Stabilized symplectic embeddings

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January 15, 2017

Abstract

We survey some symplectic embedding results focussing on the case when both domain and range are products of 4-dimensional ellipsoids or polydisks with Euclidean space. The stabilized problems have additional flexibility but some 4-dimensional obstructions persist.

1 Introduction

The symplectic embedding problem is among the easiest to state in symplectic topology. Nevertheless it provides a model situation to search for boundaries between symplectic rigidity and flexibility, and to test the power of symplectic invariants.

Problem. Let U and V be open subsets of $\mathbb{R}^{2n}$. Find the infimum of $\lambda > 0$ such that there exists a symplectic embedding $f : U \hookrightarrow \lambda V$.

Saying that f is symplectic means that $f^*\omega = \omega$, where $\omega = \sum dx_i \wedge dy_i$ is the standard symplectic form on $\mathbb{R}^{2n}$.

The only classical obstruction to symplectic embeddings is volume. Note that $\frac{1}{n!}\omega^n$ is the standard volume form and so symplectic embeddings are volume preserving. Hence if there exists a symplectic embedding $U \hookrightarrow \lambda V$ then necessarily $\mathrm{vol}(U) \leq \lambda^{2n}\mathrm{vol}(V)$.

We can say that an embedding problem is flexible if this estimate is sharp. We know of rather few nontrivial examples of flexible embedding problems. On the other hand rigidity for symplectic embeddings was discovered only in 1985 by Gromov in his seminal work on the subject.

Define a ball of capacity c by

$$B^{2n}(c) = \{\sum_{i=1}^{n} \pi(x_i^2 + y_i^2) < c\}$$

and a cylinder of capacity c by

$$Z^{2n}(c) = \{\pi(x_1^2 + y_1^2) < c\}.$$

*The author is partially supported by grant # 317510 from the Simons Foundation.

Nonsqueezing Theorem.(Gromov, [8]) *$B^{2n}(a) \hookrightarrow Z^{2n}(c)$ if and only if $a \le c$.*

There are now several proofs of the Nonsqueezing Theorem. Gromov's original proof applied his theory of pseudoholomorphic curves and the subsequent developments we describe here follow basically the same scheme.

It is well known in the field that pseudoholomorphic curves can be an especially useful tool when we work in dimension 4, due to positivity of intersection. This gives a topological criterion for the curves to be embedded. As a consequence, much more is known about symplectic embeddings in dimension 4. In particular Hutchings has developed a powerful set of embedding obstructions coming from his Embedded Contact Homology, ECH, see [15] for example.

In the current article we describe some first steps in extending 4-dimensional theorems to higher dimension. We will simply take a 4-dimensional problem and stabilize it by adding Euclidean factors to both the domain and range. In some cases the 4-dimensional rigidity generalizes directly to the stabilized case, but in others we will see that there is significant additional flexibility.

In section 2 we describe the situation for ellipsoid embeddings and in section 3 the polydisk situation. For general U and V however, the embedding problem remains broadly open, even in dimension 4.

2 Embedding ellipsoids

For given $a_i \in (0, \infty]$ we define a symplectic ellipsoid by

$$E(a_1, \ldots, a_n) = \{\sum_{i=1}^{n} \frac{\pi}{a_i}(x_i^2 + y_i^2) < 1\}.$$

Then we have $E(c, \ldots, c) = B^{2n}(c)$ and $E(c, \infty, \ldots, \infty) = Z^{2n}(c)$.

In this section we discuss the case when our domain U is an ellipsoid and the range V is either a ball or a stabilized ball. In dimension 4 there are also solutions when the range is a cube (due to Frenkel–Müller, [7]) or certain polydisks (due to Cristofaro-Gardiner–Frenkel–Schlenk, [3]) but in these cases only conjectures for the corresponding stabilized problems, see [10], Conjecture 1.19, and [3], Conjecture 1.4.

The solution to the problem of 4-dimensional ellipsoid embeddings into a ball can be expressed in the function

$$e_2(x) = \inf\{c > 0 | E(1, x) \hookrightarrow B^4(c)\}.$$

By rescaling and reordering the factors we may assume $x \ge 1$. The description of the function e_2 is due to McDuff and Schlenk and reveals both the beauty and intricate nature of the symplectic embedding problem.

To give their solution we need to fix some notation. First define the sequence $\{g_k\}_{k=0}^{\infty}$ where $g_0 = 1$ and g_k for $k \ge 1$ is the kth odd index

Fibonacci number. Hence g_k is the sequence beginning $1, 1, 2, 5, 13, 34, \ldots$. Then we can define sequences $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ by $a_k = (\frac{g_{k+1}}{g_k})^2$ and $b_k = \frac{g_{k+2}}{g_k}$. These are increasing sequences with $a_0 < b_0 < a_1 < b_1 < \ldots$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \tau^4$, where τ is the golden ratio.

Theorem 2.1 (McDuff–Schlenk, [17], Theorem 1.1.2)**.** *For $1 \le x < \tau^4$ the function $e_2(x)$ is linear on the intervals $[a_k, b_k]$ and constant on intervals $[b_k, a_{k+1}]$ with $e_2(a_k) = \sqrt{a_k}$.*

If $x \ge 8\frac{1}{36}$ then $e_2(x) = \sqrt{x}$.

In other words, the first part of the graph is an infinite staircase with ever shorter steps converging to τ^4. Meanwhile if $x \ge 8\frac{1}{36}$ the embedding problem is flexible. For brevity we have not tried to describe the graph over the interval $(\tau^4, 8\frac{1}{36})$. Here there are eight 'exotic' additional steps. It turns out that the ECH capacities give a sharp obstructions in all cases, see [18].

In McDuff and Schlenk's proof, the existence of holomorphic curves is used both to obstruct embeddings and construct the optimal embeddings. Thus the embeddings themselves are completely non-explicit. Elementary embedding constructions were the subject of the earlier book [21] of Schlenk. In terms of concrete embeddings in dimension 4 the methods Schlenk describes have not generally been improved upon (although see [19] for the volume filling embeddings $E(1, k^2) \hookrightarrow B^4(k)$ when $k \in \mathbb{N}$). These 'folded' embeddings are sufficient to read off the asymptotic behavior $\lim_{x\to\infty} \frac{e_2(x)}{\sqrt{x}} = 1$, although for $x > 2$ (that is, when the inclusion map is not optimal) they never reproduce the embeddings established in Theorem 2.1.

Given an $n \ge 3$ the stabilized embedding function is given by

$$e_n(x) = \inf\{c > 0 | E(1, x) \times \mathbb{R}^{2(n-2)} \hookrightarrow B^4(c) \times \mathbb{R}^{2(n-2)}\}.$$

By taking product embeddings we see immediately that $e_n(x) \le e_2(x)$ for all x, n and it would be natural to guess that we always have equality. This notion was disproved in a remarkable paper of Guth [9] where an explicit construction demonstrated the extra flexibility present in higher dimension. This construction was improved by the author in [10] using folding methods as in [21] to obtain the following.

Theorem 2.2 (Hind, [10], Pelayo–Ngọc, [20])**.** *$E(1, x) \times \mathbb{R}^{2(n-2)} \hookrightarrow B^4(c) \times \mathbb{R}^{2(n-2)}$ whenever $c > \frac{3x}{x+1}$.*

The paper [10] dealt only with compact subsets. To obtain embeddings of all of $E(1, x) \times \mathbb{R}^{2(n-2)}$ we use the technique from Theorem 4.3 in [20].

The graph of $\frac{3x}{x+1}$ intersects $\sqrt{x}$ precisely at $x = \tau^4$, where it also coincides with $e_x(x)$. It follows that $e_n(x) < e_2(x)$ for all $x > \tau^4$ and $n \ge 3$.

As a possible first step in extending the ECH capacities to higher dimension, Cristofaro-Gardiner established the following in collaboration with the author, showing that when $x \le \tau^4$ symplectic rigidity persists in the stabilized case.

Theorem 2.3 (Cristofaro-Gardiner–Hind, [4]). *If $x \leq \tau^4$ then $e_n(x) = e_2(x)$.*

While the holomorphic curves giving the 4-dimensional ECH obstructions remain useful in higher dimensions when $x < \tau^4$, the graph of $e_n(x)$ when $x > \tau^4$ remains mysterious. Kerman in collaboration with the author has shown that the folding construction is sharp at least asymptotically.

Theorem 2.4 (Hind–Kerman, [11], [12]). *For all $n \geq 3$, $\lim_{x\to\infty} e_n(x) = 3$.*

At the other end of the scale there is a sequence of points converging to τ^4 from above (called 'ghost stairs') at which the folded embedding from Theorem 2.2 is again sharp. To describe these points let $\{h_k\}_{k=1}^{\infty}$ be the even index Fibonacci numbers, that is, the sequence beginning $1, 3, 8, 21 \ldots$. Then let $x_k = \frac{h_{2k+3}}{h_{2k+1}}$ for $k \geq 0$.

Theorem 2.5 (Cristofaro-Gardiner–Hind–McDuff, [5]). *$e_n(x_k) = \frac{3x_k}{x_k+1}$ for all $k \geq 0$, $n \geq 3$.*

These are labelled ghost stairs because they give a staircase of obstructions which originally appeared in the paper [17] of McDuff and Schlenk, but in dimension 4 the obstructions are not sharp and so do not appear in the graph of e_2. In higher dimension we do not know if the x_k are the tips of a staircase in the graph of e_n, or alternatively if $e_n(x) = \frac{3x}{x+1}$ for all $n \geq 3$ and $x \geq \tau^4$.

3 Embedding polydisks

For given $a_i \in (0, \infty]$ we define a symplectic polydisk by

$$P(a_1, \ldots, a_n) = \{\pi(x_i^2 + y_i^2) < a_i \text{ for all } i\}.$$

In dimension 4 a solution to the embedding problem for polydisks into a ball amounts to describing the function

$$p_2(x) = \inf\{c > 0 | P(1, x) \hookrightarrow B^4(c)\}$$

for $x \geq 1$. The techniques from [17] do not apply to this case, and indeed the only embedding constructions available come from Schlenk's book [21]. The known theorems show that at least for small x folding cannot be improved.

Theorem 3.1 (Hind–Lisi, [13], Hutchings, [15], Christianson–Nelson, [2]).
If $1 \leq x \leq 2$ then $p_2(x) = 1 + x$.
If $2 \leq x \leq \frac{\sqrt{7}-1}{\sqrt{7}-2}$ then $p_2(x) = 2 + \frac{x}{2}$.

To be precise, Hind–Lisi established the case $x = 2$, Hutchings dealt with all $x \leq 12/5$ and Christianson–Nelson completed the theorem as stated.

Although the 4-dimensional case remains incomplete, we can nevertheless write down the stabilized function

$$p_n(x) = \inf\{c > 0 | P(1, x) \times \mathbb{R}^{2(n-2)} \hookrightarrow B^4(c) \times \mathbb{R}^{2(n-2)}\}.$$

Surprisingly, we can say a lot about $p_3(x)$.

Theorem 3.2. *For $x \geq 2$ we have $p_3(x) = 3$.*

We conclude with an outline of a proof of this assuming some familiarity with pseudoholomorphic curves, and in particular finite energy curves, see [6] and [1]. Detailed results about Lagrangian submanifolds will appear in a joint paper with Opshtein, [14].

First note that it suffices to prove that $p_3(2) = 3$. Indeed, Theorem 2.4 implies that $\lim_{x\to\infty} p_n(x) = 3$ and the p_n are clearly nondecreasing.

Arguing by contradiction, suppose that there exists a symplectic embedding $P(1, 2, S) \hookrightarrow B^4(c) \times \mathbb{R}^2$ for an S extremely large and $c < 3$. We will identify $P(1, 2, S)$ with its image under this embedding.

Now fix a large d, such that $3d-1 > dc$ but still with $d << S$ and an $\epsilon > 0$ such that $d\epsilon$ is small. Then we define an open subset $U = U(1, 2, S, \epsilon) \subset P(1, 2, S)$ by

$$\{1-\epsilon < \pi(x_1^2+y_1^2) < 1, 2-(3d-1)\epsilon < \pi(x_2^2+y_2^2) < 2, \frac{S}{2} < \pi(x_3^2+y_3^2) < S\}.$$

We can think of U as a tubular neighborhood of a Lagrangian torus, say

$$L = \{\pi(x_1^2 + y_1^2) = 1 - \frac{\epsilon}{2}, \pi(x_2^2 + y_2^2) = 2 - \frac{3d-1}{2}\epsilon, \pi(x_3^2 + y_3^2) = \frac{3S}{4}\}.$$

In fact U admits a symplectic embedding into T^*T^3 taking L to the zero-section.

For $\delta_2, \delta_3 < \delta_1 << \epsilon$ very small, there is another symplectic embedding $E = E(\epsilon - \delta_1, (3d-1)(\epsilon - \delta_2), (3d-1)(\epsilon - \delta_3)) \hookrightarrow U$ which extends to the closure of the ellipsoid. Composing the two we get

$$E \hookrightarrow U \hookrightarrow B^4(c) \times \mathbb{R}^2 \subset \mathbb{C}P^2(c) \times \mathbb{R}^2$$

where the last inclusion is a standard compactification of the ball factor, so we are adding an $L_\infty = l_\infty \times \mathbb{R}^2$ with l_∞ the line at infinity in $\mathbb{C}P^2$.

To study holomorphic curves we must fix an almost-complex structure on $(\mathbb{C}P^2(c)\times\mathbb{R}^2)\setminus E$ with a cylindrical end on ∂E as in [1], section 3 (identifying E with its image as usual). To control the projection of holomorphic curves to the $\mathbb{R}^2$ factor we also assume that our almost-complex structures are equal to a standard product structure on a fixed region $\{x_3^2 + y_3^2 > R\}$.

We can then study finite energy curves asymptotic to Reeb orbits on ∂E. For definitions and basic properties see [1], section 6. The Reeb orbits here are closed loops in ∂E tangent to $\ker \omega|_{\partial E}$, and for suitable δ_i these will be exactly covers of the $\gamma_i = \partial E \cap \{x_j = y_j = 0 \,\text{for}\, j \neq i\}$. Denote by γ_i^r the r-fold cover.

The analysis in [11] and [4] implies that for a generic almost-complex structure and sequence of $d \to \infty$ there exist finite energy planes asymptotic to γ_1^{3d-1} and intersecting L_∞ exactly d times, counting with multiplicity. We may assume our d lies in this sequence, so the planes exist and have area $dc - (3d-1)(\epsilon - \delta_1)$. (For the area formula note that if we compactify our planes by adding a $(3d-1)$ times cover of the (x_1, y_1) plane inside E, then the curves will project to spheres of degree d in $\mathbb{C}P^2(c)$.)

Our goal is to take a limit of such finite energy planes as we perform a neck stretching (as in [1] section 3.4) along a smoothing Σ of ∂U. Reeb orbits on Σ appear in 2-dimensional families indexed by $(k, l, m) \in \mathbb{Z}^3 \setminus \{0\}$ describing the homology class when we project the orbit in T^*T^3 to the zero-section. That is, k gives the winding about $\{0\}$ in the (x_1, y_1)-plane and so on.

The compactness theorem in [1] describes the limit as a holomorphic building, that is, a collection of finite energy holomorphic curves in completions of $(\mathbb{C}P^2(c) \times \mathbb{R}^2) \setminus U$ and $U \setminus E$ with matching asymptotic limits along Σ. (We should really also include curves mapping to the symplectization $\mathbb{R} \times \Sigma$ in our discussion, however these do not affect the argument.) An analysis as in [14] implies that the limit consists of finite energy planes in $(\mathbb{C}P^2(c) \times \mathbb{R}^2) \setminus U$ with negative ends asymptotic to Reeb orbits on Σ and a single finite energy curve in $U \setminus E$ with a number of positive ends on Σ but a single negative end asymptotic to γ_1^{3d-1} on ∂E. A potential limiting building with degree $d = 2$ is illustrated in Figure 1.

Identifying U with a subset of T^*T^3 we see that the symplectic form is exact and moreover has a primitive whose integral over a Reeb orbit of Σ in the class (k, l, m) is given (in an arbitrarily large range, and up to a small correction due to the smoothing) by

$$\mathcal{A}(k, l, m) = \frac{\epsilon}{2}|k| + \frac{(3d-1)\epsilon}{2}|l| + \frac{S}{4}|m|.$$

Now as $d << S$ and our curves have area of order dc, we see that the limiting component in U cannot have any positive ends asymptotic to Reeb orbits with $m \neq 0$.

Given this we investigate the planes in $(\mathbb{C}P^2(c) \times \mathbb{R}^2) \setminus U$. If the plane has intersection number g with L_∞ and is asymptotic to a Reeb orbit in the class $(k, l, 0)$ then the deformation index given by

$$\text{index} = 6g - 2(k + l).$$

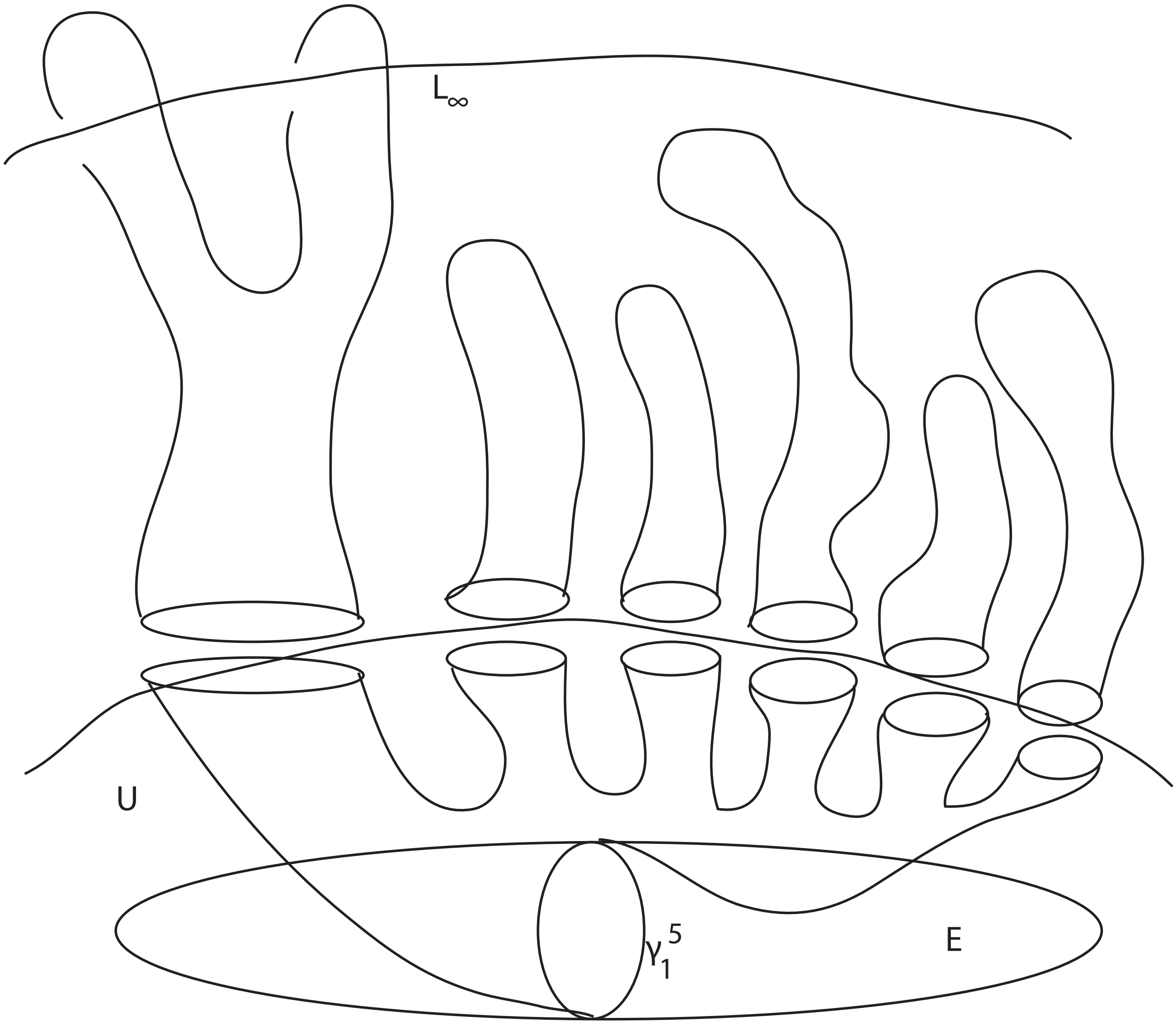

Figure 1: A limiting building with $d = 2$.

Again following [14] we can show that for a generic almost-complex structure all limiting planes have index either 0 or 2 (the index is necessarily even). The idea behind this is that genericity of our almost-complex structure can be used to exclude curves of negative index. Thus the planes have nonnegative index, but if their index exceeds the dimension of the asymptotic family of Reeb orbits then other curves in the building will be forced to have negative index. Index 0 planes are rigid but for those of index 2 the asymptotic limit will vary in the moduli space.

We also observe here the consequence that if our plane is asymptotic to an orbit of class $(k, 0, 0)$ with $k < 0$ then in fact we must have $k = -1$, the index must be 2, and the intersection number $g = 0$.

The symplectic area of such a plane is given by the area of a disk with boundary on L up to a correction of order ϵ,

$$\text{area} = cg - (k + 2l) + O(\epsilon) = (c - 3)g + (3g - k - l) - l + O(\epsilon).$$

Increasing c if necessary the area of our planes can be bounded above 0 by a constant independent of ϵ. On the other hand the first term in this formula is nonpositive, and by our index calculation the second term is either 0 or 1. Therefore we must have $l \leq 0$. But the asymptotic limits of our finite energy planes bound a cycle in U, and hence the sum of their homology classes is 0. Hence, all planes are asymptotic to orbits in classes $(k, 0, 0)$, and by the matching conditions for curves in a holomorphic building all positive asymptotic limits of our curve in U are also of this type.

Stokes' theorem now gives the area of our curve in U as

$$\frac{\epsilon}{2} \sum |k_i| - (3d - 1)(\epsilon - \delta_1)$$

where the sum is over the covering degrees of the positive limits. Again since these limits bound a cycle we have $\sum_{k_i>0} |k_i| = \sum_{k_i<0} |k_i|$ and so

$$\sum_{k_i<0} |k_i| \geq (3d - 1)(1 - \frac{\delta_1}{\epsilon}).$$

We can take δ_1 arbitrarily small, so $\sum_{k_i<0} |k_i| \geq 3d - 1$ and by the observation above our limiting building must contain $3d - 1$ planes asymptotic to orbits of class $(-1, 0, 0)$ and each having area $1 + O(\epsilon)$. The total area of the limit is equal to the symplectic area of our initial planes. Therefore we get

$$dc - (3d - 1)(\epsilon - \delta_1) \geq (3d - 1)(1 + O(\epsilon)).$$

As $dc < 3d - 1$, when ϵ is sufficiently small this gives a contradiction.

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