Embeddings of polydisks

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June 21, 2011

1 Introduction

Let $L \subset P(1,2) \subset \mathbb{C}P^2(R) \setminus \mathbb{C}P^1(\infty) \subset \mathbb{C}P^2(R)$. Here $\mathbb{C}P^2(R)$ is complex projective space equipped with the Fubini-Study symplectic form ω scaled such that the area of a line is R, and $\mathbb{C}P^1(\infty)$ is the line at infinity. The polydisk $P(1,2)$ is a symplectically embedded product of disks $D(1)\times D(2)$ of areas 1 and 2 respectively. Finally L is the Lagrangian torus $\partial D(1)\times\partial D(2) \subset \partial P(1,2)$. We will also consider the symplectic manifold $X = \mathbb{C}P^2(R)\sharp\overline{\mathbb{C}P^2(1)}$ where X is the result of blowing up the ball in $\mathbb{C}P^2(R)$ of capacity 1 centered at $(0,0) \in P(1,2)$ and $E = \overline{\mathbb{C}P^2(1)}$ is the exceptional divisor, a symplectic sphere of area 1.

To fix notation, let $(k,l) \in H_1(L)$ denote the homology class $k[\partial D(1)] + l[\partial D(2)]$.

In this note we establish the following.

Theorem 1.1. *$R \geq 3$.*

We will argue by contradiction and assume that such an embedding exists in a $\mathbb{C}P^2(R)$ with $R < 3$. Then we consider the limits of various lines as a stretching the neck operation is performed along the boundary of a tubular neighborhood of the Lagrangian torus. The various possible results are documented and all will eventually lead to a contradiction.

To perform a neck stretch we must choose a tubular neighborhood of L. By Weinstein's theorem a tubular neighborhood can be identified with a neighborhood of the zero-section in T^*L. We fix a flat metric on L and let our neighborhood be a unit cotangent disk bundle with boundary Σ. The Louville form on T^*L restricts to a contact form on Σ and the associated Reeb vector field generates the geodesic flow (identifying tangent and cotangent bundles using the metric). Then we choose a sequence of almost-complex structures J^N on $\mathbb{C}P^2(R)$. We may assume that these all coincide outside of a neighborhood of Σ but $(\mathbb{C}P^2, J^N)$ admits a biholomorphic embedding of $\Sigma \times (-N, N)$ equipped with a translation invariant complex structure mapping the Reeb vectors to the unit vectors to $(-N, N)$. A sequence of J^N holomorphic spheres of fixed degree has a limit in the sense of [2]. This is a holomorphic building whose components are finite energy curves mapping into three symplectic manifolds with cylindrical ends, diffeomorphic to $\mathbb{C}P^2 \setminus L$, T^*L and $\Sigma \times \mathbb{R}$.

2 Fredholm theory

For generic almost-complex structures finite energy curves appear in moduli spaces of dimension determined by the asymptotic limits and the relative homology class. We give the virtual indices of these moduli spaces in each of our three manifolds with cylindrical ends.

Finite energy curves in $X \setminus L$ can be compactified to give maps from Riemann surfaces with boundary, where the boundary components map to closed geodesics on L (with respect to the flat metric). There is an S^1 family of closed geodesics in each homology class (k, l) for $k, l \in \mathbb{Z}$.

Suppose that such a curve C intersects $\mathbb{C}P^1(\infty)$ in d points, counting with multiplicity (in this case we will say that it has degree d), and has s negative ends asymptotic to geodesics in the classes (k_i, l_i) respectively for $1 \leq i \leq s$.

Proposition 2.1. *The deformation index of C modulo reparameterizations is given by*

$$\text{index}(C) = s + 6d - 2 - 2(C \bullet E) + 2\sum_{i=1}^{s}(k_i + l_i).$$

Remark 2.2. For example, if $s = 0$ and $d = C \bullet E = 1$ then $\text{index}(C) = 2$. As $s = 0$ this moduli space makes sense for the almost-complex structures J^N. In this case the curves form the fibers of a nontrivial bundle $X \to \mathbb{C}P^1$.

If $s = 0$ and $C \bullet E = d - 1$ then $\text{index}(C) = 2d$. Such curves are sections of the above bundle with d 'zeros' (intersections with $\mathbb{C}P^1(\infty)$) and $d - 1$ 'poles' (intersections with E). A unique curve in the class can be specified by fixing $2d$ constraint points, for example the d zeros, $d-1$ poles and an additional point to fix the phase.

For curves in T^*L an index computation gives the following.

Proposition 2.3. *The virtual index of a finite energy curve of genus* 0 *in* T^*T^2 *is given by*

$$\mathrm{index}(C) = 2s - 2$$

where s *is the number of (positive) ends.*

Finally we have the following.

Proposition 2.4. *The virtual index of a finite energy curve of genus* 0 *in* $\Sigma \times \mathbb{R}$ *is given by*

$$\mathrm{index}(C) = 2s^+ + s^- - 2$$

where s^+ *is the number of positive ends and* s^- *the number of negative ends.*

3 Curves in T^*L and $\Sigma \times \mathbb{R}$

We can choose the almost-complex structures on T^*L and $\Sigma \times \mathbb{R}$ independently of any particular embedding and there are general results about the holomorphic curves which exist for certain choices. The following proposition follows from work of C. Wendl, [18], and observations in section 10 of [15].

Note that the action of T^2 on L by translation lifts to a T^2 action on T^*L which induces one on $\Sigma \times \mathbb{R}$. By the $\mathbb{R}$ action on $\Sigma \times \mathbb{R}$ we mean translation in the second factor.

Proposition 3.1. *There exist compatible almost-complex structures on* $\Sigma \times \mathbb{R}$ *which are invariant under the* $T^2 \times \mathbb{R}$ *action and compatible almost-complex structures on* T^*L *which are invariant under the* T^2 *action. The almost-complex structures have the property that holomorphic cylinders always have both asymptotic limits on the same geodesic, if both ends are positive then with opposite orientation. Immersed finite energy curves are automatically regular.*

Remark 3.2. All holomorphic curves considered in this paper will be either embedded or will multiply cover embedded curves.

4 A finite energy foliation

As mentioned in Remark 2.2, for each $N < \infty$ there exists a foliation of X by J_N holomorphic spheres in the class of a fiber $F = [\mathbb{C}P^1(\infty)] - [E]$ of the bundle $X \to E$. Letting $N \to \infty$ we find finite energy curves in $X \setminus L$. Let us fix a countable dense set of points $\{p_i\} \in X \setminus L$. For each N there exists a unique curve C_i^N in the fiber class intersecting p_i. Taking a diagonal subsequence of $N \to \infty$ we may assume that all C_i^N converge as $N \to \infty$ to holomorphic buildings having a component, denoted by C_i, passing through p_i. Positivity of intersection implies that these curves are either disjoint or have identical image. Taking further limits of the C_i we obtain a finite energy foliation $\mathcal{F}$ of $X \setminus L$, see [11].

Remark 4.1. Once we fix the subsequence $N \to \infty$ such that all C_i^N converge, it is necessarily the case that the limiting components in $X \setminus L$ of any convergent sequence of J^N-holomorphic spheres form leaves of $\mathcal{F}$. This also follows immediately from positivity of intersection.

Proposition 4.2. *Leaves of $\mathcal{F}$ consist of three kinds of curves.*

1. *Closed curves in the fiber class.*
2. *Planes of degree 0 asymptotic to $(1,0)$ geodesics.*
3. *Planes C of degree 1 with $C \bullet E = 1$ and asymptotic to $(-1,0)$ geodesics.*

Proof. We pick a point $p \in X \setminus L$ and look at a limit of J^N-holomorphic spheres C^N through p. The component of the limit through p is the leaf of $\mathcal{F}$ through p, see Remark 4.1. If C^N converges to a closed curve then it is of type 1. Suppose then that the limit is a holomorphic building. The sum of the areas of the components in $X \setminus L$ is $R - 1 < 2$. Curves of degree 0 have positive integral area and at most one component has degree 1. Therefore there must be exactly two components in $X \setminus L$. As there are no finite energy planes in T^*L (as there are no contractible geodesics) these components are both planes, say C_0 and C_1, where C_0 is of degree 0 and asymptotic to a (k,l) geodesic and C_1 is of degree 1 and asymptotic to a $(-k,-l)$ geodesic. We consider two cases.

Case 1**:** $C_0 \bullet E = 0$**;** $C_1 \bullet E = 1$**.**

Note that the intersection and area equalities imply that both of our curves must be simple. Therefore we can compute

$$\text{index}(C_0) = -1 + 2(k + l) \geq 0$$

$$\text{area}(C_0) = k + 2l = 1.$$

and

$$\text{index}(C_1) = 3 - 2(k + l) \geq 0$$
$$\text{area}(C_1) = R - 1 - (k + 2l) = R - 2.$$

Solving these equations gives $k = 1$, $l = 0$ and we get curves of types 2 and 3.

Case 2: $C_0 \bullet E = 1$; $C_1 \bullet E = 0$.

Now we compute

$$\text{index}(C_0) = -3 + 2(k + l) \geq 0$$

$$\text{area}(C_0) = k + 2l - 1 = 1.$$

and

$$\text{index}(C_1) = 5 - 2(k + l) \geq 0$$
$$\text{area}(C_1) = R - (k + 2l) = R - 2.$$

Solving these we get $k = 2$, $l = 0$. In particular C_0 is a, necessarily embedded, plane asymptotic to a $(2, 0)$ geodesic. However, no such planes exist. Indeed, by automatic regularity, [18], any such plane appears in a 1-parameter family asymptotic to a 1-parameter family of geodesics. But blowing down this would give us a 1-parameter family of planes in $\mathbb{C}^2 \setminus L$ asymptotic to $(2, 0)$ geodesics and all intersecting at a point. This is a contradiction as such planes have intersection number 0.

Excluding Case 2 we have established the Proposition. □

Proposition 3.1 gives the following clarification.

Corollary 4.3. *If a sequence of J^N holomorphic spheres converges to a non-trivial holomorphic building then the components in $X \setminus L$ consist of two planes asymptotic to the same geodesic (with opposite orientation).*

Lemma 4.4. *$\mathcal{F}$ contains a unique leaf of degree 1 asymptotic to each $(-1, 0)$ geodesic.*

Proof. Suppose not, then there are at least two such leaves P_1 and P_2 of degree 1 and asymptotic to the same geodesic γ. The leaves each intersect E transversally in points p_1 and p_2 respectively. Choose a circle $\sigma \subset E$ separating p_1 from p_2. Let Σ be the leaves of $\mathcal{F}$ which are components of limiting holomorphic buildings which intersect σ. We expect all but finitely many leaves of Σ to be holomorphic spheres, but by Corollary 4.3 the planes occur in pairs asymptotic to the same geodesic and so Σ can be compactified to give a hypersurface in X separating p_1 from p_2. It follows that if P_1 and P_2 can be compactified to give disks in X with the same boundary then one of them must intersect Σ, but this is impossible as Σ consists of leaves in the same foliation. □

Corollary 4.5. *There is a projection $\pi : X \to E$ whose fibers are (compactified) leaves of $\mathcal{F}$. The image of L is a circle in E and $(1,0)$ geodesics are mapped to points.*

Proof. On degree 1 leaves of $\mathcal{F}$ we define π simply to be the intersection with E. The map π extends continuously to X be requiring it to be constant on all leaves with the same asymptotic limit (with either orientation) and on the geodesic asymptotic limit itself. □

5 Degree d curves with point constraints

At least if we allow cusp-curves, then to any set of $2d$ points $\{q_i\} \in X$ there exists a J^N-holomorphic curve C^N of degree d and with $C \bullet E = d-1$ intersecting each of the points. If the points are in sufficiently general position then C^N will be embedded and unique. We note that $C^N \bullet F = 1$, see Remark 2.2. Let us fix the $2d$ points on L and take a limit of the C^N as $N \to \infty$.

The limiting holomorphic building will have multiple components in $X \setminus L$ and T^*L, we now document three lemmas describing their indices and asymptotic behaviour. The first is special to our situation, the other two are fairly general considerations given the first. By the index of a component we will mean its constrained deformation index, the virtual dimension of the moduli space of curves passing through the same point constraints.

Lemma 5.1. *Let $\tilde{C}$ be a component of our limit which multiply covers a simple finite energy curve C. Then* $\mathrm{index}(\tilde{C}) \geq \mathrm{index}(C) \geq 0$.

Proof. The fact that $\mathrm{index}(C) \geq 0$ follows by assuming that our almost-complex structures are regular. We first suppose that $\tilde{C}$ is a component in $X \setminus L$, which is a q-fold cover of C. Let $\tilde{s}$ and s be the numbers of ends of $\tilde{C}$ and C respectively, and let $k = \mathrm{index}(C)$. Then by Proposition 2.1, we see that

$$\mathrm{index}(\tilde{C}) = \tilde{s} - 2 + q(k - s + 2)$$

as the degree, intersection number with E and total multiplicity of the ends multiply by q under the cover.

Now, if $\tilde{C}$ and C have underlying domains $\tilde{\Sigma}$ and Σ respectively, where $\tilde{\Sigma}$ and Σ are punctured Riemann spheres, then by the covering map $\tilde{\Sigma} \to \Sigma$ extends to a degree q map of $\mathbb{C}P^1$. By the Riemann-Hurwitz formula this has $2(q-1)$ critical points, counted with multiplicity. Therefore $\tilde{s} \geq qs - 2(q-1)$ and so $\mathrm{index}(\tilde{C}) \geq qk$ as required.

In the case when $\tilde{C}$ maps to T^*L or $\Sigma \times \mathbb{R}$ the proof is easier since the number of point constraints stays the same under covers but the number of ends can only increase. □

Let the constrained indices of our limiting components be given by $I_1, \dots, I_n$ and the numbers of ends of the corresponding components by $s_1, \dots, s_n$. This means that in the limiting holomorphic building there are $\frac{1}{2}\sum s_i$ limiting asymptotic geodesics, that is, each is a limit for two ends from different components. Now, the virtual deformation index of the limiting holomorphic building is the same as the constrained deformation index of our closed holomorphic curves, namely 0. This gives the following.

Lemma 5.2.

$$\sum_i I_i - \frac{1}{2}\sum_i s_i = 0.$$

Finally we have the following.

Lemma 5.3. *$s_i \geq I_i$ for each i.*

Proof. Suppose that $s_k < I_k$ for some k. For each remaining component let s_i' be the number of ends which are not matched in the limit with the kth component. Then $S = \sum_{i \neq k} s_i' = \sum_i s_i - 2s_k$ and $\sum_{i \neq k} I_i = \sum_i I_i - I_k$. Therefore

$$\sum_{i \neq k} I_i - \frac{1}{2}\sum_{i \neq k} s_i' = s_k - I_k < 0.$$

We claim that such an arrangement of components (that is, all except the kth) give a holomorphic building G which generically should not exist.

We look at the moduli space of all $n-1$ tuples of finite energy curves having the same degrees and asymptotic behaviour as the components of G. As all geodesics appear in S^1 families there is a map from this moduli space to $(S^1)^S$ and the dimension of the image is bounded above by $\sum_{i \neq k} I_i$. Note that if a component happens to be a multiple cover then the ends move in the same family as the ends of the covered curve, but by Lemma 5.1 the index can only increase.

On the other hand, an element in this moduli space can fit together to give a holomorphic building if and only if all corresponding ends match. This means that the image of the above map should intersect a subset of codimension $\frac{1}{2}\sum_{i\neq k} s'_i$. Given the inequality above, by Sard's theorem this should not occur. $\square$

Let f_i be a component of our limit with image in $X \setminus L$. As above we denote its index and number of ends by I_i and s_i respectively, denote its degree and intersection with E by d_i and e_i, and the class of the asymptotic geodesics by (k^i_j, l^i_j).

Lemma 5.4. $\sum_j l^i_j = 0$.

Proof. Proposition 2.1 gives

$$I_i = 6d_i - 2 - 2e_i + 2\sum_j (k^i_j + l^i_j)$$

and the area formula is

$$\text{area}(f_i) = Rd_i - e_i + \sum_j (k^i_j + 2l^i_j).$$

As the area must be positive we can substitute for $\sum(k^i_j + l^i_j)$ to get

$$0 < Rd_i - e_i + \frac{1}{2}(I_i - s_i - 6d_i + 1 + 2e_i) + \sum_j l^i_j$$

$$= (R-3)d_i + \frac{1}{2}(I_i - s_i) + 1 + \sum_j l^i_j.$$

Now, the first term here is strictly negative (assuming $R < 3$) and by Lemma 5.3 the second term is nonpositive. Therefore, since it is integral, $\sum_j l^i_j \geq 0$. Summing over all components in $X \setminus L$, the asymptotic geodesics must represent the zero homology class in L, indeed, they bound the projection to L of components in T^*L. Hence $\sum_i \sum_j l^i_j = 0$ and it follows that for each i we have $\sum_j l^i_j = 0$. $\square$

We recall that for all N the intersection $C^N \bullet F = 1$. Therefore, by positivity of intersection, components in $X \setminus L$ either have image equal to a leaf of $\mathcal{F}$ or intersect the leaves transversally, the sum of the transversal intersections of the limiting components with any given leaf is at most a single point (it is possible that components in $X \setminus L$ could avoid isolated leaves if the intersections converge to L as $N \to \infty$). Not all limiting components are leaves of $\mathcal{F}$ as such leaves have degree equal to their intersection with E, for curves in our class these numbers are different. Therefore we can find a limiting component whose image does not lie in a single leaf. We write this as a map $f : \Sigma \to X \setminus L$, where Σ is again a punctured Riemann sphere. As the intersections with the leaves are transversal in single points, composing with π gives an embedding $u = \pi \circ f : \Sigma \to E$. Suppose that f is asymptotic to geodesics γ_i at its punctures. If the geodesic is in a class (k, l) with $l = 0$ then u approaches a point near the puncture and in fact can be extended continuously across the puncture. On the other hand if $l \neq 0$ then the image of u approaches $\pi(L)$ near the puncture. As u is an embedding it follows that there can be at most one such puncture, but in that case Lemma 5.4 gives $l = 0$. In conclusion all ends are asymptotic to geodesics in classes $(k, 0)$, the embedding $u : \Sigma \to E$ extends to a degree 1 map from $\mathbb{C}P^1$, and f is the unique component which does not lie in a leaf of $\mathcal{F}$.

The remaining components of the limiting holomorphic building fit together along their boundaries to form disks asymptotic to the γ_i. Positivity of intersection again implies that all components of the disk asymptotic to γ_i actually cover planes asymptotic to γ_i, or if they map to T^*L then cover the cylinder over γ_i. We conclude that f has at least $2d$ ends asymptotic to geodesics γ_i passing through the points q_i.

Now we study the bundle $V = \pi^{-1}(\pi(L)) \to \pi(L)$. The fibers of V consist of degree 0 and degree 1 planes meeting along the same geodesic, that is $V = V_0 \cup V_1 \cup L$ where V_0 is the union of degree 0 planes in $\mathcal{F}$ and V_1 the union of the degree 1 planes. The image of f gives a section of V away from $\Gamma = \bigcup \pi(\gamma_i)$. The degree of the plane thus gives a locally constant map $\Gamma \to \{0, 1\}$ and we claim that this map is in fact constant.

Indeed, let P_0 and P_1 be the degree 0 and 1 planes of $\mathcal{F}$ asymptotic to the geodesic covered by a γ_i. Then near the puncture the finite energy curve f is asymptotic to a cover of either P_0 (if $k_i > 0$) or P_1 (if $k_i < 0$), see [8]. Therefore close to the puncture the image of f will intersect only either degree 0 or only degree 1 planes, and so the section remains in V_0 or V_1 as we pass a point of Γ. This justifies our claim.

Now, if f if asymptotic to a geodesic in the class $(k, 0)$ then the limiting building must also contain a plane asymptotic to a cover of the plane in $\mathcal{F}$ asymptotic to the corresponding geodesic in the class $(-\mathrm{sign}(k), 0)$. As f has degree d it can therefore have at most d ends asymptotic to geodesics in the class $(k, 0)$ with $k > 0$. Therefore by the above claim and since f has at least $2d$ ends we see that all ends are asymptotic to geodesics in a class $(k, 0)$ with $k < 0$. In summary, corresponding to each puncture of f are various components of degree 0, each of which has area at least 1, and perhaps some covers of degree 1 planes. Let $L \leq d$ be the sum of the degrees of these components, then L is also the sum of the intersection numbers of these components with E. Thus f has degree $d-L$ and $f(\Sigma) \bullet E = d-1-L$.

We can now compute

$$\mathrm{index}(f) = s + 6(d-L) - 2 - 2(d-1-L) + 2\sum_{i=1}^{s} k_i = s + 4(d-L) + 2\sum_{i=1}^{s} k_i$$

and

$$\mathrm{area}(f) = (d-L)R - (d-1-L) + \sum_{i=1}^{s} k_i$$

where f has $s \geq 2d$ ends, each covering a $(1, 0)$ geodesic $k_i < 0$ times. In particular $\sum_{i=1}^{s} k_i \leq -2d$.

Solving for $\sum_{i=1}^{s} k_i$ in the first equation and substituting in the second we get

$$\begin{aligned}\mathrm{area}(f) &= (d-L)R - (d-1-L) + \frac{1}{2}(\mathrm{index}(f) - s - 4(d-L)) \\ &= (d-L)(R-3) + 1 + \frac{1}{2}(\mathrm{index}(f) - s).\end{aligned}$$

But as $\sum_{i=1}^{s} k_i \leq -2d$,

$$\frac{1}{2}(\mathrm{index}(f) - s) = 2(d-L) + \sum_{i=1}^{s} k_i \leq -2L$$

and $\mathrm{area}(f) \geq 0$, so

$$0 \leq (d-L)(R-3) + 1 + \frac{1}{2}(\mathrm{index}(f) - s) \leq (d-L)(R-3) + 1 - 2L.$$

Since $L \leq d$, if $R < 3$ this implies that we must have $L = 0$, but then $3 - R \leq \frac{1}{d}$ and taking d sufficiently large we have a contradiction.

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