MATH 13150: Freshman Seminar
Unit 1

1. How many numbers are there ... ?

1.1. **First problem.** We begin with some basic counting. Let's start with a really easy question.
QUESTION: How many numbers are there from 1 to 8, including 1 and 8?
To solve this, we can just list the numbers from 1 to 8 as

$$1, 2, 3, 4, 5, 6, 7, 8$$

We can count them and arrive at 8 as the answer. When you do this, it is pretty easy to see that you do not really have to count all the numbers. For example, if you are asked the question:
How many numbers are there from 1 to 246?
Here, and in all the later problems in this section, the numbers at the end 1 and 246 are included.
To solve this, you can imagine listing the numbers

$$1, 2, 3, \ldots, 245, 246$$

and if you think about counting these numbers, it is pretty clear you will find 246 as your answer.
More generally, if N is any number, there are N numbers from 1 to N. There is nothing hard about this, and probably many of you can see this answer without thinking through the problem. I am showing you this style of reasoning so you can use it as a tool when you are asked harder problems.
To summarize: if you don't know how to solve a problem, try to first do it in the simplest way you can imagine. If necessary, simplify the problem so that it is easy enough to solve, and then try to see a general pattern you can use on the harder problem.

1.2. **Second counting problem.** Let's now tackle a harder question.
How many numbers are there from 104 to 112?
If you don't have a better idea, you can just list the numbers to find:

LIST 1: $104, 105, 106, 107, 108, 109, 110, 111, 112$

and count and see there are 9 numbers on the list, so the answer is 9.
This approach won't work so well if the list is much longer, so let's try another approach which reduces this problem to the problem in the previous section. Let's take the list and subtract 103 from each number on the list. When we do this, we get

LIST 2: $1, 2, 3, 4, \ldots, 9$

and there are clearly 9 numbers on this list. Since there are exactly the same number of numbers in List 1 as in List 2, this means there are 9 numbers in List 2.
Let's try a harder version of this problem.
QUESTION 1: How many numbers are there from 24 to 91?

To solve this, list the numbers from 24 to 91, using dots to indicate the chain continues to get

$24, 25, 26, \ldots, 90, 91$

Note that if we subtract 23 from each number on the list, the new list is

$1, 2, 3, \ldots, 67, 68$ (we found $67 = 90 - 23$ and $68 = 91 - 23$, which you can do with a calculator if needed).

The second list clearly has 68 numbers, and the first list has the same number of elements as the second list, so we conclude that the first list has 68 numbers, so the answer to Question 1 is 68.
If you are really determined, you can write down the numbers from 24 to 91 and count them. It is not so hard to make a mistake, and much more time-consuming than using the method I described.
QUESTION 2: How many numbers are there from 231 to 1465?
To do this, begin by listing the numbers as:

$$231, 232, 233, \ldots, 1464, 1465$$

Now subtract 230, which is 1 less than the first number on the list, to get a new list with the same number of elements:

$$1, 2, 3, \ldots, 1234, 1235$$

Since this new list has 1235 elements on it, the answer to Question 2 is 1235.
You can imagine performing this procedure on any list and finding an answer. There is nothing hard about doing this. However, if it is faster and more efficient to find a general formula.
GENERAL QUESTION: Suppose k and n are any two numbers. How many numbers are there from k to n, including k and n.
We can solve this just as when we had actual numbers. List the numbers from k to n

LIST 1: $k, k+1, k+2, \ldots, n-1, n$

Subtract $k-1$ from the all numbers on the list, because $k-1$ is 1 less than k. The new list is:

$1, 2, 3, \ldots, n-1-(k-1), n-(k-1)$, which is

$1, 2, 3, \ldots, n-k, n-k+1$, using a little Algebra I to get the last two elements on the list.

This list has $n-k+1$ elements. Since this list has the same number of elements as LIST 1, it follows that LIST 1 has $n-k+1$ elements on it. We conclude:

Theorem 1.1. *There are $n-k+1$ numbers from k to n.*

This is a general fact that we can always use. We found this general fact by simply repeating what we did in a more concrete problem (one with numbers rather than letters) and doing the same thing. At some level, the reasoning we used is that the argument we were using had nothing to do with the specific numbers given to us. This may make some of you suspicious.

Let's check to see that are general formula gives the same answer as we got before.
QUESTION 2 (redone): How many numbers are there from 231 to 1465?
To solve this, apply Theorem 1 with $k = 231$ and $n = 1465$. Theorem 1 says there are $1465 - 231 + 1 = 1235$ numbers from 231 to 1465. This agrees with the answer we got before, which provides some reassurance that the Theorem is correct.

1.3. **More counting problems.** Now we'll consider other counting problems. It's a good idea to try to relate any of these problems to Theorem 1.
QUESTION 1: How many numbers are there from 21 to 84 that are divisible by 3?
To solve this, we can list the numbers from 21 to 84 divisible by 3.

LIST 1: $21, 24, 27, \ldots, 81, 84$.

If we divide each of these numbers by 3, we don't change the number of elements on the list, but we get a new list of consecutive numbers:

LIST 2: $7, 8, 9, \ldots, 27, 28$

By Theorem 1 from the last section, we see there are $28 - 7 + 1 = 22$ elements on the list. So there are 22 elements on LIST 2, so the answer to our question is 22.
QUESTION 2: How many numbers are there from 21 to 84 divisible by 5?
It isn't so easy to list the numbers divisible by 5 since the numbers at the ends are not divisible by 5. Instead, just list all numbers

$$21, 22, 23, 24, 25, 26, 27, \ldots, 78, 79, 80, 81, 82, 83, 84$$

Eliminate numbers at the front and end of each list not divisible by 5. The remaining numbers are

$$25, 26, \ldots, 79, 80$$

Now list the multiples of 5, as the non-contributing numbers at the beginning and end are gone. We get:

$$25, 30, \ldots, 80$$

and divide each number by 5 to get:

$$5, 6, \ldots, 16$$

and using Theorem 1, we see there are $16 - 5 + 1 = 12$ numers on the list. So 12 is the answer to Question 2.
EXERCISES: Explain your answer.

(1) How many numbers are there from 1 to 435?
(2) How many numbers are there from 379 to $8,351$?
(3) How many numbers from 44 to 176 are even?
(4) How many numbers from 63 to 861 are divisible by 7?
(5) How many numbers from 44 to 478 are divisible by 6?
(6) How many numbers from 43 to 175 are even?
(7) How many numbers from 44 to 176 are odd? Do you see a relation between this question and the last question?
(8) How many numbers from 10 to 300 are perfect squares?

(9) How many numbers from 13 to 20 begin with 1 and end with 9? How many numbers from 13 to 200 begin with 1 and end with 9?

Decoding: Read chapter one of Singh, "The CodeBook"

(10) Decode the message "O GZK VOFFG LUX JOTTKX", which is encoded using a Caesar shift cipher.

(11) The message "CFFBKYZJZJRDRKYSFFB" is an encoded message using a Caesar shift cipher, and spaces between letters have been removed (e.g., "GO TO CLASS" is replaced by "GOTOCLASS"). Decode this message (hint: for this, you can try all twenty six possible keys in succession. A smarter way to do this is to use frequency analysis. Figure out which letter is used most often in the encoded message. The most commonly used English letters are in order of decreasing frequency, "E, T, A, O, I, N, ...", as you can find from Table 1 on page 19 of Singh. Most likely the most common letter appearing in this message is not a very rare English letter like Q. It would make sense to first try the Caesar shift cipher with E corresponding to the most common letter in encoded message, and if that fails then let T correspond to the most common letter, etc). "