numbers we see to rewrite
$\binom{15}{10}=\frac{5\cdot 3\cdot 2\cdot 7\cdot 13\cdot 2^2\cdot 3\cdot 11}{5\cdot 2^2\cdot 3\cdot 2}$
$=\frac{2^3\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13}{2^3\cdot 3\cdot 5}=3\cdot 7\cdot 11\cdot 13,$
so the only primes dividing $\binom{15}{10}$ are $3,7,11$ and 13. In particular, 3 divides $\binom{15}{10}$ but 2 does not.
We can solve this in a more instructive way by using the following kind of reasoning.
Look at $\binom{15}{10}$, written as a fraction

$$\frac{15\cdot 14\cdot 13\cdot 12\cdot 11}{5\cdot 4\cdot 3\cdot 2\cdot 1}.$$

Look for multiples of 3. In the numerator, $15=3\cdot 5$ and $12=3\times 2^2$ are the only multiples of 3. This means that we can write the numerator as

$3\cdot 3$ times other primes besides 3.

Similarly, in the denominator, 3 is the only multiple of 3, so we can write the denominator as

3 times other primes.

It follows that we can write $\binom{15}{10} = \frac{3^2}{3}$ times a fraction which has no factors of 3 on top or bottom. Since $\frac{3^2}{3} = 3$, we can write

$\binom{15}{10} = 3$ times a product of primes besides 3. In particular, 3 divides $\binom{15}{10}$.

If we try this with 2, we see that in the numerator of $\binom{15}{10}$, only $14 = 2 \cdot 7$ and $12 = 2^2 \cdot 3$ are multiples of 2. In the denominator, only 2 and $4 = 2^2$ are multiples of 2. This means we can write:

$\binom{15}{10} = \frac{2 \cdot 2^2}{2 \cdot 2^2}$ times a fraction with no factors of 2 on top or bottom, so:

$\binom{15}{10} = \frac{2^3}{2^3}$ times a fraction with no $2's$, so

$\binom{15}{10}$ is a fraction with no $2's$, since the 2^3 in the denominator cancels the 2^3 in the numerator.

To see the advantage of the second approach to the problem, let's try a variant with bigger numbers.

PROBLEM: Does 7 divide $\binom{40}{20}$? Does 11 divide $\binom{40}{20}$? Does 17 divide $\binom{40}{20}$?

To answer the first question, write

$$\binom{40}{20} = \frac{40 \cdot 39 \cdot 38 \cdots 22 \cdot 21}{20 \cdot 19 \cdot 18 \cdots 2 \cdot 1}.$$

The multiples of 7 in the numerator are: $35, 28, 21$. Since $35 = 7 \cdot 5$, $28 = 7 \cdot 2^2$ and $21 = 7 \cdot 3$, the numerator is 7^3 times a product of primes with no 7's.

The multiples of 7 in the denominator are: $14 = 7 \cdot 2$ and 7, so the denominator is 7^2 times a product of primes besides 7. It follows that:

$\binom{40}{20} = \frac{7^3}{7^2}$ times a fraction with no 7's, so

$\binom{40}{20} = 7$ times a number not divisible by 7.

We conclude that 7 does divide $\binom{40}{20}$.

Similarly, 11 divides $\binom{40}{20}$. The multiples of 11 in the numerator are 22 and 33, so the numerator of $\binom{40}{20}$ is 11^2 times a number not divisible by 11. The only multiple of 11 in the denominator is 11, so the denominator is 11 times a number not divisible by 11. It follows that:

$\binom{40}{20} = \frac{11^2}{11}$ times a fraction with no multiples of 11, so
$\binom{40}{20} = 11$ times a number not divisible by 11, and 11 divides $\binom{40}{20}$.
Similar reasoning shows that 17 does not divide $\binom{40}{20}$. Since 34 in the numerator and 17 in the denominator are the only multiples of 17 in the fraction for $\binom{40}{20}$,
$\binom{40}{20} = \frac{17}{17}$ times a number not divisible by 17, so
$\binom{40}{20} = 1$ times a number not divisible by 17. All the 17's were cancelled when we wrote $\frac{17}{17} = 1$.

APPLICATION: Does 77 divide $\binom{40}{20}$? Does 119 divide $\binom{40}{20}$?
To answer this question, note that it follows from our divisibilty criterion that a product of two primes divides a number exactly when each of the primes divides the number. For example, $77 = 7 \cdot 11$ divides a number exactly when 7 and 11 both divide the number. It follows that 77 divides $\binom{40}{20}$ since 7 and 11 both do. On the other hand, $119 = 7 \cdot 17$ does not divide $\binom{40}{20}$ since although 7 does divide $\binom{40}{20}$, 17 does not.

1.7. **Which numbers are squares of fractions?** Suppose we want to know whether a number like 2 is the square of a fraction, or equivalently, whether $\sqrt{2}$ is a fraction. How do we answer this kind of question?
A fraction is written in reduced form when no prime divides both the numerator and denominator. For example,
$\frac{2^3 \cdot 3^2 \cdot 11}{2^4 \cdot 5 \cdot 13}$ is not in reduced form, since 2 divides the numerator and denominator.
On the other hand, $\frac{3^2 \cdot 11}{2 \cdot 5 \cdot 13}$ is in reduced form. Clearly, we can always put any fraction into reduced form by cancelling.
So let's take a fraction in reduced form, such as:
$\frac{2^2 \cdot 3^5 \cdot 5}{7^3 \cdot 11^4}$.
If we square this fraction by multiplying it by itself, a moment's calculation shows that the square is given by doubling each of the exponents:
$\frac{2^4 \cdot 3^{10} \cdot 5^2}{7^6 \cdot 11^8}$.

Note that each exponent appearing in the fraction is even. This is always true:

FACT: The square of any fraction in reduced form is a fraction in reduced form where every prime occurs to an even power.

$\frac{2^3 \cdot 3^4}{5^2 \cdot 7^6}$ is not the square of a fraction. It can't be, because it is in reduced form and the prime 2 appears with an odd exponent.

FACT: Any fraction in reduced form where each prime occurs to an even power is the square of a fraction.

To see why this is true, consider an example, like
$\frac{2^4 \cdot 3^2 \cdot 5^6}{7^4 \cdot 13^8}$.
Note all exponents are even. To see this is the square of a fraction, just divide all the exponents by 2 to get:
$\frac{2^2 \cdot 3 \cdot 5^3}{7^2 \cdot 13^4}$.
A moment's calculation shows that:
$(\frac{2^2 \cdot 3 \cdot 5^3}{7^2 \cdot 13^4})^2 = \frac{2^4 \cdot 3^2 \cdot 5^6}{7^4 \cdot 13^8}$.
If a fraction is not in reduced form, just put it in reduced form. For example,
$\frac{2^5 \cdot 3^3 \cdot 5^4}{2^9 \cdot 3 \cdot 11^4} = \frac{3^2 \cdot 5^4}{2^4 \cdot 11^4}$
is in reduced form and all exponents are even, so it is the square of a fraction. In fact, it is the square of
$\frac{3 \cdot 5^2}{2^2 \cdot 11^2}$.
Now to answer the question we began with, let's see that 2 is not the square of a fraction. In reduced form $2 = \frac{2^1}{1}$. Since 2 appears to an odd exponent, 2 is not the square of a fraction. The 1 appears in the denominator just to make 2 look like a fraction.
EXERCISES: Explain your answer.
FIRST TWO PROBLEMS WERE MISSTATED IN ORIGINAL VERSION, fixed 3-4, 7 pm

(1) Find two numbers a and b so that 33 divides $a \cdot b$, but 33 does not divide a or b.
(2) Find two numbers a and b so that 60 divides $a \cdot b$, but 60 does not divide a or b.
(3) For the following pairs of numbers m and n, decide whether m divides n, and if so, compute $\frac{n}{m}$.
    (a) $m = 2^4 \cdot 3^3 \cdot 7$ and $n = 2^4 \cdot 3^5 \cdot 5^3 \cdot 7^5 \cdot 11$.
    (b) $m = 2^2 \cdot 3^2 \cdot 5^2 \cdot 101$ and $n = 2 \cdot 3^3 \cdot 5^{23} \cdot 101^2$

   (c) $m = 2^3 \cdot 3^3 \cdot 5^3$ and $n = 2^4 \cdot 3^5 \cdot 5^3$
   (d) $m = 2^3 \cdot 3^3 \cdot 5^3$ and $n = 2^4 \cdot 3^5 \cdot 5^2$

(4) Determine how many numbers divide each of the following numbers:
   (a) $2^5 \cdot 3^3 \cdot 5^7$
   (b) $2 \cdot 3 \cdot 5 \cdot 13$
   (c) 68
   (d) 420
   (e) 3080

(5) Exactly one of the following numbers is divisible by 101. Decide which one is divisible by 101 and explain why it is divisible by 101 and why the others are not divisible by 101.
   (a) 2^{437}
   (b) $\binom{100}{48}$
   (c) $\binom{101}{47}$.

(6) For each of the following pairs of numbers m and n, compute $\gcd(m, n)$, $\operatorname{lcm}(m, n)$ and $m \cdot n$. Verify that $\gcd(m, n) \cdot \operatorname{lcm}(m, n) = m \cdot n$.
   (a) $m = 2^4 \cdot 3^3 \cdot 7$ and $n = 2^4 \cdot 3^5 \cdot 5^3 \cdot 7^5 \cdot 11$.
   (b) $m = 2^2 \cdot 3^2 \cdot 5^2 \cdot 101$ and $n = 2 \cdot 3^3 \cdot 5^{23} \cdot 101^2$
   (c) $m = 384$ and $n = 864$
   (d) $m = 3080$ and $n = 5,336,100$.

(7) Compute $\gcd(3080, 5, 336, 100)$ using the Euclidean algorithm, and verify that the answer is the same as in Part (d) of the last problem.

(8) Let $m = 2^4 \cdot 3^3 \cdot 7$ and $n = 2^4 \cdot 3^5 \cdot 5^3 \cdot 7^5 \cdot 11$. How many numbers divide both m and n? (hint: dividing both m and n is the same as dividing $\gcd(m, n)$).

(9) Decide which of the following numbers divides $\binom{50}{31}$.
   (a) 11
   (b) 2
   (c) 2^2
   (d) 7
   (e) 13
   (f) 14
   (g) 26

(10) Which of the following numbers is the square of a fraction? Explain your answer.
   (a) 3
   (b) $\dfrac{2^4 \cdot 3^5}{5^2 7^8}$
   (c) $\dfrac{2^4 \cdot 7^4}{3^6 \cdot 5^{10}}$
   (d) $\dfrac{2^8 \cdot 7^3}{3^4 \cdot 7^5}$

(e) $3^4 \cdot 13^6$

(11) Is $\dfrac{2^6 \cdot 3^9}{5^3 \cdot 7^6}$ the cube (3rd power) of a fraction? If so, which fraction is it the cube of?

(12) Is $\dfrac{2^8 \cdot 3^9}{5^2 \cdot 7^3}$ the cube of a fraction. If so, which fraction is it the cube of?