SYMPLECTIC HYPERSURFACES IN $\mathbb{C}P^3$

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ABSTRACT. We establish the uniqueness of the symplectic 4-manifolds which admit low degree symplectic embeddings into $\mathbb{C}P^3$. We also discuss the uniqueness of the fundamental group of the complement of such embeddings into arbitrary symplectic 6-manifolds.

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1. INTRODUCTION

B. Siebert and G. Tian in [10] showed that the symplectic hypersurfaces (symplectic submanifolds of codimension two) in $\mathbb{C}P^2$ of a fixed degree $d \leq 17$ are all equivalent up to symplectic isotopy.

In higher dimensions no analogous results are known. S. Donaldson in [2] gave a general construction of symplectic hypersurfaces in integral symplectic manifolds Poincaré dual to large multiples of the symplectic form, and the hypersurfaces arising in this construction are unique up to symplectic isotopy for large enough multiples, see [1]. However it is unknown if such a construction generates all symplectic hypersurfaces or how large a multiple is needed for the uniqueness.

In this note we begin a study of the symplectic hypersurfaces of small degree inside $\mathbb{C}P^3$, establishing their uniqueness up to symplectomorphism although not necessarily global symplectic isotopy. We look also at the topology of the complements of such hypersurfaces in general symplectic 6-manifolds. One difficulty is that, unlike the algebraic case (and Donaldson's situation), it is unclear whether or not the complement of a symplectic hypersurface has the structure of a Stein manifold. Thus one cannot apply the Leftschetz hyperplane theorem.

We remark that the study of symplectic submanifolds of codimension greater than two can be understood in terms of the h-principle (see [4]).

Here are our results. Let ω be the standard Fubini-Study symplectic form on $\mathbb{C}P^3$ Poincaré dual to the hyperplane H. Suppose that Σ is a 4-manifold and $i : \Sigma \to \mathbb{C}P^3$ is an embedding. Then $i_*[\Sigma] = d[H] \in H_4(\mathbb{C}P^3; \mathbb{Z})$ where d is called the degree of the embedding. The embedding is symplectic if $\tilde{\omega} := i^*\omega$ is a symplectic form, in this case we will simply say that $\Sigma \subset \mathbb{C}P^3$ is a symplectic hypersurface.

Theorem 1.1. *Let $\Sigma \subset \mathbb{C}P^3$ be a symplectic hypersurface.*

(i) If $d = 1$ then $(\Sigma, \tilde{\omega})$ is symplectomorphic to the standard $\mathbb{C}P^2$ with $\int_L \tilde{\omega} = 1$ for lines $L \subset \mathbb{C}P^2$.

(ii) If $d = 2$ then $(\Sigma, \tilde{\omega})$ is symplectomorphic to $S^2 \times S^2$ with a split symplectic form satisfying $\int_{S^2 \times x} \tilde{\omega} = \int_{x \times S^2} \tilde{\omega} = 1$.

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We also address the uniqueness question for such embeddings by considering their complements in general symplectic 6-manifolds (M, ω). Now $i : \Sigma \to M$ denotes a symplectic hypersurface, that is, i is an embedding and $\tilde{\omega} := i^*\omega$ is symplectic. In the case when $\Sigma \subset M$ looks locally like a symplectic embedding of degree 1 or 2 in $\mathbb{C}P^3$ we can say the following about the fundamental group of the complement.

Theorem 1.2. *Let $\Sigma \subset (M^6, \omega)$ be a symplectic hypersurface. We assume that $\omega|_{H_2(M,\mathbb{R})}$ attains its minimum positive value on the image of $i_* : H_2(\Sigma) \to H_2(M)$. Suppose that the normal bundle of Σ has first Chern class $[d\tilde{\omega}] = [di^*\omega]$.*

(i) If $d = 1$ and $(\Sigma, \tilde{\omega})$ is symplectomorphic to the standard $\mathbb{C}P^2$ then $M \setminus \Sigma$ is simply connected.

(ii) If $d = 2$ and $(\Sigma, \tilde{\omega})$ is symplectomorphic to $S^2 \times S^2$ with a split symplectic form then $\pi_1(M \setminus \Sigma) = \mathbb{Z}_2$.

Case (i) here is due to Y. Eliashberg but in fact more is true. By a theorem of D. McDuff, see [7], $M \setminus \Sigma$ is diffeomorphic to the ball B^6. Our proof for case (ii) also applies to show that all homology groups of $M \setminus \Sigma$ are isomorphic to those of $\mathbb{R}P^3$ and a natural conjecture here is that $M \setminus \Sigma$ is diffeomorphic to $T^*\mathbb{R}P^3$, the complement of an algebraic hypersurface of degree 2 in $\mathbb{C}P^3$.

Theorem 1.1(i) is proved in section 2, Theorem 1.1(ii) in section 3 and Theorem 1.2(ii) in section 4. The results are applications of the theory of pseudoholomorphic curves. The necessary techniques are developed in the papers [3] and [5], see also the books [8] or [9].

2. $d = 1$

Recall that $\Sigma \subset (\mathbb{C}P^3, \omega)$ is a symplectic embedding, which we now assume to be of degree $d = 1$ (or equivalently, homologous to a hyperplane H). Then $\tilde{\omega} = \omega|_\Sigma$ is a symplectic form. For convenience we will also write ω and $\tilde{\omega}$ for the corresponding cohomology classes in $H^2(\mathbb{C}P^3)$ and $H^2(\Sigma)$ respectively.

As Σ is symplectic, we can find an almost-complex structure J on $\mathbb{C}P^3$ which is tamed by ω and such that Σ becomes an almost-complex submanifold. Then the tangent bundle $T\Sigma$ and normal bundle ν of Σ in $\mathbb{C}P^3$ become complex vector bundles over Σ. As Σ is Poincaré dual to ω, the first Chern class $c_1(\nu) = \tilde{\omega}$. Write $c_i = c_i(T\Sigma)$ for $i = 1, 2$. We have $T\Sigma \oplus \nu = T(\mathbb{C}P^3)|_\Sigma$. Therefore the total Chern classes satisfy

$$c(T\Sigma)c(\nu) = c(T(\mathbb{C}P^3)|_\Sigma).$$

That is,

$$(1 + c_1 + c_2)(1 + \tilde{\omega}) = (1 + \tilde{\omega})^4.$$

Solving this gives $c_1 = 3\tilde{\omega}$ and $c_2 = 3\tilde{\omega}^2$.

Now, with respect to the standard complex structure i, given any two points in $\mathbb{C}P^3$ there exists a unique holomorphic sphere passing through the two points and homologous to a line $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^3)$. Therefore, using the theory of pseudoholomorphic curves, see for instance [3] or [8], we can also find a J-holomorphic sphere C in $\mathbb{C}P^3$ passing through any two points p, q. To do this, very roughly, let J_t be a family of tame almost-complex structures with $J_0 = i$ and $J_1 = J$. Then one looks at the moduli space of pairs (u, t) where $0 \le t \le 1$ and u is a J_t-holomorphic sphere passing through p and q and homologous to $[\mathbb{C}P^1]$, modulo the reparameterization group $G = PSL(2, \mathbb{C})$. The complex structure i is regular and so if everything is chosen generically this moduli space has dimension

$$\text{ind} = 6 + 2c_1(\mathbb{C}P^3)[\mathbb{C}P^1] - \dim(G) + 1 - 8 = 1.$$

It is compact since $[\mathbb{C}P^1]$ does not decompose into a sum of homology classes of positive symplectic area and so no bubbling is possible. Thus we have a cobordism between the one point moduli space of J_0-holomorphic curves and the space of J_1-holomorphic curves which is therefore nonempty. If J itself is not regular, to make the above argument valid one may have to perturb it slightly first, but then we still find a J-holomorphic sphere by taking limits of the spheres between p and q which are holomorphic with respect to the perturbed structures.

Choosing p and q to lie in Σ then the (image of the) J-holomorphic curve, C, must also lie in Σ. For, by positivity of intersections [6], otherwise the sphere must have intersection number at least 2 with Σ (coming from the two intersection points p and q). But $[\mathbb{C}P^1] \bullet [\Sigma] = 1$, a contradiction.

Let $r \in \mathbb{C}P^3 \setminus \Sigma$ and define $\mathcal{M}$ to be the moduli space of (parameterized) J-holomorphic curves passing through r and C and whose image is homologous to $[\mathbb{C}P^1]$. Perturbing J in a neighborhood of r, the moduli space $\mathcal{M}$ can be assumed to be a manifold of the expected dimension $\text{ind} = 6+2c_1(\mathbb{C}P^3)[\mathbb{C}P^1]-6 = 8$. There is a natural evaluation map $e : \mathcal{M} \times \mathbb{C}P^1 \to \mathbb{C}P^3$. The group $G = PSL(2, \mathbb{C})$ acts on $\mathcal{M} \times \mathbb{C}P^1$ by $g.(f, z) = (f \circ g^{-1}, g(z))$ and by Gromov's compactness theorem [3] the quotient $B = \mathcal{M} \times_G \mathbb{C}P^1$ is compact. In fact, it is a compact oriented 4-manifold. Therefore e reduces to a map between compact manifolds $e : B \to \mathbb{C}P^3$. Again by pseudoholomorphic curve theory, the homology class of the image, $e_*[B]$, is independent of the choice of J and can be calculated to be the class of a hyperplane H when $J = i$. By positivity of intersections, the image of each curve in $\mathcal{M}$ can intersect Σ only once, and so $e(B) \cap \Sigma = C$ both as a set and a homology class. In particular its orientation as an intersection coincides with its complex orientation. Furthermore, as a homology class in Σ, $e(B) \cap \Sigma$ is Poincaré dual to $\tilde{\omega}$ and so $C \bullet C = \int_C \omega = 1$. We can now calculate the virtual genus of the holomorphic curve C in Σ to be

$$g = \frac{1}{2}(2 + C \bullet C - c_1(T\Sigma)(C)) = 0.$$

Since this coincides with the actual genus, by the adjunction formula our curve C must be embedded.

To conclude this section we apply the classification of rational and ruled symplectic 4-manifolds due to McDuff [5], see also [3]. This says that since $(\Sigma, \tilde{\omega})$ contains an embedded symplectic sphere of self-intersection 1, it must be a blow-up of $\mathbb{C}P^2$, with its standard symplectic structure, and the sphere represents the class of a line. In our case $c_2 = 3$ and so Σ is $\mathbb{C}P^2$ itself.

3. $d = 2$

Using the same notation as above, Σ now denotes a symplectic hypersurface of degree 2 and again we choose an almost-complex structure J making Σ into an almost-complex submanifold. Its normal bundle has first Chern class $c_1(\nu) = 2\omega|_\Sigma = 2\tilde{\omega}$. The identity $c(T\Sigma)c(\nu) = c(T(\mathbb{C}P^3)|_\Sigma)$ then gives the equation

$$(1 + c_1 + c_2)(1 + 2\tilde{\omega}) = (1 + \tilde{\omega})^4$$

for the Chern classes c_i of $T\Sigma$. Solving this we find that $c_1 = 2\tilde{\omega}$ and $c_2 = 2\tilde{\omega}^2$.

Following the same scheme as for the case of degree 1, we plan to find an embedded J-holomorphic curve in Σ and then apply the classification theorem.

Lemma 3.1. *Σ contains a J-holomorphic sphere C homologous to $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^3)$.*

Proof We can find an immersed surface $F \subset \Sigma$ with $c_1(\nu)[F] \neq 0$. We can assume that the surface F is smooth although not necessarily symplectic. Pick a $p \in \Sigma \setminus F$ and look at the moduli space $\tilde{\mathcal{M}}$ of holomorphic spheres passing through p and F and homologous to $[\mathbb{C}P^1]$, modulo reparameterizations. This is generically (after perhaps perturbing J near p) an oriented manifold and we calculate $\dim\tilde{\mathcal{M}} = 2$.

Now, if the lemma fails then each curve in $\tilde{\mathcal{M}}$ must intersect Σ transversally at p and exactly one point on F. For otherwise such a curve would either lie in Σ or have intersection number greater than two with Σ. Therefore we have a well-defined map $e : \tilde{\mathcal{M}} \to F$. The degree of this map is independent of the choice of J and consideration of the standard complex structure i shows that it has degree 1. Let E be the complex vector bundle over $\tilde{\mathcal{M}}$ with fiber $E_C := T_{C\cap F}C$ over a curve $C \in \tilde{\mathcal{M}}$ (generically the curves will be nonsingular at points of F). Since each C intersects Σ transversally there exists a bundle isomorphism $E \to \nu|_F$ covering e. For example, if $\pi : T(\mathbb{C}P^3)|_\Sigma \to \nu$ denotes a projection with kernel $T\Sigma$ then we can map $v \in E_C \mapsto \pi(v) \in \nu_{C\cap F}$. On the other hand $\pi : T_p(C) \to \nu|_p$ is also an isomorphism by transversality at p and so we can canonically parameterize each curve $C \in \tilde{\mathcal{M}}$ by a holomorphic map $u : \mathbb{C}P^1 \to \mathbb{C}P^3$ requiring that 0 map to p, 1 map to F and $\pi \circ u'(0)$ map 1 to a fixed vector $v \in \nu|_p$. The bundle E is then trivialized by $u'(1)(1)$, giving a contradiction since $E \cong e^*\nu|_F$ has nonzero Chern class $e^*c_1(\nu|_F)$. $\Box$

We would like to show that the curve C from Lemma 3.1 is embedded and in order to do this we will compute its self-intersection number. Towards this end, we look at the moduli space $\tilde{\mathcal{N}}$ of unparameterized J-holomorphic spheres homologous to $[\mathbb{C}P^1]$ and passing through C and a fixed point $q \in \mathbb{C}P^3 \setminus \Sigma$. Again $\tilde{\mathcal{N}}$ is generically an oriented manifold of dimension 2 and the images of the spheres in $\tilde{\mathcal{N}}$ generate a cycle in $\mathbb{C}P^3$ homologous to a hyperplane H.

By positivity of intersections each curve in $\tilde{\mathcal{N}}$ either intersects Σ transversally in two distinct points with exactly one of those points on C, intersects Σ at two distinct points on C, or intersects Σ tangentially at a single point lying on C. Note that we do not expect any J-holomorphic spheres passing through q to be tangent to C, even if we vary J in a 1-parameter family. Also, all but a finite number of spheres S in $\tilde{\mathcal{N}}$ should intersect Σ in a point off C. Therefore we can define two maps $\tilde{\mathcal{N}} \setminus S \to \Sigma$ by in the first case taking the intersection of the sphere with C and in the second case the intersection of the sphere with $\Sigma \setminus C$. These maps can be extended smoothly over S to give maps $e, f : \tilde{\mathcal{N}} \to \Sigma$ with images C and, say, D respectively. Both maps are of degree 1 and since $C \cup D$ represents $\Sigma \bullet H$ inside $\mathbb{C}P^3$ we must have $[D] = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^3)$ and inside Σ the union $C \cup D$ is Poincaré dual to $\tilde{\omega}$. If we perturb C then D is similarly perturbed and so as surfaces inside Σ we have $C \bullet C = D \bullet D$.

To calculate $C \bullet D$ there are two kinds of points to consider, those coming from spheres in $\tilde{\mathcal{N}}$ intersecting C twice and those coming from spheres tangent to Σ along C. Since there are no spheres in $\tilde{\mathcal{N}}$ tangent to C these two kinds of points are distinct and remain isolated as J is varied in a one parameter family. Therefore we can count them separately.

In the case when the complex structure is the standard one and C is a genuine holomorphic curve there are no intersection points of the first kind. Deforming the almost-complex structure we see that such points appear and vanish in pairs and in our case such intersections contribute zero to the intersection number.

To compute the contribution of the second set of points we look at two complex line bundles Y and Z over $\tilde{\mathcal{N}}$ where the fiber of Y over a curve Q is given by $Y_Q := T_q Q$ and $Z = e^*(\nu|_C)$. Since $\tilde{\mathcal{N}}$ is cobordant via a moduli space of J_t-holomorphic curves to the space of holomorphic curves through q and a complex line we have that $c_1(Y) = 1$ and since e is of degree 1 we have $c_1(Z) = \int_C 2\tilde{\omega} = 2$. We define a bundle map $m : Y \to Z$. To do this we fix a Hermitian form h on the complex vector space $(T_q(\mathbb{C}P^3), J)$, an involution $A = \frac{1}{z}$ on $\mathbb{C}P^1$ and a projection $\pi : T(\mathbb{C}P^3)|_\Sigma \to \nu$ as before. Then for $Q \in \tilde{\mathcal{N}}$ parameterized by a map u such that $u(0) = q$, $u(\infty) = e(Q)$ and $h(u'(0)(1)) = 1$, and for $v \in T_q Q = Y_Q$ with $h(v) = 1$ we set $m(v) = \pi(uAu^{-1})'(v)$ and extend linearly. Thus we find a section of $Y^* \otimes Z$ with zeros at curves $Q \in \tilde{\mathcal{N}}$ which are tangent to Σ. The indices must coincide and so $C \bullet D = c_1(Y^* \otimes Z) = 1$.

Putting all of this together we have that

$$2 = \int_{C+D} \omega = (C + D)^2 = 2C \bullet C + 2C \bullet D$$

and so $C \bullet C = 0$. The virtual genus of C is then

$$g = \frac{1}{2}(2 + C \bullet C - c_1(T\Sigma)(C)) = 0$$

and so C is an embedded curve.

Repeating the whole argument (replacing the surface F in the proof of Lemma 3.1 with C) we can find another embedded holomorphic curve C' with zero self-intersection but with positive intersection with C. Applying the classification [5], after blowing-down both C and C' can be realized as fibers of symplectic S^2-bundles. This implies that $(\Sigma, \tilde{\omega})$ is a blow-up of $S^2 \times S^2$ with its split symplectic form. But $c_2 = 4$ and so we conclude that Σ is indeed symplectomorphic to $S^2 \times S^2$.

4. Fundamental group

Here we prove case (ii) of Theorem 1.2. Now Σ is a symplectic hypersurface with normal bundle having first Chern class $2\tilde{\omega}$. We assume that $(\Sigma, \tilde{\omega})$ is symplectomorphic to $S^2 \times S^2$ with a split symplectic form.

We choose J as before making Σ an almost-complex submanifold and then the theory of pseudoholomorphic curves, see [3], implies that the J-holomorphic spheres in Σ in the classes $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$ form two foliations. In fact, by Weinstein's symplectic neighborhood theorem we can choose J such that a neighborhood U of Σ in M is biholomorphic to a neighborhood of a standard quadric in $\mathbb{C}P^3$. This guarantees regularity for J-holomorphic spheres close to Σ.

Choose two distinct spheres Q_1 and Q_2 lying in Σ which are small perturbations of a J-holomorphic sphere in the class $[\mathrm{pt} \times S^2]$. We will identify the normal bundle ν to Σ in M with a tubular neighborhood of Σ and the restriction $\nu|_{Q_1}$ with a suitable submanifold. The Chern class $c_1(\nu|_{Q_1}) = \int_{Q_1} 2\tilde{\omega} = 2$ and so the complement of the zero-section in $\nu|_{Q_1}$, say η, is homotopic to $\mathbb{R}P^3$. The projection map $\pi : T(M)|_{Q_1} \to \nu|_{Q_1}$ can be extended to a submersion from a tubular neighborhood of Q_1 in M onto $\nu|_{Q_1}$.

We will study the moduli space $\mathcal{M}$ of J'-holomorphic spheres in M, where J' is a small perturbation of J, which are homologous to $[S^2 \times \mathrm{pt}]$ and whose images intersect both Q_1 and Q_2. For generic J', $\mathcal{M}$ is a manifold of dimension 10 and $B = \mathcal{M} \times_G \mathbb{C}P^1$ is a compact oriented 6-manifold with a natural evaluation map $e : B \to M$. For compactness we need to exclude bubbling, this follows from our hypothesis on $\omega|_{H_2(M)}$.

If $p \in \Sigma \setminus (Q_1 \cup Q_2)$ and Q_1 and Q_2 are sufficiently small perturbations of J-holomorphic spheres in the class $[\mathrm{pt} \times S^2]$ then the unique J-holomorphic sphere in Σ through p in the class $[S^2 \times \mathrm{pt}]$ will intersect both Q_1 and Q_2 and the unique sphere through p in the class $[\mathrm{pt} \times S^2]$ will be disjoint from Q_1 and Q_2. The intersection number of J-holomorphic spheres homologous to $[S^2 \times \mathrm{pt}]$ with Σ is 2. Therefore, by positivity of intersections any J-holomorphic sphere in M passing through Q_1, Q_2 and p must lie in Σ and therefore there exists a unique J-holomorphic sphere passing through p. Similarly there exists a unique J'-holomorphic sphere through p, provided that J' is sufficiently close to J. For by compactness all such J'-holomorphic spheres must lie in the neighborhood U of Σ and thus there can exist only one by the regularity of the integrable complex structure (and all nearby almost-complex structures) on $\mathbb{C}P^3$. (For this, see for example Lemma 3.5 in [7]). Assuming that e is a submersion at p we deduce that e is of degree 1 and for any point q sufficiently close to p the set $e^{-1}(q)$ will also consist of a single point.

We now construct a bijection from $\pi_1(M \setminus \Sigma, q)$ to $\pi_1(\mathbb{R}P^3) \cong H_1(\mathbb{R}P^3) = \mathbb{Z}_2$. Let $\gamma : [0,1] \to M \setminus \Sigma$ be a loop with $\gamma(0) = \gamma(1) = q$. Perturbing γ such that it lies transverse to e, then the component of $e^{-1}(\gamma)$ mapping surjectively onto γ is a connected compact 1-dimensional submanifold N of B (since there exists a unique curve through q). We will also use a coordinate $t \in [0,1]/0 \sim 1$ for points of N. We may also suppose now that $J' = J$ in a neighborhood of Σ away from γ. The curves C_t corresponding to $t \in N$ intersect Σ in two distinct points and can be uniquely parameterized by a holomorphic map u_t such that $u_t(0) \in Q_1$, $u_t(\infty) \in Q_2$ and $u_t(1) \in \gamma$. We can homotope γ (fixing its image) onto the curve $\sigma(t) = u_t(1)$. Next we apply the homotopy $\sigma_s(t) = u_t(s)$, $0 < s \leq 1$, so $\sigma_1 = \sigma$. Again by positivity of intersections the u_t intersect Σ transversally only at 0 and ∞. Therefore the loops σ_s lie in $M \setminus \Sigma$ and the transversal intersection along Q_1 implies that for $s = \epsilon$ sufficiently small $\pi(\sigma_\epsilon(t)) \in \eta$. It is easy to find a homotopy from σ_ϵ to the loop $\pi(\sigma_\epsilon(t)) \in \eta$. Thinking of this as an element of $\pi_1(\mathbb{R}P^3)$ gives our map $\phi : \pi_1(M \setminus \Sigma, q) \to \pi_1(\mathbb{R}P^3)$.

The map is well-defined. In fact any two homologous loops γ_1 and γ_2 in $M \setminus \Sigma$ will be mapped to the same element of $\pi_1(\mathbb{R}P^3) \cong H_1(\mathbb{R}P^3)$. To see this, let $F \subset M \setminus \Sigma$ be a surface with boundary $\gamma_1 \coprod (-\gamma_2)$. Then $e^{-1}(F)$ is typically 2-dimensional with boundary $e^{-1}(\gamma_1) \coprod (-e^{-1}(\gamma_2))$. Each curve $C \in \mathcal{M}$ corresponding to a point in $e^{-1}(F)$ can be parameterized as above and used to give a map $e^{-1}(F) \to \eta$. The image is a cycle bounded by $\phi(\gamma_1) - \phi(\gamma_2)$ as required.

The map ϕ is injective by construction. Next, let C be the curve in $\mathcal{M}$ through q again parameterized by a map u such that $u(0) \in Q_1$, $u(\infty) \in Q_2$ and $u(1) = q$. Then $u(e^{2\pi i t})$ is a loop γ in $M \setminus \Sigma$ such that $\phi(\gamma)$ is homotopic to the nontrivial loop in a single fiber of η. Thus ϕ is surjective and hence an isomorphism.

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