Strongly MDS Convolutional Codes, A New Class of Codes with Maximal Decoding Capability[1]

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Abstract — **A new class of rate 1/2 convolutional codes called strongly MDS convolutional codes are introduced and studied. These are codes having optimal column distances. Properties of these codes are given and a concrete construction is provided.**

An $[n,k,\delta]$ convolutional code is called MDS if its free distance is maximal among all rate k/n convolutional codes of degree δ under the bound: $d_{free} \le (n-k)(\lfloor \delta/k \rfloor + 1) + \delta + 1$, see [2, 3]. Strongly MDS codes are a subclass of MDS codes which have a remarkable decoding capability. We show in this paper that a $[2,1,\delta]$ strongly MDS code can correct up to δ errors in any *sliding window* of $4\delta+2$ code symbols. This compares to an MDS block code with parameters $[n, n/2]$, $n = 4\delta+2$, which corrects up to δ errors in any *slotted window* (block) of length $4\delta+2$.

Let $\mathcal{C}$ be a rate 1/2 convolutional code over a field $\mathbb{F}$, generated by $G(D) = [\ a(D) \quad b(D)\ ]$, with $a(D) = a_0 + \ldots + a_\delta D^\delta, b(D) = b_0 + \ldots + b_\delta D^\delta$, $a_0 \ne 0$ or $b_0 \ne 0$, and $a(D), b(D)$ coprime.

A parity check matrix for $\mathcal{C}$ is given by $H(D) = [\ -b(D) \quad a(D)\ ]$. We expand the matrix $H(D)$ into $H(D) = H_0 + \ldots + H_\delta D^\delta, H_j \in \mathbb{F}^{1\times 2}, j = 0, \ldots, \delta$. Let

$$H_j^c = \begin{bmatrix} H_0 & & & \\ H_1 & H_0 & & \\ \vdots & \vdots & \ddots & \\ H_j & H_{j-1} & \ldots & H_0 \end{bmatrix} \in \mathbb{F}^{(j+1)\times 2(j+1)} \qquad (1)$$

and let

$$d_j^c = \min_{v_0 \ne 0} \{ \mathrm{wt}((v_0, \ldots, v_j)) \,|\, (v_0, \ldots, v_j) \in \ker H_j^c \},$$

be the jth column distance of the code $\mathcal{C}$. We have the following natural bound on the d_j^c.

Theorem 1 *A convolutional code of rate 1/2 has the jth column distance bounded above by* $d_j^c \le j+2$. *We also have* $d_{free} \le 2\delta + 2$.

Definition 2 A code with $d_{free} = 2\delta + 2$ will be called MDS convolutional code.

Corollary 3 *The index* $j = 2\delta$ *is the earliest step at which a rate 1/2 MDS convolutional code can attain equality* $d_j^c = d_{free}$ *in the distance inequality:*

$$d_0^c \le d_1^c \le \ldots \le d_\infty^c = d_{free} = 2\delta + 2.$$

Definition 4 A rate 1/2, degree δ, convolutional code is called *strongly MDS* if $d_{2\delta}^c = 2\delta + 2 = d_{free}$.

Theorem 5 *Let* $\mathcal{C}$ *be a 1/2 rate convolutional code of degree* δ. *Let A be the submatrix of* $H_{2\delta}^c$ *consisting of the columns with indices* $1, 3, \ldots, 2\delta+1$ *and denote by B the remaining submatrix. Put* $T := B^{-1}A$. *The following statements are equivalent:*
1) The code $\mathcal{C}$ *is strongly MDS;*
2) $d_{2\delta}^c = 2\delta + 2 = d_{free}$*;*
3) The first column of the matrix $[T, I]$ *is not a linear combination of any* 2δ *other columns of that matrix;*
4) The matrix $T = \begin{bmatrix} h_0 & & & \\ h_1 & h_0 & & \\ \vdots & & \ddots & \\ h_{2\delta} & h_{2\delta-1} & \ldots & h_0 \end{bmatrix}$ *has the property that all its square submatrices* $A_{j_1,\ldots,j_r}^{i_1,\ldots,i_r}$ *formed by the* $i_1, \ldots, i_r$ *rows and* $j_1, \ldots, j_r$ *columns of T, are invertible, for all* $1 \le r \le 2\delta+1$ *and all indices* $1 \le i_1 < \ldots < i_r \le 2\delta+1, 1 \le j_1 < \ldots < j_r \le 2\delta+1$ *which satisfy* $j_\nu \le i_\nu$ *for* $\nu = 1, \ldots, r$.

Example 6 Let $n = 2\delta$ and consider the $(n+1)\times(n+1)$ matrix

$$X = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}. \qquad (2)$$

Then $T := X^n$ is a *totally positive matrix*, i.e. T satisfies Property 4) over large enough prime fields, see [1].

Decoding: Let $(y(D), z(D)) \in (\mathbb{F}[D])^2$ be a received message and let $(v(D), w(D)) \in \mathcal{C}$ the transmitted vector and $(f(D), e(D)) \in (\mathbb{F}[D])^2$ the error vector, $y(D) = v(D) + f(D), z(D) = w(D) + e(D)$.

Suppose we have corrected all the components received before y_0, z_0. Assuming that the weight of the error $[\ f_0 \ \ldots \ f_{2\delta} \ e_0 \ \ldots \ e_{2\delta}\ ]^T$ in this $4\delta+2$ window is at most δ, we find an algorithm that computes f_0 and e_0. Knowing f_0 and e_0 we update our received message, and move one step further.

The following theorem tells that such an algorithm exists.

Theorem 7 *Let* $f = (f_0, \ldots, f_{2\delta})^T, e = (e_0, \ldots, e_{2\delta})^T$ *be two vectors in* $\mathbb{F}^{2\delta+1}$ *such that* $\mathrm{wt}[\ f \quad e\ ]^T \le \delta$. *Let*

$$[\ T \quad I\ ][\ f \quad e\ ]^T = [\ s_0 \quad s_1 \quad \ldots \quad s_{2\delta}\ ]^T. \qquad (3)$$

If $[\ \tilde{f} \quad \tilde{e}\ ]^T$ *is another solution of the equation (3) with* $\mathrm{wt}[\ \tilde{f} \quad \tilde{e}\ ]^T \le \delta$ *then* $f_0 = \tilde{f}_0, e_0 = \tilde{e}_0$.

References

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[1] Authors were supported in part by NSF grant DMS-00-72383.