The International Diversification Puzzle Revisited

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Abstract

This paper revisits the international portfolio diversification puzzle using a more general version of the Heathcote and Perri (2004) model. We show that the assumption of a unit-elasticity aggregator function and of an i.i.d. endowment process is not completely innocuous. Since the terms of trade provide full insurance against endowment shocks whenever the elasticity of substitution is one, the optimal portfolio shares of domestic and foreign assets do not depend on the state of nature in the planning solution. We show that only under Cobb-Douglas and linear aggregator functions can the optimal portfolio share be characterized by a simple 'golden rule' that does not depend on the endowment realization. This implies that if we assume a more general CES aggregator function the optimal portfolio is a more complicated object than one is led to believe by Heathcote and Perri's (2004) analytical results.

1 Introduction

One of six major puzzles in international macroeconomics – to quote the title of Obstfeld and Rogoff's contribution to the NBER Macroeconomics Annual (2000) – is the surprising preference for domestic assets in equity portfolios. Tesar and Werner (1998), for instance, document that foreign assets accounted for approximately 10% of equity portfolios in the United States in the mid-1990s. Japan, Canada, Germany, and the UK had corresponding shares of 4%, 11%, 19%, and 23%, respectively.[1]

[1] French and Poterba (1991) and Warnock (2002) find foreign asset shares in roughly the same neighborhood.

Many models of international risk sharing, on the other hand, imply foreign asset shares of 50% or more. Moreover, there is some disagreement on how 'bad' the diversification puzzle really is.

Cole and Obstfeld (1991) argue that – depending on certain assumptions – international financial markets offer no or relatively few diversification opportunities above and beyond what domestic asset markets can offer. In particular, in a simple exchange economy, the market equilibrium under financial autarky (which implies trade balance in all states of nature) is isomorphic to the planning solution, provided the welfare weights satisfy certain conditions.[2] In this case, relaxing the financial autarky assumption is inconsequential and one may argue that the international diversification puzzle really is not so bad after all.[3]

In a model where human capital plays a critical role, Baxter and Jermann (1997) come to a very different conclusion. Using data for four OECD countries, they observe that the returns to human and physical capital are highly correlated within countries. Since (a) one cannot diversify away from returns to human capital and (b) labor's share in national income is in the neighborhood of 60%, hedging against labor income risk entails taking a *short* position in domestic assets! In stark contrast to Cole and Obstfeld (1991) they argue that the diversification puzzle is *worse* than you think!

Most recently, Heathcote and Perri (2004) offered yet another model with optimal portfolio shares that track the empirical data more closely than previous attempts. The assumption of home bias in the consumption good aggregator turns out to be critical for their results. The assumption, of course, is closely related to home bias in trade, which also made it into the top-six list of major puzzles by Obstfeld and Rogoff (2000). In what follows, we make the same home bias assumption, even though we could have tinkered with trade costs instead. While

[2] See Cole and Obstfeld (1991), p. 8 for details.

[3] Even in their investment and production economy they find that international portfolio diversification offers limited additional risk sharing opportunities compared to those available in domestic markets. Since our paper uses a simple exchange model to derive the key results, we shall not pursue this line of thought any further at this point.

such a model would have been more intricate, the basic insight we gain from our simple framework would not have changed in any critical respect.

The rest of the paper is organized as follows. Section 2 sets up the basic planning problem and solves for optimal portfolio shares. Since we solve the problem for a CES aggregator (with a limited, yet reasonable range of substitution elasticities) instead of Heathcote and Perri's (2004) unit-elasticity Cobb-Douglas aggregator, we will supplement our analytical results with numerical examples. This will enable us discuss the results in a far more intuitive manner. Section 3 derives how the optimal allocations change in response to (small) endowment shocks. Section 4 concludes and suggests directions for future research.

2 One-Period Planning Problem with no Uncertainty: Solving for Optimal Portfolio Shares

Let us recall the basic setup of our model. Home receives an endowment A of, say, aluminum and Foreign receives an endowment B of, say, bricks. Home uses a and b to build houses according to a CES production function of the form

$$c = \Big(\omega a^\rho + (1-\omega) b^\rho\Big)^{1/\rho}$$

while Foreign builds homes using a^* and b^* using technology

$$c^* = \Big((1-\omega)(a^*)^\rho + \omega (b^*)^\rho\Big)^{1/\rho}$$

The optimal allocations and the corresponding implied prices obviously depend on A and B. In order to keep the notation as simple as possible, however, we shall write c instead of $c(A, B)$ and so forth, unless required otherwise.

The social planner chooses the optimal allocation of bricks and aluminum in both countries subject to the aggregate resource constraints for these inputs a (a^*) and b (b^*):

$$\begin{aligned} A &= a + a^* \\ B &= b + b^* \end{aligned}$$

Moreover, instead of picking allocations directly, he chooses shares λ and λ^f such that

$$\begin{aligned} a &= \lambda A \\ b &= \lambda^f B \\ a^* &= (1-\lambda)A \\ b^* &= (1-\lambda^f)B \end{aligned}$$

where a, b denote inputs in Home, a^*, b^* denote inputs in Foreign, λ denotes the share of A allocated to the domestic agent, and λ^f denotes the share of B (Foreign's endowment of bricks) allocated to the same domestic agent.

2.1 Analytical Results

With this information in mind, and assuming log preferences, we can formalize the planning problem.

$$\max_{\lambda,\lambda^f} \log\left\{\Big(\omega(\lambda A)^\rho+(1-\omega)(\lambda^f B)^\rho\Big)^{1/\rho}\right\}+\log\left\{\Big((1-\omega)\big((1-\lambda)A\big)^\rho+\omega\big((1-\lambda^f)B\big)^\rho\Big)^{1/\rho}\right\} \tag{1}$$

FOC w/r/t λ:

$$\begin{aligned} &\Big(\omega(\lambda A)^\rho + (1-\omega)(\lambda^f B)^\rho\Big)^{-1}\omega\lambda^{\rho-1} \\ =& \Big((1-\omega)\big((1-\lambda)A\big)^\rho + \omega\big((1-\lambda^f)B\big)^\rho\Big)^{-1}(1-\omega)(1-\lambda)^{\rho-1} \end{aligned} \tag{2}$$

FOC w/r/t λ^f:

$$\begin{aligned} &\Big(\omega(\lambda A)^\rho + (1-\omega)(\lambda^f B)^\rho\Big)^{-1}(1-\omega)(\lambda^f)^{\rho-1} \\ =& \Big((1-\omega)\big((1-\lambda)A\big)^\rho + \omega\big((1-\lambda^f)B\big)^\rho\Big)^{-1}\omega(1-\lambda^f)^{\rho-1} \end{aligned} \tag{3}$$

Combining FOCs, we get:

$$\left(\frac{1-\omega}{\omega}\right)^2 = \left(\frac{\lambda^f(1-\lambda)}{\lambda(1-\lambda^f)}\right)^{\rho-1} \tag{4}$$

If $\omega = 1 - \omega$ (i.e. no home bias), then

$$\begin{aligned} (1-\lambda^f)\lambda &= \lambda^f(1-\lambda) \\ \Leftrightarrow \frac{1-\lambda^f}{\lambda^f} + 1 &= \frac{1}{\lambda} \\ \Leftrightarrow \frac{1}{\lambda^f} &= \frac{1}{\lambda} \\ \Leftrightarrow \lambda^f &= \lambda \end{aligned} \tag{5}$$

Substituting (5) into (2) and combining FOCs the solution $\lambda = \lambda^f = 1 - \lambda$ is trivial and implies

$$\lambda = \lambda^f = \frac{1}{2}, \text{ for any } \rho \leq 1 \tag{6}$$

If, on the other hand $\omega > 1 - \omega$ (i.e. with home bias), then $\left(\frac{1-\omega}{\omega}\right)^2 < 1$ and

$$\begin{aligned} \frac{\lambda^f(1-\lambda)}{\lambda(1-\lambda^f)} &< 1 \\ \Leftrightarrow \frac{1}{\lambda} &< \frac{1-\lambda^f}{\lambda^f} + 1 = \frac{1}{\lambda^f} \\ \Leftrightarrow \lambda &> \lambda^f \end{aligned} \tag{7}$$

In other words, with home bias, the optimal share of domestic assets rises relative to the share of foreign assets.[4] This illustrates nicely how we can shed some light on the diversification puzzle simply by assuming home bias in preferences (or, as in this case, in the final consumption good aggregator).

Even though the planner does not care about prices, let us see what the implied terms of trade are, with and without home bias. Recall that Home's terms-of-trade q is the ratio of marginal products, evaluated at the planner's allocation of inputs

[4]Note, however, that in the planning solution we cannot conclude that $\lambda^f = 1 - \lambda$. The interested reader may want to consult the appendix where we pin down λ^f analytically for the two-period competitive equilibrium with uncertainty under certain symmetry assumptions.

between Home and Foreign:[5]

$$\begin{aligned} q &= \frac{\omega}{1-\omega}\left(\frac{a}{b}\right)^{\rho-1} \\ &= \frac{\omega}{1-\omega}\left(\frac{\lambda A}{\lambda^f B}\right)^{\rho-1} \end{aligned} \tag{9}$$

With no home bias ($\omega = 1-\omega$), we have already shown that $\lambda = \lambda^f$ and it follows that

$$q = \left(\frac{A}{B}\right)^{\rho-1} \tag{10}$$

With home bias, on the other hand, the terms of trade are

$$q = \frac{\omega}{1-\omega}\left(\frac{\lambda}{\lambda^f}\right)^{\rho-1}\left(\frac{A}{B}\right)^{\rho-1} \tag{11}$$

2.2 Numerical Results

Whenever $A = B$, the planning solution is somewhat uninteresting in that the real exchange rate and terms of trade are constant and equal to one, $\lambda + \lambda^f = 1$, and $\frac{a}{b} = \frac{b^*}{a^*}$. Moreover, consumption and GDP are equalized across countries. The more interesting cases are $A \neq B$. Without loss of generality, let $A < B$. Then the following results are noteworthy:

Without Home Bias : The numerical results are fairly straightforward and intuitive for the most part.

First of all, $\omega = \lambda = \lambda^f = 0.5$, for all values of ρ. This is, in fact, in fact what Cole and Obstfeld (1991) referred to in foonote 7. As a result, $\frac{a}{b} = \frac{a^*}{b^*}$ and $c = c^*$, no matter what the elasticity of substitution. In other words, the optimal allocation entails perfect risk sharing. The real exchange rate is equal to one for all values of ρ. In the special case $\frac{1}{1-\rho} = 1$, we have $q = \frac{A}{B}$ and $GDP = GDP^*$. In general, however, GDP's are not equalized.

[5] Foreign's terms of trade are the symmetric opposite, namely

$$q^* = \frac{1-\omega}{\omega}\left(\frac{a^*}{b^*}\right)^{\rho-1}. \tag{8}$$

Obviously, any optimal allocation will satisfy $q = q^*$.

With Home Bias : The optimal allocation with home bias differs from our previous results in several regards.

We have already shown that $\lambda > \lambda^f$ and, more importantly, $\lambda^f \neq 1 - \lambda$, unless $A = B$. Moreover, λ is increasing in $\frac{1}{1-\rho}$, as is $\lambda + \lambda^f$ (see Figure 1).

Welfare is not equalized across countries in the sense that $c \neq c^*$. However, in terms of the numeraire (the Home consumption good), consumption expenditure is equalized, that is $c = rx \times c^*$, for all ρ (see Figure 2). That is, the planning solution implements **perfect risk sharing**, no matter what the elasticity of substitution between a and b.

For the unit elasticity case, $\lambda = \omega$ and $q = \frac{B}{A}$. The intuition for this allocation is that the terms of trade provide full insurance against endowment fluctuations. As a result, the home bias parameter alone determines the optimal domestic portfolio share.

The interested reader may also want to consult the appendix, where we solve for the two-period competitive equilibrium with uncertainty. While the results are instructive, we shall not discuss the market solution in any detail here since it is inconsequential for what follows.

3 One-Period Planning Problem with no Uncertainty: Computing Marginal Changes in Optimal Portfolio Shares

In the previous section, we have characterized the planning solution for a particular state of nature, i.e. for a particular realization of A and B. The example enabled us to illustrate some of the key features of the optimally diversified portfolio with and without home bias as well as the corresponding degree of risk-sharing.

As will become apparent shortly, the following alternative approach allows us to characterize the optimal portfolio in more detail. In particular, we shall see that

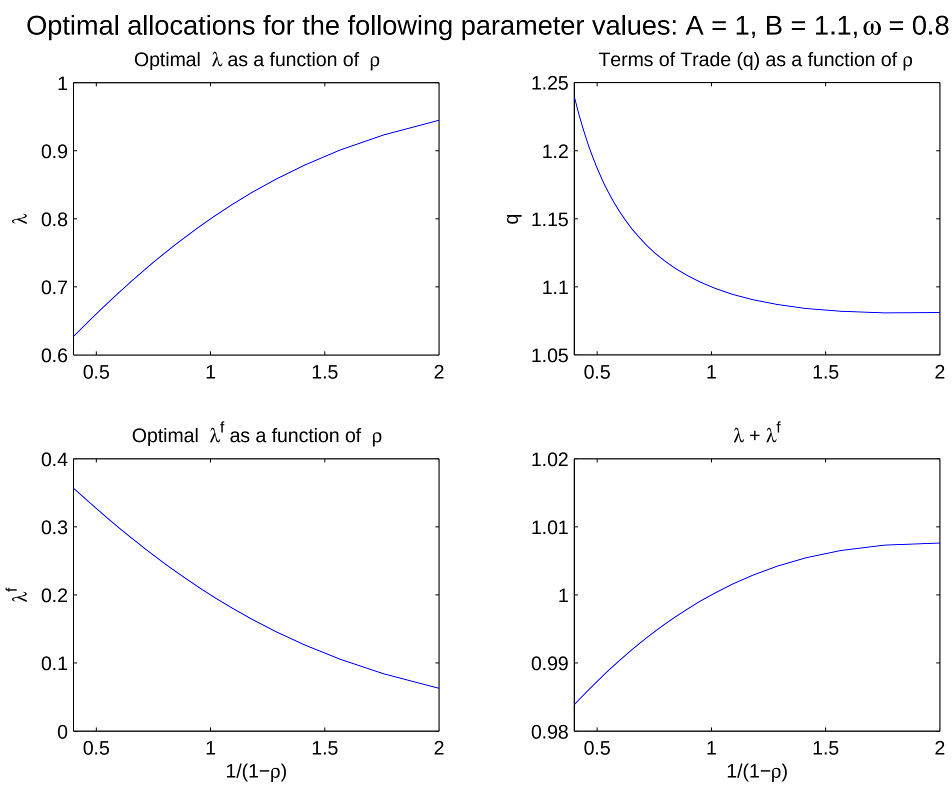

Figure 1: Optimal shares and terms-of-trade

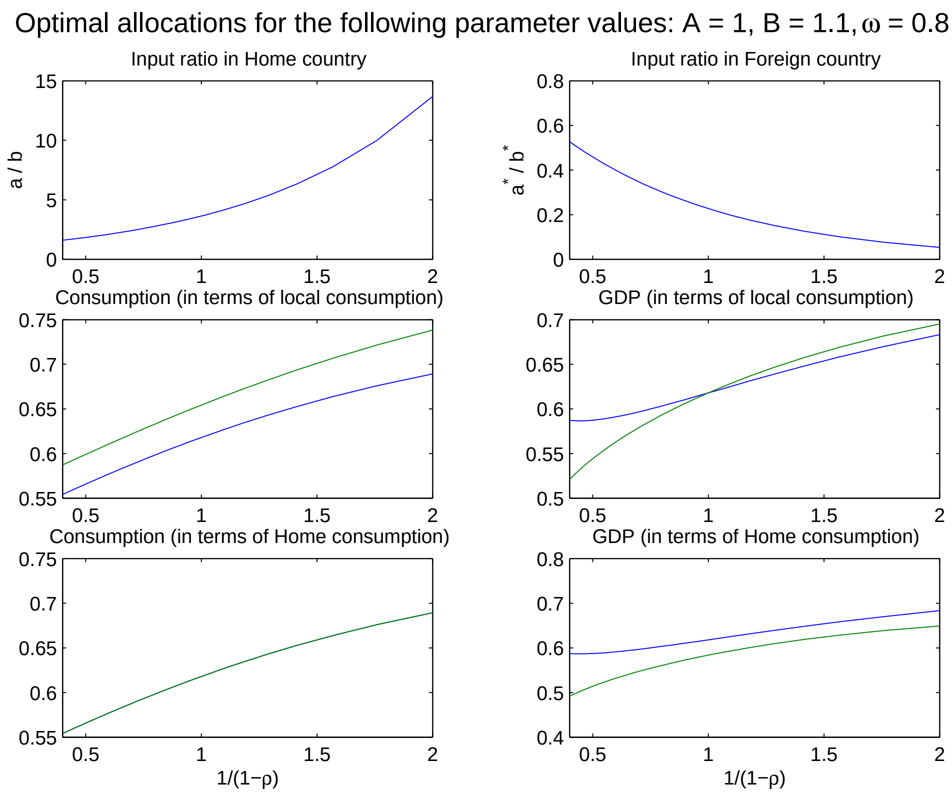

Figure 2: Input ratios, consumption, and GDP

the assumption of a Cobb-Douglas aggregator is very restrictive in some regards.[6]

First, we solve the by now familiar one-period planning problem:

$$\begin{aligned} \max_{a,a^*,b,b^*} \frac{1}{2}u(c) \quad &+ \quad \frac{1}{2}u(c^*) \\ \text{subject to} \quad & c = G(a,b) \\ & c^* = G^*(a^*,b^*) \\ & a + a^* = A && (12) \\ & b + b^* = B && (13) \end{aligned}$$

Let preferences be logarithmic. Obviously, the optimal allocation is identical to the previous planning solution, except that we do not characterize it in terms of portfolio allocations λ and λ^f. The reasons for this change will become apparent shortly.

Let us characterize the optimal allocation as sharply as we can for CES production functions $G(\cdot,\cdot)$.

The optimal allocation a is defined implicitly by

$$\frac{(1-\omega)(A-a)^\rho + \omega\left(B - \frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}}\right)^\rho}{\omega a^\rho + (1-\omega)\left(\frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}}\right)^\rho} = \frac{(1-\omega)(A-a)^{\rho-1}}{\omega a^{\rho-1}} \tag{14}$$

We can solve for b using the solution to (14) and

$$b = \frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}} \tag{15}$$

Equations (12) and (13) then yield optimal allocations for a^* and b^*.

Before we proceed, let's derive the prices that enable us to implement the planning solution as a decentralized equilibrium.[7]

[6] We have already seen that the optimal optimal portfolio is a function of the elasticity of substitution in previous sections. In what follows, we demonstrate that Cobb-Douglas is restrictive in yet another dimension.

[7] In the decentralized problem, prices and allocations are obviously state-contingent. In order to prevent 'over-crowding' we suppress the *histories* notation for now.

Let Home consumption c be the numeraire and define the remaining prices as follows:

rx : price of consumption abroad (c^*);

q_A : price of A;

q_A^* : price of A abroad, in terms of c^*;

q_B : price of B;

q_B^* : price of B abroad, in terms of c^*.

Note that the law of one price implies $q_A = rx \times q_A^*$ and $q_B = rx \times q_B^*$.

The implied terms-of-trade can be derived from the following expressions, evaluated at the optimal (planning) allocation.

$$\begin{aligned} q_A &= \partial G/\partial a \\ q_B &= \partial G/\partial b \\ q_A^* &= \partial G^*/\partial a^* \\ q_B^* &= \partial G^*/\partial b^* \end{aligned}$$

Home's terms of trade are:

$$q = \frac{q_A}{q_B}$$

Foreign's terms of trade are:

$$q^* = \frac{q_A^*}{q_B^*}$$

Using the notation from the previous section, we can write the consumers' implied budget constraints:

$$
\begin{aligned}
c &= q_A \times \lambda A + rx \times q_B^* \times B\lambda^f && (16) \\
c^* &= q_B^* \times (1-\lambda^f)B + \frac{q_A}{rx} \times A(1-\lambda) && (17)
\end{aligned}
$$

So far, there is nothing new under the sun. The following steps, however, will shed some light on how the optimal portfolio rule responds to small endowment shocks.

The idea is to take partial derivatives of the consumers' budget constraints and eventually solve the system of two equations for $\frac{\partial \lambda}{\partial A}$ and $\frac{\partial \lambda^f}{\partial A}$.[8]

To that effect, we begin by computing the partial derivatives of the left hand sides of the budget constraints: $\frac{\partial c}{\partial A}$, $\frac{\partial c^*}{\partial A}$, $\frac{\partial c}{\partial B}$, and$\frac{\partial c^*}{\partial B}$.

Note, in particular that:

$$
\frac{\partial c}{\partial A} = \frac{\partial c}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}A} + \frac{\partial c}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}A} \tag{18}
$$

and similarly for $\frac{\partial c^*}{\partial A}$.

To compute the differentials **not** involving c or c^*, it is convenient to rearrange (14)

$$
\left(\frac{(1-\omega)(A-a)^\rho + \omega\left(B - \frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}}\right)^\rho}{\omega a^\rho + (1-\omega)\left(\frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}}\right)^\rho}\right)^{\frac{1}{\rho-1}} \frac{a}{A-a} = \left(\frac{1-\omega}{\omega}\right)^{\frac{1}{\rho-1}} \tag{19}
$$

and differentiate implicitly to obtain (implicit) expressions for $\frac{\mathrm{d}a}{\mathrm{d}A}$ and $\frac{\mathrm{d}a}{\mathrm{d}B}$.

To ease the notation somewhat , let us define the following expressions:

[8] Since the problem is perfectly symmetric between Home and Foreign, we can take derivatives with respect to A or B and gain the same insight. Recall that Home receives a period endowment A while Foreign (denoted by *) is endowed with B.

$$
\begin{aligned}
X = \frac{X_1}{X_2} &= \frac{(1-\omega)(A-a)^\rho}{\omega a^\rho + (1-\omega)\left(\frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}}\right)^\rho} \\
Y = \frac{Y_1}{Y_2} &= \frac{\omega\left(B - \frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}}\right)^\rho}{\omega a^\rho + (1-\omega)\left(\frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{A-a}{aB}+\frac{1}{B}}\right)^\rho} \\
X_1 &= (1-\omega)(A-a)^\rho \\
X_2 = Y_2 &= \omega a^\rho + (1-\omega)\left(\frac{1}{Z}\right)^\rho \\
Y_1 &= \omega\left(B - \frac{1}{Z}\right)^\rho \\
Z &= (\frac{1-\omega}{\omega})^{\frac{2}{\rho-1}}\frac{A-a}{aB} + \frac{1}{B}
\end{aligned}
$$

The total differential with respect to A then can be written as:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}A}\left((X+Y)^{\frac{1}{\rho-1}}\frac{a}{A-a}\right) \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}A}\left(\frac{a}{A-a}\right)(X+Y)^{\frac{1}{\rho-1}} \\
&\qquad\quad + \frac{a}{A-a}\frac{1}{\rho-1}(X+Y)^{\frac{2-\rho}{\rho-1}}\left(\frac{\mathrm{d}}{\mathrm{d}A}X + \frac{\mathrm{d}}{\mathrm{d}A}Y\right) \\
&\qquad = 0,
\end{aligned} \tag{20}
$$

where

$$\frac{\mathrm{d}}{\mathrm{d}A}\Big(\frac{a}{A-a}\Big) = \frac{\mathrm{d}a}{\mathrm{d}A}\frac{A}{(A-a)^2} - \frac{a}{(A-a)^2} \tag{21}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}A}X &= \frac{(1-\omega)\rho(A-a)^{\rho-1}\big(1-\frac{\mathrm{d}a}{\mathrm{d}A}\big)}{X_2} - \frac{X_1}{(X_2)^2} \\ &\times\bigg(\omega\rho a^{\rho-1}\frac{\mathrm{d}a}{\mathrm{d}A} - (1-\omega)\rho Z^{-(\rho+1)}\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\frac{1}{B}\Big(\frac{1}{a}-\frac{\mathrm{d}a}{\mathrm{d}A}\frac{A}{a^2}\Big)\bigg)\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}A}Y &= \frac{\rho\omega\Big(B-\frac{1}{Z}\Big)^{\rho-1}Z^{-2}\frac{1}{B}\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\Big(\frac{1}{a}-\frac{\mathrm{d}a}{\mathrm{d}A}\frac{A}{a^2}\Big)}{Y_2} - \frac{Y_1}{(Y_2)^2} \\ &\times\bigg(\rho\omega a^{\rho-1}\frac{\mathrm{d}a}{\mathrm{d}A} - (1-\omega)\rho Z^{-(\rho+1)}\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\frac{1}{B}\Big(\frac{1}{a}-\frac{\mathrm{d}a}{\mathrm{d}A}\frac{A}{a^2}\Big)\bigg)\end{aligned} \tag{23}$$

Substituting (22) and (23) into (20) we can solve for $\frac{\mathrm{d}a}{\mathrm{d}A}$.[9]

We can compute the total differential with respect to B analogously.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}B}\bigg((X+Y)^{\frac{1}{\rho-1}}\frac{a}{A-a}\bigg) \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{a}{A-a}\Big)(X+Y)^{\frac{1}{\rho-1}} \\ &\quad\quad+\frac{a}{A-a}\frac{1}{\rho-1}(X+Y)^{\frac{2-\rho}{\rho-1}}\Big(\frac{\mathrm{d}}{\mathrm{d}B}X+\frac{\mathrm{d}}{\mathrm{d}B}Y\Big) \\ &\quad= 0,\end{aligned} \tag{24}$$

[9]Since the expression is fairly complex, we refrain from computing an analytical solution for $\frac{\mathrm{d}a}{\mathrm{d}A}$ and solve the expression numerically instead.

where

$$\frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{a}{A-a}\Big) = \frac{\mathrm{d}a}{\mathrm{d}B}\frac{A}{(A-a)^2} \tag{25}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}B}X = & \frac{-(1-\omega)\rho(A-a)^{\rho-1}\big(\frac{\mathrm{d}a}{\mathrm{d}B}\big)}{X_2} - \frac{X_1}{(X_2)^2} \\ & \times\Bigg(\omega\rho a^{\rho-1}\frac{\mathrm{d}a}{\mathrm{d}B} - (1-\omega)\rho Z^{-(\rho+1)}\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}} \\ & \times\Big(-\frac{1}{B}\frac{\mathrm{d}a}{\mathrm{d}B}\frac{A}{a^2} - \frac{1}{B^2}\frac{A-a}{a}\Big) - \frac{1}{B^2}\Bigg)\Bigg)\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}B}Y = & \frac{\rho\omega\Big(B-\frac{1}{Z}\Big)^{\rho-1}\Bigg(1+Z^{-2}\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\Big(\frac{1}{B}\frac{A}{a^2}\frac{\mathrm{d}a}{\mathrm{d}B} - \frac{1}{B^2}\frac{A-a}{a}\Big) - \frac{1}{B^2}\Bigg)\Bigg)}{Y_2} \\ & -\frac{Y_1}{(Y_2)^2}\Bigg(\rho\omega a^{\rho-1}\frac{\mathrm{d}a}{\mathrm{d}B} - (1-\omega)\rho Z^{-(\rho+1)}\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}} \\ & \times\Big(-\frac{1}{B}\frac{\mathrm{d}a}{\mathrm{d}B}\frac{A}{a^2} - \frac{1}{B^2}\frac{A-a}{a}\Big) - \frac{1}{B^2}\Bigg)\Bigg)\end{aligned} \tag{27}$$

As before, we can solve numerically for $\frac{\mathrm{d}a}{\mathrm{d}B}$.

Using (15) and the results for $\frac{\mathrm{d}a}{\mathrm{d}A}$ and for $\frac{\mathrm{d}a}{\mathrm{d}B}$ we can easily compute $\frac{\mathrm{d}b}{\mathrm{d}A}$ and $\frac{\mathrm{d}b}{\mathrm{d}B}$.

$$\frac{\mathrm{d}b}{\mathrm{d}A} = -\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{1}{\rho-1}}\frac{A-a}{aB} + \frac{1}{B}\Bigg)^{-2}\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{1}{\rho-1}}\frac{1}{B}\Big(\frac{1}{a} - \frac{\mathrm{d}a}{\mathrm{d}A}\frac{A}{a^2}\Big)\Bigg) \tag{28}$$

$$\frac{\mathrm{d}b}{\mathrm{d}B} = -\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{1}{\rho-1}}\frac{A-a}{aB} + \frac{1}{B}\Bigg)^{-2}\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{1}{\rho-1}}\Big(-\frac{1}{B}\frac{\mathrm{d}a}{\mathrm{d}B}\frac{A}{a^2} - \frac{1}{B^2}\frac{A-a}{a}\Big) - \frac{1}{B^2}\Bigg) \tag{29}$$

Proceeding analogously, we can find expressions for $\frac{\mathrm{d}a^*}{\mathrm{d}A}$, $\frac{\mathrm{d}a^*}{\mathrm{d}B}$, $\frac{\mathrm{d}b^*}{\mathrm{d}A}$, and $\frac{\mathrm{d}b^*}{\mathrm{d}B}$.[10]

[10] The details and the results are in the appendix. Note that $\frac{\mathrm{d}a}{\mathrm{d}B} + \frac{\mathrm{d}a^*}{\mathrm{d}B} \neq 0$ and $\frac{\mathrm{d}b}{\mathrm{d}B} + \frac{\mathrm{d}b^*}{\mathrm{d}B} \neq 1$

Next, we need to differentiate the right hand sides of the consumers' budget constraint with respect to A (or B).[11] For now, let us consider a disturbance to Foreign's endowment, that is, a small change in B.

$$\begin{aligned}\frac{\partial c}{\partial B} &= \frac{\partial q_A}{\partial B}\lambda A + q_A\frac{\partial \lambda}{\partial B}A + \frac{\partial rx}{\partial B}q_B^*B\lambda^f \\ &\quad + rx\frac{\partial q_B^*}{\partial B}B\lambda^f + rxq_B^*\frac{\partial \lambda^f}{\partial B}B + rxq_B^*\lambda^f \qquad (30)\\ \frac{\partial c^*}{\partial B} &= \frac{\partial q_B^*}{\partial B}\big(B(1-\lambda^f)\big) + \frac{\partial q_A}{\partial B}\Big(\frac{A}{rx}(1-\lambda)\Big) + q_B^*\Big(1-\frac{\partial \lambda^f}{\partial B}B - \lambda^f\Big)\\ &\quad -\frac{\partial rx}{\partial B}\Big(\frac{q_A A}{rx^2}(1-\lambda)\Big) - \frac{\partial \lambda}{\partial B}\frac{q_A A}{rx}, \qquad (31)\end{aligned}$$

where

$$\frac{\partial q_A}{\partial B} = \frac{\partial q_A}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}B} + \frac{\partial q_A}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}B} \qquad (32)$$

$$\frac{\partial q_B^*}{\partial B} = \frac{\partial q_B^*}{\partial a^*}\frac{\mathrm{d}a^*}{\mathrm{d}B} + \frac{\partial q_B^*}{\partial b^*}\frac{\mathrm{d}b^*}{\mathrm{d}B} \qquad (33)$$

Once we have computed all relevant total differentials and partial derivatives, equations (30) and (31) form a simple system of two (linear) equations in two unknowns: $\frac{\partial \lambda}{\partial B}$ and $\frac{\partial \lambda^f}{\partial B}$.

Incidentally, there is a short-cut to computing $\frac{\partial \lambda}{\partial B}$ and $\frac{\partial \lambda^f}{\partial B}$ using only a handful of expressions derived earlier. Recall that λ and λ^f are Home's investment shares chosen by the social planner, namely $\lambda = \frac{a}{A}$ and $\lambda^f = \frac{b}{B}$.

It follows immediately that

unless $\frac{1}{1-\rho} = 1$. The inequalities, of course, are due to the fact that we only compute first-order approximations of the change in optimal allocations in response to endowment shocks. The reasons why the first-order approximations are indistinguishable from the fully characterized change in the unit-elasticity case will become apparent shortly.
Incidentally, as we approach a linear production function (i.e. $\rho \to 1$), higher-order effects disappear and the first-order effects again fully characterize the change in allocation.
In general, however, we need to compute all relevant differentials in order to fully characterize the change in the optimal allocation in response to (small) endowment shocks.

[11] Since we have assumed that the process governing endowment shocks is independent and identical between Home and Foreign, we can limit ourselves, without loss of generality, to considering a shock to one of the endowments.

$$\frac{\partial\lambda}{\partial B} = \frac{\mathrm{d}a}{\mathrm{d}B}\frac{1}{A}, \text{ and} \tag{34}$$
$$\frac{\partial\lambda^f}{\partial B} = \frac{\mathrm{d}b}{\mathrm{d}B}\frac{1}{B} - \frac{b}{B^2} \tag{35}$$

Figure 3 illustrates the change in investment shares in response to a (small) change in B. As expected, the partial derivative is zero whenever $\rho = 0$, i.e. when the elasticity of substitution $\frac{1}{1-\rho} = 1$.[12] What's more, unit elasticity implies that the terms of trade provide full insurance against endowment shocks, i.e. $\frac{\partial q}{\partial B} = 1$. This, of course, is Heathcote and Perri's (2004) result and it implies that the optimally diversified portfolio shares depend on the parameters of the problem but **not** on the particular realization of the endowment.

If, however, $\frac{1}{1-\rho} \neq 1$ the terms of trade no longer provide full insurance against endowment shocks and we no longer have a 'golden rule' for the asset share in an optimally diversified portfolio. Rather, λ and λ^f depend on the parameters of the problem as well as the particular realization of the endowment.

While our model certainly is a gross simplification of even the most stylized facts, it does highlight the possibility that λ and λ^f are slightly more complicated objects.

4 Conclusion and Future Research

The results in the previous section establish that in a simple exchange economy, the assumption of a unit-elasticity aggregator is somewhat misleading in that it suggests the existence of *a* 'golden rule' for portfolio diversification. In this respect, the terms of trade play a critical role since they insure fully against risks that are not already 'covered' by the optimal portfolio itself.

[12] See appendix for a simple proof.
Note that $\lim_{\rho\to 0}\frac{\partial\lambda}{\partial B} = 0$ and $\lim_{\rho\to 0}\frac{\partial\lambda^f}{\partial B} = 0$. However, since such extreme elasticities are outside the 'reasonable' range we do not present detailed results and we omit a discussion of the economic intuition.

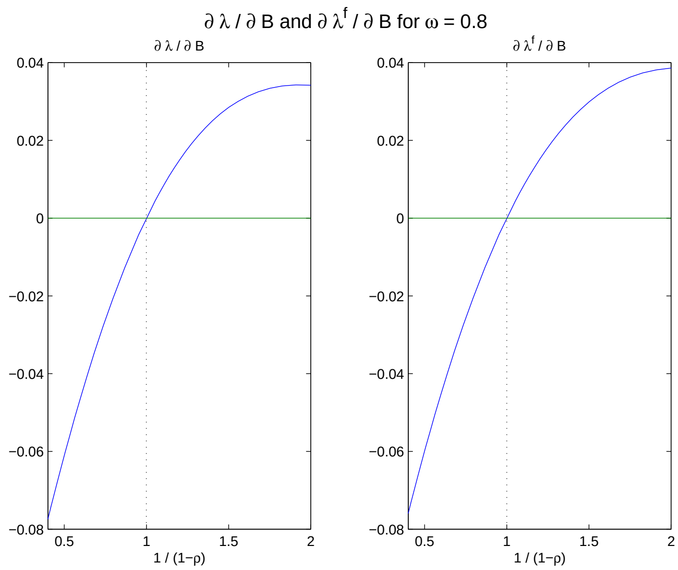

Figure 3: Sensitivity of investment shares to (small) endowment shocks

If, however, the elasticity of substitution is different from one, terms of trade no longer offer full insurance and characterizing the optimal portfolio is more intricate.

Our results are obviously limited to the planning problem and they do not carry over one-for-one to a decentralized economy with uncertainty. In appendix A we solve for such an economy. There, assuming an i.i.d. stochastic process for the endowments, it turns out that the optimal portfolio shares are **not** state-contingent. Once we relax the i.i.d. assumption, *ex ante* prices of claims to A and B are no longer equalized across states of nature and the optimal portfolio shares **are** state-contingent.

While none of this is groundbreaking, it dispels the notion of a 'one-size-fits-all' allocation rule suggested by Heathcote and Perri (2004). The combined effect of a unit-elasticity aggregator and of an i.i.d. endowment process yields constant asset shares λ and λ^f in the planning solution as well as the decentralized (two-period) equilibrium with uncertainty. Relaxing the unit-elasticity assumption overturns this result in the planning problem while relaxing the i.i.d. assumption does the same in the decentralized economy.

Finally, since the endowment economy does not entail any intertemporal decisions, it is essentially a static problem. This, however, seems grossly inadequate when analyzing optimal portfolio choices driven by risk-sharing motives. Appendix C introduces a production model with inelastic labor supply, which we plan to use as a starting point for future research on the diversification puzzle.[13]

In particular, one may ask how the introduction of non-diversifiable labor income risk affects the optimal portfolio and how sensitive these allocations are to different aggregate labor income shares. Baxter and Jermann's (1997) results suggest that the optimal foreign portfolio shares increase in the presence of human capital risk. One may argue, then, that this actually 'worsens' the diversification puzzle. Alternatively, we may introduce trade costs, additional intermediate goods (in order for some goods not to be traded at all) and differences in country size (labor force) in an effort to generate trade volumes and optimal portfolios, which reflect the stylized fact that smaller countries are both more open (in terms of trade volumes) and diversified (in terms of foreign asset holdings).

Baxter el al. (1998) find that in an n-country general equilibrium model with non-traded and traded consumption goods (rather than intermediate goods, as in Heathcote and Perri) the optimal portfolio of traded goods exhibits no home bias. Moreover, consumers may wish to hold more or less than 100% of their country's non-traded endowment and possibly non-zero amounts of other countries' non-traded endowments. The optimal share of the non-traded domestic endowment depends critically on the elasticity of substitution; in fact, the share is a discontinuous(!) function of the substitution elasticity. In sum, whether or not the complete portfolio (of traded and non-traded endowments) exhibits home bias depends critically on the substitution parameter ρ.

In conclusion, whether or not a carefully calibrated model with human cap-

[13]The appendix also provides the sketch of a proof for the existence and uniqueness of a non-stochastic steady state around which we can log-linearize the model, if necessary.

ital risk and trade costs ‘delivers’ home bias in the optimal asset portfolio is an open question and the results presented here merely corroborate that substitution elasticities matter a great deal, even in the simplest possible model of optimal portfolio choice.

Appendix

A Two-Period Competitive Equilibrium with Uncertainty

As in the previous section we have a two-good, two-country endowment economy with home bias in final production. The model again is symmetric. With respect to endowment shocks this means that A and B are drawn from the same distribution. For simplicity, assume the shocks are independent across countries.[14] Since the distributions are identical, the *ex ante* equilibrium price of a claim to A and B must be the same. Let P_A denote the price of a claim to one unit of A; P_B is the price of a claim to B. To keep the notation simple we assume a discrete S-dimensional state space and denote different realizations of the endowment shock by $s = 1, \dots, S$.

Before we formalize the problem, let us adopt the following convention: λ is *Home's* share of A (its own endowment), λ^f is *Home's* share of B; λ^* is *Foreign's* share of B, and λ^{*h} is *Foreign's* share of A.

In period 0, before the endowment shock is realized, the domestic agent picks λ and λ^f, knowing the distribution of A and B. Once the uncertainty has been resolved in period 1, the agent receives his allocation characterized by equations (38) and (39).

Domestic agents solve the following optimization problem:

$$\max_{\lambda,\lambda^f} \sum_{s \in S} \pi(s)\big(\omega a(s)^\rho + (1-\omega) b(s)^\rho\big)^{1/\rho} \tag{36}$$

subject to:

$$P_A = \lambda P_A + \lambda^f P_B \tag{37}$$

$$a(s) = \lambda A(s) \tag{38}$$

$$b(s) = \lambda^f B(s) \tag{39}$$

[14] Since we are only looking at a single realization, we need not (yet) worry about the distribution across time.

where $\pi(s)$ is the probability of state s. Taking first order conditions with respect to λ, λ^f, $a(s)$, $b(s)$ and combining them we obtain:

$$\left(\frac{\lambda}{\lambda^f}\right)^{\rho-1} = \frac{1-\omega}{\omega}\frac{P_B}{P_A}\frac{\sum_{s\in S}\pi(s)B(s)^{\rho}\left(\omega\left(\lambda A(s)\right)^{\rho}+(1-\omega)\left(\lambda^f B(s)\right)^{\rho}\right)^{\frac{1-\rho}{\rho}}}{\sum_{s\in S}\pi(s)A(s)^{\rho}\left(\omega\left(\lambda A(s)\right)^{\rho}+(1-\omega)\left(\lambda^f B(s)\right)^{\rho}\right)^{\frac{1-\rho}{\rho}}} \quad (40)$$

Recall that we assume the distributions of $A(s)$ and $B(s)$ to be identical and independent across countries. This assumption simplifies our solution considerably. First of all, the ratio of probability-weighted sums turns out to be one. Moreover, it must be that $P_A = P_B$, since *ex ante* the two assets (i.e. claims to endowments A and B) are identical. Then it follows from the budget constraint (37) that $\lambda^f = 1 - \lambda$.[15] This, incidentally, is the key difference between the one-period planning solution and the two-period competitive equilibrium. Recall that in the planning solution, $\lambda + \lambda^f \neq 0$, unless $\frac{1}{1-\rho} = 1$ (unit elasticity).

Hence, the characterization of the optimal λ turns out to be fairly straightforward:

$$\lambda = \frac{1}{\left(\frac{\omega}{1-\omega}\right)^{\frac{1}{\rho-1}}+1} \quad (41)$$

As in the planning solution, λ is an increasing function of ρ, as illustrated in figure 4. In other words, the lower the elasticity of substitution (e.g. on the Leontief side of Cobb-Douglas), the more diversified is the optimal portfolio (i.e. the lower is λ). Figure 5 shows input ratios, terms of trade, consumption and GDP for a particular realization $\left(A(s), B(s)\right)$.

While the competitive equilibrium is very similar to the planning solution, it does differ in some critical respects.[16] First of all, perfect risk sharing only obtains in the unit-elasticity case, where terms of trade fluctuations offer insurance against

[15] By symmetry it follows that $\lambda^{*h} = 1 - \lambda^*$ for the foreign agent.

[16] Note, however, that the market equilibrium and the planning solution are not comparable, strictly speaking. The planning problem is set up as a one-period problem while the decentralized economy is a two-period model.

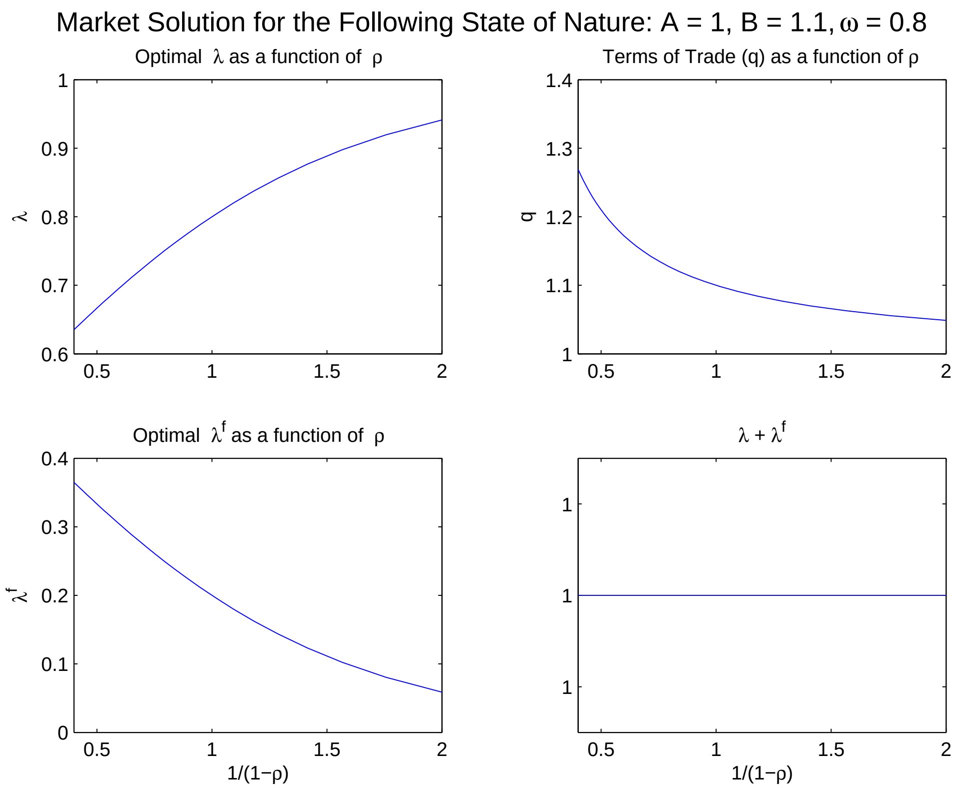

Figure 4: Optimal Portfolio Share in Competitive Equilibrium

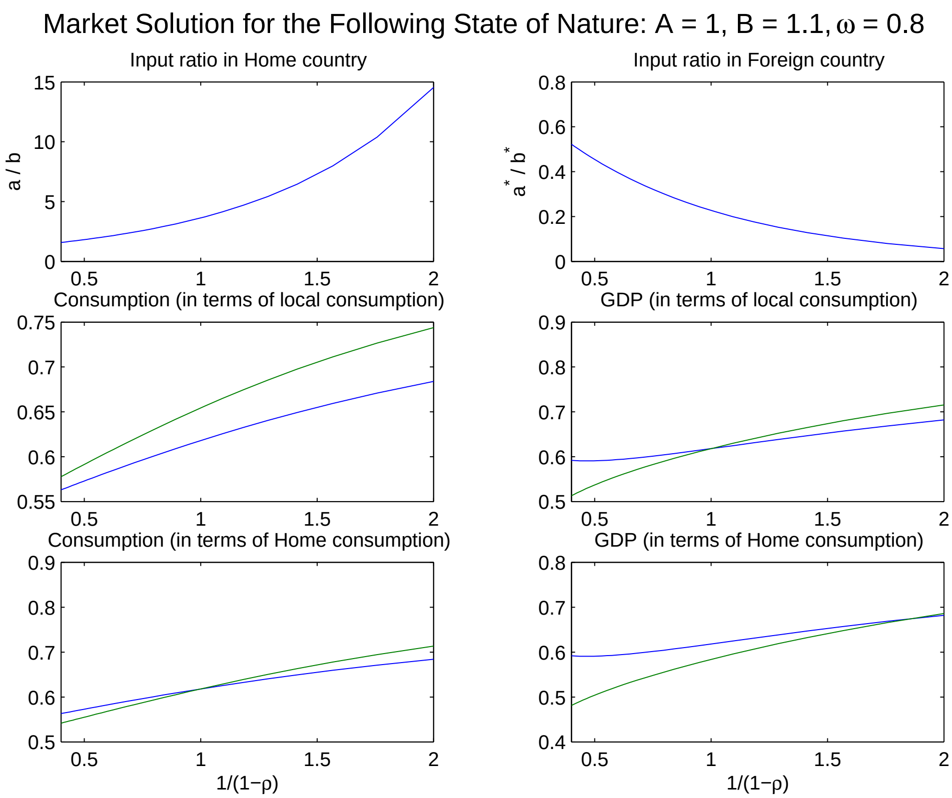

Figure 5: Equilibrium Allocations

risk not already 'covered' by the portfolio itself. For $\frac{1}{1-\rho} \neq 1$, the insurance offered by the terms of trade is incomplete and there is no perfect risk sharing. Secondly, $\lambda + \lambda^f = 1$ for all rho. Recall that in the planning solution $\lambda + \lambda^f \neq 1$ unless the elasticity of substitution is one.

B Total Differentials for Foreign

In what follows, we compute expressions for $\frac{\mathrm{d}a^*}{\mathrm{d}A}$, $\frac{\mathrm{d}a^*}{\mathrm{d}B}$, $\frac{\mathrm{d}b^*}{\mathrm{d}A}$, and $\frac{\mathrm{d}b^*}{\mathrm{d}B}$. We follow the same procedure as for the Home country.

Rearrange (14) using (12):

$$
\begin{aligned}
&\left(\frac{(1-\omega)(a^*)^\rho + \omega\Big(B - \frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{a^*}{A-a^*}\frac{1}{B}+\frac{1}{B}}\Big)^\rho}{\omega(A-a^*)^\rho + (1-\omega)\Big(\frac{1}{\left(\frac{1-\omega}{\omega}\right)^{\frac{2}{\rho-1}}\frac{a^*}{A-a^*}\frac{1}{B}+\frac{1}{B}}\Big)^\rho}\right)^{\frac{1}{\rho-1}}\frac{A-a^*}{a^*} \\
= \quad & (X^* + Y^*)^{\frac{1}{\rho-1}}\frac{A-a^*}{a^*} \\
= \quad & \left(\frac{X_1^*}{X_2^*} + \frac{Y_1^*}{Y_2^*}\right)^{\frac{1}{\rho-1}}\frac{A-a^*}{a^*} \\
= \quad & \left(\frac{1-\omega}{\omega}\right)^{\rho-1}
\end{aligned} \tag{42}
$$

where X_1^*, X_2^*, Y_1^*, and Y_2^* are defined for ease of notation in what follows.

$$
\begin{aligned}
X_1^* &= (1-\omega)(a^*)^\rho \\
X_2^* = Y_2^* &= \omega(A-a^*)^\rho + (1-\omega)\Big(\frac{1}{Z^*}\Big)^\rho \\
Y_1^* &= \omega\Big(B - \frac{1}{Z^*}\Big)^\rho \\
Z^* &= \Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\frac{a^*}{A-a^*}\frac{1}{B} + \frac{1}{B}
\end{aligned}
$$

Differentiating (42) implicitly with respect to A yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}A}\Big(X^* + Y^*\Big)^{\frac{1}{\rho-1}}\frac{A-a^*}{a^*} &= \frac{\mathrm{d}}{\mathrm{d}A}\Big(\frac{A-a^*}{a^*}\Big)\Big(X^*+Y^*\Big)^{\frac{1}{\rho-1}} \\
&\quad + \frac{A-a^*}{a^*}\frac{1}{\rho-1}\Big(X^*+Y^*\Big)^{\frac{2-\rho}{\rho-1}}\Big(\frac{\mathrm{d}}{\mathrm{d}A}X^* + \frac{\mathrm{d}}{\mathrm{d}A}Y^*\Big) \\
&= 0, \qquad (43)
\end{aligned}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}A}\Big(\frac{A-a^*}{a^*}\Big) = \frac{1}{a^*} - \frac{\mathrm{d}a^*}{\mathrm{d}A}\frac{A}{a^{*2}} \qquad (44)$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}A}X^* &= \frac{(1-\omega)\rho a^{*\rho-1}\frac{\mathrm{d}a^*}{\mathrm{d}A}}{X_2^*} - \frac{X_1^*}{X_2^{*2}} \\
&\quad \times\Big(\omega(A-a^*)^{\rho-1}\Big(1-\frac{\mathrm{d}a^*}{\mathrm{d}A}\Big) \\
&\quad -(1-\omega)\rho Z^{-(\rho+1)}\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}A}\Big(\frac{a^*}{A-a^*}\Big)\Big) \qquad (45)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}A}Y^* &= \omega\rho\Big(B-\frac{1}{Z^*}\Big)^{\rho-1}\frac{1}{Z^{*2}Y_2^*}\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}A}\Big(\frac{a^*}{A-a^*}\Big) \\
&\quad -\frac{Y_1^*}{Y_2^{*2}}\Big(\omega\rho(A-a^*)^{\rho-1}(1-\frac{\mathrm{d}a^*}{\mathrm{d}A})^{\rho-1} \\
&\quad -(1-\omega)\rho Z^{*-(\rho+1)}\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}A}\Big(\frac{a^*}{A-a^*}\Big) \qquad (46)
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}A}\Big(\frac{a^*}{A-a^*}\Big) = \frac{\mathrm{d}a^*}{\mathrm{d}A}\frac{A}{(A-a^*)^2} - \frac{a^*}{(A-a^*)^2} \qquad (47)$$

Similarly, differentiating (42) implicitly with respect to B yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}B}\Big(X^* + Y^*\Big)^{\frac{1}{\rho-1}}\frac{A-a^*}{a^*} &= \frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{A-a^*}{a^*}\Big)\Big(X^*+Y^*\Big)^{\frac{1}{\rho-1}} \\
&\quad + \frac{A-a^*}{a^*}\frac{1}{\rho-1}\Big(X^*+Y^*\Big)^{\frac{2-\rho}{\rho-1}}\Big(\frac{\mathrm{d}}{\mathrm{d}B}X^* + \frac{\mathrm{d}}{\mathrm{d}B}Y^*\Big) \\
&= 0, \qquad (48)
\end{aligned}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{A-a^*}{a^*}\Big) = -\frac{A}{a^{*2}}\frac{\mathrm{d}a^*}{\mathrm{d}B} \tag{49}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}B}X^* = {} & \frac{(1-\omega)\rho a^{*\rho-1}\frac{\mathrm{d}a^*}{\mathrm{d}B}}{X_2^*} - \frac{X_1^*}{X_2^{*2}} \\ & \times\Bigg(-\omega(A-a^*)^{\rho-1}\Big(1-\frac{\mathrm{d}a^*}{\mathrm{d}B}\Big) \\ & -(1-\omega)\rho Z^{*-(\rho+1)}\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\Big(\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{a^*}{A-a^*}\Big) \\ & -\frac{a^*}{B^2(A-a^*)}\Big)-\frac{1}{B^2}\Big)\Bigg)\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}B}Y^* = {} & \omega\rho\Big(B-\frac{1}{Z^*}\Big)^{\rho-1}\Bigg(1+Z^{*2}\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\Bigg(\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{a^*}{A-a^*}\Big) \\ & -\frac{a^*}{B^2(A-a^*)}\Bigg)-\frac{1}{B^2}\Bigg)\Bigg)\frac{1}{Y_2^*} \\ & -\frac{Y_1^*}{Y_2^{*2}}\times\Bigg(-\omega(A-a^*)^{\rho-1}\frac{\mathrm{d}a^*}{\mathrm{d}B}+\rho(1-\omega)Z^{*-(\rho+1)} \\ & \times\Bigg(\Big(\frac{1-\omega}{\omega}\Big)^{\frac{2}{\rho-1}}\Big(\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{a^*}{A-a^*}\Big)-\frac{a^*}{B^2(A-a^*)}\Big)-\frac{1}{B^2}\Bigg)\Bigg)\end{aligned} \tag{51}$$

$$\frac{\mathrm{d}}{\mathrm{d}B}\Big(\frac{a^*}{A-a^*}\Big) = \frac{A}{(A-a^*)^2}\frac{\mathrm{d}a^*}{\mathrm{d}B} \tag{52}$$

We can solve these equations numerically for $\frac{\mathrm{d}a^*}{\mathrm{d}A}$ and $\frac{\mathrm{d}a^*}{\mathrm{d}B}$.

Expressions for $\frac{\mathrm{d}b^*}{\mathrm{d}A}$ and $\frac{\mathrm{d}b^*}{\mathrm{d}B}$ follow from the results above and

$$b^* = B - \frac{1}{\big(\frac{1-\omega}{\omega}\big)^{\frac{2}{\rho-1}}\frac{a^*}{A-a^*}\frac{1}{B}+\frac{1}{B}} \tag{53}$$

C Constant Portfolio Shares under Cobb-Douglas

From the FOCs and the constraints it follows that the optimal allocation of inputs in Home country is:

$$\begin{aligned} a &= \omega A \\ b &= (1-\omega)B \end{aligned}$$

Knowing that $\lambda = \frac{a}{A}$ and $\lambda^f = \frac{b}{B}$ we can easily compute the partial derivatives of the portfolio shares with respect to B.

$$\begin{aligned} \frac{\partial \lambda}{\partial B} &= \frac{\partial (a/A)}{\partial B} = \frac{1}{A}\frac{\partial a}{\partial B} \\ \frac{\partial \lambda^f}{\partial B} &= \frac{\partial (b/B)}{\partial B} = \frac{1}{B}\frac{\partial b}{\partial B} - \frac{b}{B^2} \end{aligned} \tag{54}$$

Since $\frac{\partial a}{\partial B} = \frac{\partial \omega A}{\partial B} = 0$, it follows that

$$\frac{\partial \lambda}{\partial B} = 0$$

Similarly, knowing that $\frac{\partial b}{\partial B} = \frac{\partial (1-\omega)B}{\partial B} = 1-\omega$, it follows that

$$\begin{aligned} \frac{\partial \lambda^f}{\partial B} = \frac{1}{B}\frac{\partial b}{\partial B} - \frac{b}{B^2} &= \frac{1-\omega}{B} - \frac{b}{B^2} \\ &= \frac{1-\omega}{B} - \frac{(1-\omega)B}{B^2} \\ &= 0 \end{aligned} \tag{55}$$

Q.E.D.

D Production Economy with Inelastic Labor Supply

D.1 The Basic Model

The basic model consists of representative household, a perfectly competitive intermediate sector, and a perfectly competitive final goods sector in each country.

D.1.1 The Household's Problem

The representative household makes utility-maximizing choices over consumption $c(s^t)$, the investment share in the domestic intermediate firm $\lambda(s^t)$, and the investment share in the foreign intermediate firm λ^f. Investments at home and abroad are motivated by risk diversification in the following sense:

Each country's intermediate sector produces a single good. Using the stylized example of section 2, let's assume that Home produces aluminum $a(s^t)$ and that Foreign produces bricks $b(s^t)$. Each industry uses labor (supplied inelastically) and capital as inputs and is subject to productivity shocks. As before, let * denote allocations and prices abroad, while variables without asterisk stand for domestic allocations and prices. Since final consumption (and investment) is the output of a CES aggregator with inputs $a(s^t)$ and $b(s^t)$, households diversify their portfolios to insure against unfavorable productivity shocks in the intermediate industries.

Finally, let the numeraire at home be domestic final consumption while the numeraire abroad is foreign final consumption.

Formally, households solve the following constrained optimization problem:

$$\begin{aligned}
\max_{\left(c(s^t),\lambda(s^t),\lambda^f(s^t)\right)} U\big(c(s^t)\big) &= \sum_{t=0}^{T}\sum_{s^t\in S}\beta^t\pi(s^t)u\big(c(s^t)\big) \\
\text{subject to} \quad & \sum_{t=0}^{T}\sum_{s^t\in S}\beta^t\pi(s^t)\mu(s^t)\Big(q_a(s^t)w(s^t)+\lambda(s^{t-1})d(s^t) \\
& +\lambda^f(s^{t-1})d^*(s^t)rx(s^t)-P(s^t)\big(\lambda(s^t)-\lambda(s^{t-1})\big) \\
& -rx(s^t)P^*(s^t)\big(\lambda^f(s^t)-\lambda^f(s^{t-1})\big)-c(s^t)\Big), \quad (56)
\end{aligned}$$

where $rx(s^t)$ denotes the real exchange rate, i.e. the price of foreign consumption relative to consumption at home, $q_a(s^t)$ is the price of $a(s^t)$ in terms of $c(s^t)$, $w(s^t)$ is the wage in the intermediate sector in terms of its own output, $d(s^t)$ are domestic dividend payments, and $P(s^t)$ is the price of shares in the domestic firm; $d^*(s^t)$ and $P^*(s^t)$ are the foreign counterparts.

In addition to the constraint, the FOCs are:

$$\begin{aligned} u_c\big(c(s^t)\big)P(s^t) &= \beta\sum_{s^{t+1}}\pi(s^{t+1}|s^t)u_c\big(c(s^{t+1})\big) \\ &\quad\times\big(d(s^{t+1})+P(s^{t+1})\big) \qquad (57) \\ u_c\big(c(s^t)\big)rx(s^t)P^*(s^t) &= \beta\sum_{s^{t+1}}\pi(s^{t+1}|s^t)u_c\big(c(s^{t+1})\big)rx(s^{t+1}) \\ &\quad\times\big(d^*(s^{t+1})+P^*(s^{t+1})\big) \qquad (58) \end{aligned}$$

Note that the Foreign household's problem is analogous. The corresponding FOCs are:

$$\begin{aligned} u_c\big(c^*(s^t)\big)P^*(s^t) &= \beta\sum_{s^{t+1}}\pi(s^{t+1}|s^t)u_c\big(c^*(s^{t+1})\big) \\ &\quad\times\big(d^*(s^{t+1})+P^*(s^{t+1})\big) \qquad (59) \\ \frac{u_c\big(c(s^t)\big)P(s^t)}{rx(s^t)} &= \beta\sum_{s^{t+1}}\frac{\pi(s^{t+1}|s^t)u_c\big(c^*(s^{t+1})\big)}{rx(s^{t+1})} \\ &\quad\times\big(d(s^{t+1})+P(s^{t+1})\big) \qquad (60) \end{aligned}$$

Next, let us consider the intermediate firm's problem.

D.1.2 The Intermediate Firm's Problem

We assume that firms own their capital and households purchase capital shares, which earn capital gains and dividends in each period and state of nature.

Let dividends be determined by:

$$d(s^t) = q_a(s^t)\Big(F\big(k(s^{t-1}), z(s^t), 1\big) - w(s^t)\Big) - \big(k(s^t) - (1-\delta)k(s^{t-1})\big) \qquad (61)$$

Let $Q(s^t)$ denote the value of $d(s^t)$ in terms of date-0 final consumption. Then, the domestic firms solves:

$$\max_{k(s^t)} \sum_{t=0}^{T} \sum_{s^t \in S} Q(s^t) d(s^t) \tag{62}$$

$$\begin{aligned} \text{where } Q(s^t) &= \gamma(s^t) \frac{\pi(s^t)\beta^t u_c\big(c(s^t)\big)}{u_c\big(c(s^0)\big)} \\ &\quad + \big(1-\gamma(s^t)\big) \frac{\pi(s^t)\beta^t u_c\big(c^*(s^t)\big) rx(s^0)}{u_c\big(c^*(s^0)\big) rx(s^t)} \end{aligned} \tag{63}$$

As for the household, the foreign intermediate firm's problem is analogous.

The FOCs at home and abroad are:

$$Q(s^t) = \sum_{s^{t+1}} Q(s^{t+1}) \Big(q_a(s^t) F_k\big(k(s^t), z(s^{t+1}), 1\big) + 1 - \delta \Big) \tag{64}$$

$$Q^*(s^t) = \sum_{s^{t+1}} Q^*(s^{t+1}) \Big(q_b^*(s^t) F_k^*\big(k^*(s^t), z^*(s^{t+1}), 1\big) + 1 - \delta \Big) \tag{65}$$

Even though labor is supplied inelastically, we need to formally describe the equilibrium wage as a function of capital and productivity shocks:

$$w(s^t) = F_l\big(k(s^{t-1}), z(s^t), 1\big) \tag{66}$$

and analogously for the intermediate producer in Foreign.

Lastly, let's take a look at the final good producer's problem.

D.1.3 The Final Good Producer's Problem

The final good is produced by aggregating aluminum and bricks with a CES technology.

$$\max_{a(s^t), b(s^t)} \big(\omega a(s^t)^\rho + (1-\omega) b(s^t)^\rho\big)^{\frac{1}{\rho}} - q_a(s^t) a(s^t) - q_b(s^t) b(s^t) \tag{67}$$

The corresponding FOCs for the domestic firm are:

$$q_a(s^t) = \big(\omega a(s^t)^\rho + (1-\omega) b(s^t)^\rho\big)^{\frac{1-\rho}{\rho}} \omega a(s^t)^{\rho-1} \tag{68}$$

$$q_b(s^t) = \big(\omega a(s^t)^\rho + (1-\omega) b(s^t)^\rho\big)^{\frac{1-\rho}{\rho}} (1-\omega) b(s^t)^{\rho-1} \tag{69}$$

The foreign producer's problem is analogous with CES aggregator

$$G^*\big(a^*(s^t), b^*(s^t)\big) = \big((1-\omega)a^*(s^t)^\rho + \omega b^*(s^t)^\rho\big)^{\frac{1}{\rho}}.$$

Note the difference in home bias between Home and Foreign.

In addition to the FOCs and constraints above, any equilibrium allocation must satisfy the following goods and asset market-clearing conditions:

$$a(s^t) + a^*(s^t) = F\big(k(s^{t-1}), z(s^t), 1\big) \quad (70)$$

$$b(s^t) + b^*(s^t) = F^*\big(k^*(s^{t-1}), z^*(s^t), 1\big) \quad (71)$$

$$\lambda(s^t) + \lambda^{*h}(s^t) = 1 \quad (72)$$

$$\lambda^f(s^t) + \lambda^*(s^t) = 1 \quad (73)$$

$$c(s^t) + k(s^t) - (1-\delta)k(s^{t-1}) = G\big(a(s^t), b(s^t)\big) \quad (74)$$

$$c^*(s^t) + k^*(s^t) - (1-\delta)k^*(s^{t-1}) = G^*\big(a^*(s^t), b^*(s^t)\big) \quad (75)$$

The following section sketches the proof for existence and uniqueness of the non-stochastic steady state.

D.1.4 Characterizing the Non-Stochastic Steady State

Let $E\big(z(s^t)\big) = E\big(z^*(s^t)\big) = z = z^* = 1$; $k(s^t) = k, \forall t, s^t$; and $k^*(s^t) = k^*, \forall t, s^t$.

To keep the notation simple, we suppress the *histories* notation for steady-state variables, unless they fluctuate in steady-state.

In what follows, we will try to characterize the steady-state allocations as sharply as possible.

$$a + a^* = F(k, 1, 1) \quad (76)$$

$$b + b^* = F^*(k^*, 1, 1) \quad (77)$$

Clearly, output of bricks and aluminum is constant. It can be shown easily that not only the output of intermediate goods is constant in steady state, but so are the optimal allocations a, a^*, b, and b^*. The proof is available upon request.

It follows from the above that the prices of inputs in Home and Foreign (in terms of local final consumption) must also be constant.

$$\begin{aligned} q_a &= \left(\omega a^{\rho} + (1-\omega)b^{\rho}\right)^{\frac{1-\rho}{\rho}} \omega a^{\rho-1} && (78) \\ q_b &= \left(\omega a^{\rho} + (1-\omega)b^{\rho}\right)^{\frac{1-\rho}{\rho}} (1-\omega) b^{\rho-1} && (79) \\ q_a^* &= \left((1-\omega)a^{*\rho} + \omega b^{*\rho}\right)^{\frac{1-\rho}{\rho}} (1-\omega) a^{*\rho-1} && (80) \\ q_b^* &= \left((1-\omega)a^{*\rho} + \omega b^{*\rho}\right)^{\frac{1-\rho}{\rho}} \omega b^{*\rho-1} && (81) \end{aligned}$$

This, in turn, implies by equation (61) that dividend payments are constant too.

$$\begin{aligned} d &= q_a\big(F(k,1,1) - w\big) - \delta k && (82) \\ d^* &= q_b^*\big(F^*(k^*,1,1) - w^*\big) - \delta k^* && (83) \end{aligned}$$

It follows also that

$$rx = \frac{q_a}{q_a^*} \qquad (84)$$

is constant.

From equation (63) and its counterpart for the intermediate producer abroad, we can see that $\frac{Q}{\beta^t}$ and $\frac{Q^*}{\beta^t}$ are constant.

$$\begin{aligned} \frac{Q}{\beta^t} &= \gamma \frac{u_c(c)}{u_c(c_0)} + (1-\gamma) \frac{rx_0 u_c(c^*)}{rx u_c(c_0^*)} && (85) \\ \frac{Q^*}{\beta^t} &= \gamma^* \frac{u_c(c^*)}{u_c(c_0^*)} + (1-\gamma) \frac{rx u_c(c)}{rx_0 u_c(c_0)} && (86) \end{aligned}$$

Using the fact that $\frac{Q}{\beta^t}$ is constant, as well as equations (64) and (65) we can pin down the level of q_a and q_b^*:

$$\frac{1}{\beta} = q_a F_k(k,1,1) + 1 - \delta \tag{87}$$

$$\frac{1}{\beta} = q_b^* F_k(k^*,1,1) + 1 - \delta \tag{88}$$

Since consumption and dividends are constant in steady-state, we can iterate forward equations (57) (or, equivalently, equation (60)) and solve for P:

$$P = \frac{\beta}{1-\beta} d \tag{89}$$

Similarly, using (58) or (59) we can solve for P^*:

$$P^* = \frac{\beta}{1-\beta} d^* \tag{90}$$

This completes the demonstration of the existence and uniqueness of the non-stochastic steady state.

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