

# Lagrangian unknottedness in Stein surfaces

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April 2, 2007

**Abstract** *We show that the space of Lagrangian spheres inside the cotangent bundle of the 2-sphere, with its canonical symplectic structure, is contractible. We then discuss the phenomenon of Lagrangian unknottedness in other Stein surfaces. There exist homotopic Lagrangian spheres which are not Hamiltonian isotopic, but we show that in a typical case all such spheres are still equivalent under a symplectomorphism.*

## 1 Introduction

Studying the space of Lagrangian submanifolds is a fundamental problem in symplectic topology. Lagrangian spheres appear naturally in the Leftschetz pencil picture of symplectic manifolds.

In this paper we demonstrate the uniqueness up to Hamiltonian isotopy of the Lagrangian spheres in some 4-dimensional Stein symplectic manifolds. The most important example is the cotangent bundle of the 2-sphere,  $T^*S^2$ , with its standard symplectic structure. In this case we will go on to study the space of all Lagrangian spheres in  $T^*S^2$ , showing that it is contractible.

Finally, we study an example of a Stein manifold in which a particular homotopy class (even isotopy class) contains Lagrangian spheres which are not Hamiltonian isotopic. We show that the spheres in this class are still unknotted

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\*Supported in part by NSF grant DMS-0204634.

in a weaker sense, namely they are all equivalent under a global (non Hamiltonian) symplectomorphism.

We recall that if a convex symplectic manifold has a boundary of contact-type, then we can perform surgery operations on the manifold by adding handles to the boundary. In the 4-dimensional case these handles can be of index 1 or 2. Our first examples are symplectic manifolds formed by adding 1-handles to a unit cotangent bundle  $T^1S^2$ . Questions regarding Lagrangian isotopy classes are independent of which metric we use to define a unit tangent bundle or of any choices involved in adding 1-handles.

**Theorem 1** *Let  $M$  be  $T^*S^2$  or the result of adding any number of 1-handles to  $T^1S^2$  and  $L \subset M$  be a Lagrangian sphere. Then there exists a Hamiltonian diffeomorphism of  $M$  mapping  $L$  onto the zero-section.*

We will establish this theorem by utilizing an existence result for almost-complex structures on  $S^2 \times S^2$  with convenient properties, taken from [16], and a fact about diffeomorphisms of the 2-sphere.

In fact more is true. We let  $\mathcal{L}$  denote the space of Lagrangian spheres in  $T^*S^2$  endowed with the topology of smooth convergence.

**Theorem 2** *The topological space  $\mathcal{L}$  is contractible.*

It is a consequence of a general theorem of J. Coffey [3], combined with the result of [16], that the space of parameterized Lagrangian spheres in  $S^2 \times S^2$  is homotopic to  $SO(3) \times SO(3)$ . A theorem of Y. Eliashberg and L. Polterovich, see [10], says that the space of Lagrangian planes in a standard  $\mathbb{R}^4$ , equal to a fixed plane outside of a compact set, is also contractible. The proof here involves parameterized versions of the arguments in Theorem 1. In both cases we need a result about diffeomorphisms of the 2-sphere.

**Theorem 3** *The subset of fixed-point free maps contained in the diffeomorphism group of  $S^2$  is contractible.*

In section 2 we prove our result on the diffeomorphisms of  $S^2$ . In section 3, by using the conclusions of [16], we reduce our theorem in the case of  $M = T^*S^2$  to the statements in section 2. In section 4 we will deal with the addition of handles. This involves slightly generalizing the results from [16] so we will review them again there.

We now consider the addition of 2-handles. Let  $W$  be the Stein manifold formed by adding to  $T^1S^2$  a single 2-handle along the Legendrian curve in a single fiber of the boundary. As a Stein manifold it carries a symplectic structure which has a conformally expanding vector field whose flow exists for all time. The symplectic structure is the Kähler form associated to a plurisubharmonic exhaustion function and all such forms are equivalent up to symplectomorphism (see [9]). Alternatively  $W$  can be realized as the plumbing of two copies of  $T^1S^2$ . The resulting symplectic manifold  $W$  has two Lagrangian spheres  $L_1$  and  $L_2$  coming from the zero-sections in the  $T^1S^2$  (or the original zero-section and the stable manifold of the index 2 critical point in the added handle). Again we will establish a uniqueness result for Lagrangian spheres in  $W$ .

**Theorem 4** *Let  $L$  be a Lagrangian sphere in  $W$ , the plumbing of two copies of  $T^*S^2$ , which is homotopic to one of the zero-sections  $L_1$ . Then there exists a symplectomorphism  $\phi$  of  $W$  such that  $\phi(L) = L_1$ .*

The proof combines Theorem 1 with some previous work of the author and is described in section 5.

Thus any Lagrangian spheres which are homotopic to  $L_1$  but are knotted in the Hamiltonian sense must arise from global symplectomorphisms applied to  $L_1$ . Such symplectomorphisms do indeed exist. Recall that associated to any Lagrangian sphere  $L$  is a compactly supported symplectomorphism  $\tau_L$  called a generalized Dehn Twist. It is well-defined up to Hamiltonian symplectomorphism. The square  $\tau_L^2$  is smoothly but not necessarily symplectically isotopic to the identity. Thus  $\tau_{L_2}^{2r}(L_1)$  is a Lagrangian sphere in  $W$  which is smoothly isotopic to  $L_1$  for any integer  $r$ . However, as demonstrated by P. Seidel in [25], a Floer homology computation shows that none of the  $\tau_{L_2}^{2r}(L_1)$  are Hamiltonian

isotopic.

In a forthcoming work we will address the question of whether Dehn twists give the only examples of this phenomenon, that is, whether all Lagrangian spheres homotopic to  $L_1$  are generated from  $L_1$  by iterates of Dehn twists up to Hamiltonian diffeomorphism. Actually, not too much is known about the components of the symplectomorphism groups of Stein manifolds. As far as the author is aware, the only known symplectomorphisms which are homotopically trivial but do not lie in the identity component are generated by Dehn twists.

Similar methods generalize the unknottedness result of Theorem 4 to a larger class of Stein manifolds, but it is unclear whether or not the result is true in general.

As yet we are unable to prove any similar results for Lagrangian surfaces of genus at least 2, but A. Ivrii has established some similar results for Lagrangian tori. Here we make some remarks about the case of  $\mathbb{R}P^2$ . The results of [16] show that any Lagrangian sphere  $L$  in  $S^2 \times S^2$  homotopic to the antidiagonal  $\bar{\Delta}$  is in fact Lagrangian isotopic to  $\bar{\Delta}$ . In this paper we will show that if  $L$  is disjoint from the diagonal  $\Delta$  then the Lagrangian isotopy can be chosen to lie in  $S^2 \times S^2 \setminus \Delta$ . Now, the involution  $\sigma$  of  $S^2 \times S^2$  interchanging the two factors has fixed-point set equal to  $\Delta$  and restricts to the antipodal map on  $\bar{\Delta}$ . If  $L$  is invariant under  $\sigma$  then the isotopy can also be chosen to be  $\sigma$ -equivariant. Now, quotienting out by  $\sigma$ , we observe that  $S^2 \times S^2 \setminus \Delta$  is a double-cover of a unit cotangent bundle of  $\mathbb{R}P^2$  and Lagrangian projective planes in  $T^*\mathbb{R}P^2$  homotopic to the zero-section therefore correspond to  $\sigma$ -invariant Lagrangian spheres in  $S^2 \times S^2 \setminus \Delta$  homotopic to  $\bar{\Delta}$ . Hence we have the following corollary.

**Corollary 5** *A Lagrangian  $\mathbb{R}P^2$  homotopic to the zero-section in  $T^*\mathbb{R}P^2$  must be Hamiltonian isotopic to the zero-section.*

A natural compactification of the (unit) cotangent bundle of  $\mathbb{R}P^2$  is  $\mathbb{C}P^2$ . Again the Lagrangian is unique.

**Theorem 6** *Let  $L$  be a Lagrangian  $\mathbb{R}P^2$  in  $\mathbb{C}P^2$ . Then there exists a Hamiltonian isotopy taking  $L$  onto the standard embedding.*

Perhaps our methods can be extended to cover this case, but the theorem can be established by other methods. For example, the surgery technique described by M. Symington in [27] replaces a Lagrangian  $\mathbb{R}P^2$  by a symplectic sphere, transforming  $\mathbb{C}P^2$  into an  $S^2 \times S^2$ . But the symplectic spheres in  $S^2 \times S^2$  have been classified up to Hamiltonian isotopy by B. Siebert and G. Tian in [26].

The author would like to thank Alex Ivrii for some helpful comments.

## 2 Diffeomorphisms of the two-sphere

In this section we let  $f$  denote a diffeomorphism of the 2-sphere  $S^2$  and for a point  $x \in S^2$  we denote its antipodal point by  $-x$ .

We say that a diffeomorphism  $f$  has the property  $(*)$  if  $f(x) \neq -x$  for all  $x \in S^2$ .

The aim of the section is to prove the following theorem.

**Theorem 7** *Suppose that a smooth family of diffeomorphisms  $f_p$  depending upon a parameter  $p \in S^k$ ,  $k \geq 0$ , have the property  $(*)$  and  $f_1 = \text{id}$  for a point  $1 \in S^k$ . Then there exists a family of isotopies  $f_{p,t}$ ,  $0 \leq t \leq 1$ , with  $f_{p,0} = f_p$  and  $f_{p,1} = \text{id}$  for all  $p$ ,  $f_{1,t} = \text{id}$  for all  $t$  and such that  $f_{p,t}$  has property  $(*)$  for all  $p, t$ .*

### Proof of theorem

Let  $E$  denote an equator on  $S^2$ . The complement of  $E$  consists of two open disks  $H_1$  and  $H_2$  with  $-H_1 = H_2$ .

We observe that any diffeomorphism  $g$  with property  $(*)$  and which preserves  $E$  is indeed isotopic to the identity through diffeomorphisms  $G_t$  also satisfying  $(*)$ . To construct such an isotopy, we first isotope  $g$  to the identity in a neighbourhood of  $E$ . Now the resulting map restricts to a compactly supported diffeomorphism of  $H_1$  and  $H_2$ . But compactly supported diffeomorphisms of the disk are isotopic to the identity (see for instance [28], page 205). Combining these isotopies we get the required isotopy of  $g$ . It satisfies  $(*)$  since  $-H_1 = H_2$ . This construction also applies in the case of parameterized maps  $f_p$ .

Hence it suffices to find a suitable family of isotopies from  $f_p$  to diffeomorphisms preserving an equator  $E$ .

We construct our isotopies by applying the following lemma.

**Lemma 8** *Let  $\Phi_p : [-1, 1] \times S^1 \rightarrow S^2$  be a family of smooth embeddings and  $L_{p,s} = \Phi_p(\frac{2}{\pi} \arctan(s) \times S^1)$ ,  $-\infty < s < \infty$  be a foliation of  $\Phi_p((-1, 1) \times S^1)$  by circles. For any  $N, K$  there exists a family of isotopies  $f_{p,t}$  satisfying (\*) such that  $f_{p,0} = f_p$  and  $f_{p,t}(z) \in f_p(L_{p,s+tN})$  for all  $z \in L_{p,s}$ ,  $0 \leq t \leq 1$ ,  $s > -K$ . Further  $f_{p,t}(z) = f_p(z)$  for all  $z$  outside of the image of  $\Phi_p$ .*

**Proof**

As the condition (\*) on our isotopy is an open one, we may assume any necessary genericity properties for the diffeomorphisms  $f_p$  with respect to the foliation  $L_{p,s}$ . Specifically for any  $p, s, r$  we will assume that  $f_p(L_{p,r}) \cap -L_{p,s}$  consists of an isolated set of points and any tangencies are of finite order. For economy of notation we will omit the subscript  $p$  from the maps and circles described below, but will be careful throughout to ensure that all constructions apply to the parameterized situation.

Suppose that  $N > 0$ . For  $r \in \mathbb{R}$ , let  $a_r$  be a diffeomorphism of  $S^2$  such that  $a_r(L_s) = L_{s+r}$  and  $a_r$  extends as the identity outside of the image of  $\Phi$ . Then we will define  $f_t$  on the image of  $\Phi$  by

$$f_t(z) = h_t f a_{Nt}(z)$$

where  $h_t$  is a diffeomorphism of the image of  $f\Phi$  which preserves the foliation  $\{f(L_s)\}$  and extends by the identity to a diffeomorphism of  $S^2$ . We set  $h_{t,s} = h_t|_{f(L_{s+Nt})}$ .

Then we need to find smoothly varying  $h_{t,s}$  such that  $h_{t,s}(f(a_{Nt}(z))) \neq -z$  for all  $z \in L_s$ ,  $s$  and  $0 \leq t \leq 1$ .

For  $s$  very large and  $z \in L_s$  we notice that  $a_{Nt}(z)$  is very close to  $z$  (and hence away from  $-z$ ) and so we may choose  $h_{t,s} = \text{id}$ . It is required to show that we can extend these diffeomorphisms for all parameters  $s$ .

Again since property (\*) is an open condition, we observe that once we have defined the  $h_{t,s_0}$  for some  $s_0$  we can smoothly extend the functions to define  $h_{t,s}$  for  $s$  slightly less than  $s_0$ .

The following lemma will be useful.

**Lemma 8.1** *Suppose that there exist  $h_{t,s}$  for all  $0 \leq t \leq 1$ , all  $s \geq s'$  and for all parameters  $p$  such that the corresponding maps  $f_t$  restricted to  $\bigcup_{s \geq s'} L_s$  have the property (\*). Then the isotopies  $f_t$  can be extended to isotopies of  $S^2$  still satisfying (\*) and mapping the circles  $\{L_s\}$  into the circles  $\{f(L_s)\}$ .*

We remark that the maps  $f_t$  are not required to map  $L_s$  into  $f(L_{s+tN})$ . One application of the lemma is that it allows us to conclude the proof of Lemma 8 once the  $h_{t,s}$  have been defined for  $s > -K$ .

**Proof of Lemma 8.1** For  $x \in L_{s'}$ , let  $X(f_t(x)) = \frac{d}{dt}(f_t(x))$  define a vector field on  $\bigcup_{s'+N \geq s \geq s'} f(L_s)$ . Then  $X$  can be extended to all of  $S^2$  by setting  $X = 0$  outside of  $\bigcup_{s \geq s'-\epsilon} f(L_s)$  and defining  $X$  over  $\bigcup_{s' \geq s \geq s'-\epsilon} f(L_s)$  such that the corresponding flow  $\phi_t$  takes the circles  $f(L_s)$  into other such circles. The lemma will be established by setting  $f_t = \phi_t \circ f$  once we check that such an isotopy satisfies the condition (\*).

Again let  $x \in L_{s'}$ . Then  $f_t(x) \neq -x$  for all  $0 \leq t \leq 1$  and so in fact there exists a  $\delta$  such that  $f_t(x) \notin B_\delta(-x)$ , a  $\delta$ -ball about  $-x$ , for all  $0 \leq t \leq 1$ . (By compactness, the same  $\delta$  can be chosen for all  $x \in L_{s'}$  and for all parameters  $p$ .) Now  $-B_\delta(-x)$  is a ball about  $x$  and we may assume that if  $\epsilon$  is chosen sufficiently small then for all  $t > 0$  we have  $f^{-1}\phi_t^{-1}(f(x)) = f_t^{-1}(f(x)) \in -B_\delta(-x)$ . Thus the points which flow through  $f(x)$  also avoid their antipodal points and the extension of  $f_t$  has property (\*) as required. This completes the proof of Lemma 8.1.

For any  $s$  (and parameter  $p$ ), as  $t$  increases from 0 to 1 there is a varying collection of points  $I_{t,s} = f(L_{s+tN}) \cap -L_s$ . The diffeomorphisms  $h_{t,s}$  can be extended arbitrarily once they define the inverse image of these intersections.

For a fixed value of  $s$  and parameter  $p$  the  $I_{t,s}$  will consist of a set of points varying with  $t$ . The points are distinct for distinct values of  $t$  and therefore as  $t$  varies the only qualitative changes in  $I_{t,s}$  are pairs of points appearing or

vanishing.

Now, fixing a typical  $s'$  and parameter  $p$  the families of points  $I_{t,s}$ ,  $0 \leq t \leq 1$ , will vary continuously with  $s$  for  $s$  close to  $s'$  in a manner which we will now make precise. (Since the condition  $(*)$  is an open one it suffices to work in the continuous category.) This can be guaranteed if for instance  $f(L_{s'+tN})$  intersects  $-L_{s'}$  with order at most 2 for all  $t$  and transversally if  $t = 0$  or  $t = 1$ . Continuous variation with  $s$  means that for  $s$  close to  $s'$  we have smooth families of diffeomorphisms

$$\begin{aligned}\phi_s &: [0, 1] \rightarrow [0, 1] \\ g_{t,s} &: f(L_{s+tN}) \rightarrow f(L_{s'+\phi_s(t)N}) \\ b_{t,s} &: L_s \rightarrow L_{s'}\end{aligned}$$

such that  $\phi_{s'} = \text{id}$ ,  $g_{t,s'} = \text{id}$ ,  $b_{t,s'} = \text{id}$  and  $g_{0,s}(f(z)) = f(b_{0,s}(z))$ . They can be chosen such that  $g_{t,s}(I_{t,s}) = I_{\phi_s(t),s'}$  and if  $-z \in I_{t,s}$ , then  $g_{t,s}(-z) = -b_{t,s}(z)$ . The existence of such diffeomorphisms implies that if we have defined suitable  $h_{t,s'}$  then for an  $s$  close to  $s'$  we may define the  $h_{t,s}$  by

$$h_{t,s}(f(a_{tN}(z))) = g_{t,s}^{-1} h_{\phi_s(t),s'}(f(a_{\phi_s(t)N} b_{t,s}(z))).$$

We observe that this extension is canonical modulo a collection of diffeomorphisms of intervals, thus it can be carried out continuously in families.

In summary, if the  $h_{t,s}$  are defined for all  $t$  and for all  $s \geq s'$  then they can also be defined for such  $s < s'$  and parameters  $p$  for which the  $I_{t,s}$  vary continuously between  $s$  and  $s'$ .

For each fixed parameter  $p$  there are a finite collection of parameters  $s_i$  at which the pattern of intersections  $I_{t,s_i}$  will change from nearby values. Assuming  $f$  to be generic, the change will occur (fixing the parameter  $p$  and the  $s_i$ ) only near a single point in  $S^2$  for a single  $t$  parameter. It remains to show that the  $h_{t,s}$  can still be defined near such critical parameters. We can then extend their definition to all  $s$ .

Suppose that  $s''$  is such a critical parameter. In the first case we consider the situation when  $f(L_{s''})$  and  $f(L_{s''+N})$  are transverse to  $-L_{s''}$ . Then since

$s''$  is critical there exists a  $\sigma$  with  $0 < \sigma < 1$  and  $f(L_{s''+\sigma N})$  tangent to high order with  $-L_{s''}$ , say at a point  $z$ . By this we mean that we can choose local coordinates  $(x, y)$  in  $S^2$  about  $z$  such that  $-L_{s''} = \{y = 0\}$  and  $f(L_{s''+\sigma N}) = \{y = x^n\}$  for some integer  $n > 1$ , the order of the tangency.

Nevertheless  $I_{\sigma, s''}$  still consists of an isolated set of points. We suppose that  $h_{t, s'}$  can be defined for some  $s' > s''$  and  $s''$  the the largest critical parameter less than  $s'$ .

If the tangency is of even order then we notice that in fact the  $I_{t, s}$  do in fact vary continuously with  $s$  for  $s$  close to  $s''$ . Therefore we will assume that the order is odd.

We choose a small neighborhood  $U$  of  $z$  such that  $L_s \cap -U$  consists of a small interval for  $s'' \leq s \leq s'$  and contains all of the points in  $I_{t, s}$  for  $t$  close to  $\sigma$  and  $s$  close to  $s''$  which converge to  $p$  as  $t \rightarrow \sigma$  and  $s \rightarrow s''$ . We note that for all such  $t$  the set  $U \cap I_{t, s}$  is nonempty by the assumption of odd order. We will arrange that  $f_t(L_{s'} \cap -U)$  is disjoint from  $U$  for all  $t$ . To do this, we can suppose that  $f(L_{s'} \cap -U)$  is disjoint from  $U \cap -L_{s'}$  and as  $t$  increases the intervals  $f_t(L_{s'} \cap -U)$  may intersect but can never cross  $U \cap -L_{s'}$ . Thus by deforming the  $h_{t, s'}$  we can arrange that  $f_t(L_{s'} \cap -U)$  stays disjoint from  $U \cap -L_{s'}$ . It is important that the deformations of  $h_{t, s'}$  can have compact support near  $f_t(L_{s'} \cap -U)$  and are canonical modulo compactly supported diffeomorphisms, in particular, if the intervals do intersect then there is only one direction in which they can be deformed apart. Thus this construction can be carried out continuously in the parameter  $p$ . After this a further small adjustment can keep  $f_t(L_{s'} \cap -U)$  away from  $U$ .

For an  $\epsilon > 0$  and  $s' - s''$  sufficiently small the diffeomorphisms  $\phi_s, g_{t, s}$  and  $b_{t, s}$  can be defined as before for  $s'' \leq s \leq s'$  and satisfy the required properties for  $|t - \sigma| > \epsilon$ . For all  $t$  we can choose the maps such that  $g_{t, s}(I_{t, s} \setminus U) = I_{\phi_s(t), s'} \setminus U$  while  $g_{t, s}$  preserves  $U$  and  $b_{t, s}$  preserves  $-U$ . Then the maps  $h_{t, s}$  can be defined as before.

Finally suppose that  $-L_{s''}$  is tangent to  $f(L_{s''})$  or  $f(L_{s''+N})$  (for a subset of parameters  $p$ ). We may now assume that the diffeomorphisms  $g_{t, s}$  and  $b_{t, s}$  will

exist as before for  $t$  away from 0 and 1. This is already enough in the case of a tangency with  $f(L_{s''})$  since we can set  $h_{t,s''} = \text{id}$  for  $t$  close to 0, so we suppose that  $-L_{s''}$  is tangent to  $f(L_{s''+N})$ . Using Lemma 8.1, for  $s' > s''$  we can extend the isotopy  $f_t$  on  $\bigcup_{s \geq s'} L_s$  to an  $\tilde{f}_t$  on  $S^2$  which preserves the circles  $\{f(L_s)\}$  and such that  $\tilde{f}_t = f$  on  $\bigcup_{s \leq s''} L_s$ . As  $s$  decreases from  $s'$  to  $s''$  the (foliating) family of circles  $\{f(L_s + N)\}$  is extended (with a different parameterization) by the family  $\{\tilde{f}_1(L_s)\}$ . Thus for some  $s > s''$  there will also be a tangency of  $\tilde{f}_1(L_s)$  and  $-L_s$  at which either a point in  $\tilde{f}_t(L_s) \cap -L_s$  appears or two points in  $\tilde{f}_t(L_s) \cap -L_s$  vanish as  $t$  converges to 1. But  $\tilde{f}_t$  satisfies the condition (\*). We can extend the maps  $f_t$  by modelling the situation with  $\tilde{f}_t$  as follows. As  $\tilde{f}_t$  can be defined continuously over parameters  $p$  so can our extensions of  $f_t$ . In the first case, where a new point appears, we define  $h_{t,s}$  such that the new point in  $I_{t,s''}$  lies on the same side of its antipodal point as in the case of  $\tilde{f}_t$ . In the second case, where two points in  $I_{t,s''}$  converge, we can extend  $f_t$  continuously over  $L_{s''}$  since if  $I_{t,s''}$  contains distinct points  $-x(t), -y(t)$  for  $t < 1$  then so does  $I_{t,s'}$  and  $\tilde{f}_t(L_{s'})$  for suitable  $s'$ . As  $t$  converges to 1 the points  $-x(t)$  and  $-y(t)$  converge and if the contracting interval  $[-x(t), -y(t)]$  in  $f(L_{s''+tN})$  were to contain the interval  $[f_t(x(t)), f_t(y(t))]$  (or vice versa) then an extension would be impossible. But such an arrangement of intervals would also exist in  $f(L_{s'+tN}) = \tilde{f}_t(L_{s'})$  (for the corresponding points) which gives a contradiction since the corresponding points also converge in the  $\tilde{f}_t(L_{s'})$  and  $\tilde{f}_t$  satisfies (\*). This completes the proof of Lemma 8.

We apply Lemma 8 in various situations to complete the proof of Theorem 7.

In the case when  $p \in S^0$ ,  $f_{-1}$  must have at least one fixed point and without loss of generality we assume that  $f_{-1}(N) = N$ , where  $N$  is the north pole.

When  $p \in S^k$ ,  $k \geq 1$ , we perform a family of isotopies to arrange that  $f_p(N) = N$  for all  $p$ . To do this, we fix a small circle  $C$  around  $N$ . Then  $f_p(C)$  is a small circle around  $f_p(N)$ . By an application of Lemma 8 to a cylinder between  $f_p(C)$  and a circle  $D$  around the south pole  $S$  enclosing both  $C$  and

$f_p(C)$  we can find an isotopy of  $f_p$  to a diffeomorphism satisfying  $f_p(C) = D$ . After a further application of the lemma to the cylinder between  $C$  and  $D$  we may assume that  $f_p(C) = C$ . Also, the interior of  $C$  must map to itself and from here, since  $C$  and  $-C$  are disjoint, it is easy to isotope the  $f_p$  to maps satisfying  $f_p(N) = N$ .

The same procedure can be used to find isotopies of  $f_p$  to maps satisfying  $f_p(S) = S$  for all  $p$ . This family of isotopies leaves  $f_p(N) = N$  fixed. Thus we may assume that  $f_p(N) = N$  and  $f_p(S) = S$  for all  $p$ , and in fact that  $f_p = \text{id}$  in small neighborhoods of  $N$  and  $S$ .

We can conclude as follows. Write the complement of smaller neighborhoods of  $N$  and  $S$  in  $S^2$  as  $\mathbb{R} \times S^1$  and let  $L_s = \{s\} \times S^1$ . The equator  $E = \{0\} \times S^1$ . First we apply the lemma to find isotopies to new diffeomorphisms, still denoted by  $f_p$ , such that  $f_p(z) \in L_K$  for all  $z \in E$ , for  $K$  large. Let  $G_p = f_p^{-1}(E)$ . Then  $G_p$  is disjoint from  $E$  and so we can form other foliations  $L'_{p,s}$  which include the circles  $G_p$  and  $E$ . Hence after another application of the lemma we can isotope the  $f_p$  to diffeomorphisms now satisfying  $f_p(z) \in E$  for all  $z \in E$  and  $p \in S^k$ .

### 3 Lagrangian spheres in $T^*S^2$

Let  $L$  be a Lagrangian sphere in  $T^*S^2$ . This has self-intersection number  $-2$  and so must be homotopic to the zero-section. By scaling in the fibers we may assume that  $L \subset T^1S^2$ . We will identify  $T^1S^2$  with the complement of the diagonal  $\Delta$  in  $S^2 \times S^2$  with its standard split symplectic form  $\omega = \omega_0 \oplus \omega_0$ . Under this identification, the zero-section in  $T^1S^2$  becomes the antidiagonal  $\bar{\Delta}$ . Thus Theorem 1 in this case is equivalent to the following.

**Theorem 9** *Given a Lagrangian sphere  $L \subset S^2 \times S^2 \setminus \Delta$  homotopic to  $\bar{\Delta}$ , there exists a Hamiltonian isotopy of  $S^2 \times S^2$  which fixes  $\Delta$  and maps  $L$  onto  $\bar{\Delta}$ .*

Given an almost-complex structure  $J$  on  $S^2 \times S^2$  tamed by  $\omega$ , Gromov showed in [11] that there exist unique foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  by  $J$ -holomorphic curves in the classes  $[S^2 \times \text{pt}]$  and  $[\text{pt} \times S^2]$ . With respect to the standard almost-complex

structure  $J_0 = i \oplus i$ , these foliations are exactly  $S^2 \times \text{pt}$  and  $\text{pt} \times S^2$ . The key lemma which we need from [16] is the following.

**Lemma 10** *There exists a tame almost-complex structure  $J$  on  $S^2 \times S^2$  such that each curve in the corresponding foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  intersects  $L$  transversally in a single point. The almost-complex structure  $J$  can be taken to agree with  $J_0$  near  $\Delta$ .*

The second statement was not included in [16] but is clearly true from the proof.

There exists a family of tame almost-complex structures  $J_t$ ,  $0 \leq t \leq 1$  on  $S^2 \times S^2$  with  $J_1 = J$  and, for all  $t$ ,  $J_t = J_0 = i \oplus i$  near  $\Delta$ . In particular,  $\Delta$  is a  $J_t$ -holomorphic curve for all  $t$ . By the positivity of intersections for  $J_t$ -holomorphic curves, each holomorphic curve in the foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  intersects  $\Delta$  transversally in a single point.

We define a diffeomorphism  $f : \Delta \rightarrow \Delta$  by  $f(x) = y$ , where  $y \in \Delta$  is the unique point such that the  $J$ -holomorphic curve in  $\mathcal{F}_1$  through  $y$  intersects the  $J$ -holomorphic curve in  $\mathcal{F}_0$  through  $x$  on  $L$ . Then  $f(x) \neq x$  for all  $x \in \Delta$ .

As in the previous section, for a point  $x \in \Delta$  we denote its image under the antipodal map by  $-x$ . Then the  $J_0$ -holomorphic curve in  $\mathcal{F}_0$  through  $x$  intersects the  $J_0$ -holomorphic curve in  $\mathcal{F}_1$  through  $-x$  on  $\bar{\Delta}$  for all  $x \in \Delta$ .

We can apply the theorem of section 2 without the parameter  $p$  (or in the case  $k = 0$ ) to get the following.

**Lemma 11** *There exists an isotopy  $g_t : \Delta \rightarrow \Delta$ ,  $0 \leq t \leq 1$ , with  $g_0 = \text{id}$ ,  $g_1 = -f^{-1}$  and  $g_t(x) \neq -x$  for all  $t$  and  $x \in \Delta$ .*

We now define maps  $\phi_t : S^2 \times S^2 \rightarrow S^2 \times S^2$  by requiring that  $\phi_t$  maps the  $J_t$ -holomorphic curves in  $\mathcal{F}_0$  and  $\mathcal{F}_1$  to the corresponding  $J_0$ -holomorphic foliations, the  $J_t$ -holomorphic curve in  $\mathcal{F}_0$  through  $x \in \Delta$  maps to the  $J_0$ -holomorphic curve in  $\mathcal{F}_0$  through  $x$  and the  $J_t$ -holomorphic curve in  $\mathcal{F}_1$  through  $x$  maps to the  $J_0$ -holomorphic curve in  $\mathcal{F}_1$  through  $g_t(x)$ .

Then  $\phi_0 = \text{id}$ ,  $\phi_1(L) = \bar{\Delta}$  and  $\phi_t(\Delta)$  is disjoint from  $\bar{\Delta}$  for all  $t$ . Let  $L_t = \phi_t^{-1}(\bar{\Delta})$ , so  $L_t$  gives a smooth isotopy from  $L$  to  $\bar{\Delta}$  in  $S^2 \times S^2 \setminus \Delta$ .

Also,  $\phi_{t*}(J_t)$  is tamed by the split form  $\omega$ , and we see from this that  $\phi_t(\Delta)$  is a symplectic submanifold for all  $t$ .

For fixed  $t$ , set  $\omega_s = s\phi_t^*(\omega) + (1-s)\omega$ . This is a symplectic form for all  $0 \leq s \leq 1$ . It is clearly closed and is symplectic since it tames  $J_t$ . We note that  $\Delta$  is symplectic for all  $\omega_s$  and, if  $t = 0$  or  $t = 1$ ,  $L_t$  is Lagrangian with respect to all  $\omega_s$ . Hence by an application of Moser's theorem we can find a diffeomorphism  $\psi_t$  of  $S^2 \times S^2$  such that  $\psi_t^*(\omega) = \phi_t^*(\omega)$ . The  $\psi_t$  can be chosen to vary smoothly with  $t$ , to fix  $\Delta$  and such that  $\psi_0 = \text{id}$  and  $\psi_1$  fixes  $L$ . To see this, we recall that Moser's method involves writing  $\omega_s = \omega_0 + d\alpha_s$  and studying the flow of the vectorfield  $X_s$  defined by  $X_s \lrcorner \omega_s = \frac{d\alpha_s}{ds}$ . The definition implies that  $\mathcal{L}_{X_s} \omega_s = d(\frac{d\alpha_s}{ds}) = \frac{d\omega_s}{ds}$ . We have the freedom in this construction to add any smooth family of exact 1-forms  $\beta_s$  to the  $\alpha_s$ . These  $\beta_s$  can be chosen such that  $\alpha_s + \beta_s$  vanishes on the symplectic normal bundle to  $\Delta$  and, if  $t = 0$  or  $t = 1$ , on the tangent bundle to  $L_t$ . Then the flow fixes  $\Delta$  and, if  $t = 0$  or  $t = 1$ , also fixes  $L_t$ .

Thus  $\psi_t(L_t)$  is a Lagrangian isotopy from  $L$  to  $\bar{\Delta}$  inside  $S^2 \times S^2 \setminus \Delta$  as required.

To show that the space  $\mathcal{L}$  of Lagrangian spheres is contractible, by applying a result of R. S. Palais [24] it suffices to show that  $\pi_k(\mathcal{L}) = 0$  for all integers  $k \geq 0$ . Thus Theorem 2 reduces to the following.

**Theorem 12** *Given a family of Lagrangian spheres  $L_p \subset S^2 \times S^2 \setminus \Delta$  for  $p \in S^k$  there exists a family of Hamiltonian isotopies of  $S^2 \times S^2$  which fix  $\Delta$  and map  $L_p$  onto  $\bar{\Delta}$ .*

This follows exactly as Theorem 1 for  $T^*S^2$  by applying the full parameterized version of Theorem 7 once we establish the analogue of Lemma 10, that is, we need to show the following.

**Lemma 13** *There exists a family of tame almost-complex structures  $J_p$  on  $S^2 \times$*

$S^2$  such that each curve in the corresponding foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  intersects  $L_p$  transversally in a single point. The almost-complex structures  $J_p$  can be taken to agree with  $J_0$  near  $\Delta$ .

**Proof of lemma** We briefly recall the construction of the almost-complex structures in [16]. Associated to each  $p \in S^k$  and positive integer  $N$  there exists a tame almost-complex structure  $J_{p,N}$  on  $S^2 \times S^2$  which corresponds to stretching the neck to length  $N$  along the boundary of a small tubular neighborhood of  $L_p$ . It is easy to arrange that the  $J_{p,N}$  vary smoothly with  $p$ . For fixed  $p$  it was shown in [16] that after taking a subsequence as  $N \rightarrow \infty$  reparameterizations of  $J_{p,N}$ -holomorphic spheres in the corresponding foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  converge smoothly to finite energy planes in  $T^*L_p$ . For a suitable choice of the  $J_{p,N}$  these finite energy planes must be transverse to  $L_p$ , in particular the  $J_{p,N}$  holomorphic foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are transverse to  $L_p$  for  $N$  sufficiently large. We claim that there exists an  $N$  such that the  $J_{p,N}$ -holomorphic foliations are transverse to  $L_p$  for all  $p$ , thus establishing the lemma.

Suppose that the claim is false. Then for all  $j$  there exists a point  $q_j \in S^k$  and a  $J_{q_j,j}$ -holomorphic sphere  $C_j$  tangent somewhere to  $L_{q_j}$ . A subsequence of  $\{q_j\}$  will converge to some  $p \in S^k$ . Now, there exist diffeomorphisms  $a_j : S^2 \times S^2 \rightarrow S^2 \times S^2$  such that  $a_j(L_{q_j}) = L_p$  and  $a_j$  is an  $(J_{q_j,j}, J_{p,j})$ -biholomorphism on the tubular neighborhood of  $L_{q_j}$ . Furthermore, after taking the subsequence, the  $a_j$  can be chosen to converge  $C^\infty$  uniformly to the identity and so  $I_j = a_{j*}(J_{q_j,j})$  is a sequence of almost-complex structures on  $S^2 \times S^2$  agreeing with  $J_{p,j}$  near  $L_p$  and which are tame for  $j$  large. We apply the compactness theorem from [2] exactly as in [16] to the  $I_j$ -holomorphic foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$ . The same proof shows that reparameterizations converge to finite energy planes in  $T^*L_p$  transverse to  $L_p$ . But this gives a contradiction as required since the  $I_j$  holomorphic spheres  $a_j(C_j)$  are tangent to  $L_p$ .

## 4 Manifolds with 1-handles

We will now consider the class of convex symplectic manifolds constructed by adding 1-handles to the unit cotangent bundle  $T^1S^2$  in order to establish Theorem 1 in this case. Our first observation is that any such manifold  $M$  can be symplectically embedded in  $(S^2 \times S^2, \omega)$ , after perhaps scaling the symplectic form. This follows from the methods of [9]. We can arrange that the zero-section in  $T^1S^2$  again becomes identified with  $\overline{\Delta}$  and the boundary of  $M$  is a smooth hypersurface  $\Sigma$  of contact-type in  $S^2 \times S^2$ . More precisely one can think of  $M$  as a Stein manifold having a bounded plurisubharmonic exhaustion function which is zero on the zero-section in  $T^*S^2$  and whose other critical points are nondegenerate and have Morse index 1. The symplectic form on  $M$  is the Kähler form of the plurisubharmonic exhaustion. Now, as in [6] or [7], 2-handles can be added to  $M$  to cancel the 1-handles and produce a Stein manifold symplectomorphic to  $T^1S^2 = S^2 \times S^2 \setminus \Delta$ . We will later use the fact that  $\Sigma$  is now a level-set of plurisubharmonic exhaustion on  $S^2 \times S^2 \setminus \Delta$ .

We plan to find families of almost-complex structures  $J_t$  on  $S^2 \times S^2$  and diffeomorphisms  $f_t : \Delta \rightarrow \Delta$  such that the  $J_t$ -holomorphic curves in  $\mathcal{F}_0$  through points  $x$  and the  $J_t$ -holomorphic curves in  $\mathcal{F}_1$  through  $f_t(x)$  intersect on embedded spheres  $L_t \subset M$  with  $L_0 = \overline{\Delta}$  and  $L_1 = L$ . The almost-complex structures can be constructed by deforming  $J_0$  in a neighbourhood of  $\Sigma$  and, for  $t$  close to 0 or 1, also in a neighbourhood of  $\overline{\Delta}$  or  $L$ .

Suppose that we perform the operation of stretching-the-neck along  $\Sigma$ . That is, we symplectically identify a neighbourhood of  $\Sigma$  in  $S^2 \times S^2$  with  $((-\epsilon, \epsilon) \times \Sigma, d(e^t\alpha))$ , where  $\alpha$  is a fixed contact form on  $\Sigma$ . We can then produce a manifold  $A_N$  by replacing this neighbourhood by  $(-N, N) \times \Sigma$ . Our original almost-complex structure can be extended over  $(-N, N) \times \Sigma$  to be translation invariant and the symplectic form can be extended over  $(-N, N) \times \Sigma$  such that  $A_N$  is symplectomorphic to  $(S^2 \times S^2, \omega)$  via a symplectomorphism equal to the identity outside  $(-N, N) \times \Sigma$  (for this see [21]). Under this symplectomorphism we can think of stretching the neck as studying a family of almost-complex

structures  $J_N$  on  $S^2 \times S^2$  which degenerate along  $\Sigma$  as  $N \rightarrow \infty$ .

At the same time, we can deform the almost-complex structure along the boundary of tubular neighborhoods  $U_0$  or  $U_1$  of  $L_0 = \overline{\Delta}$  or  $L_1 = L$  respectively. Stretching to length  $N_1$  and  $N_2$  on the contact hypersurfaces  $\Sigma$  and  $\partial U_i$  respectively we obtain almost-complex structures  $J_{N,0}$  and  $J_{N,1}$ , where  $N = (N_1, N_2)$ . There exist smooth families of almost-complex structures  $J_{N,t}$  connecting  $J_{N,0}$  and  $J_{N,1}$  which are fixed on the tubular neighborhoods of  $\Sigma$  and in the complement of  $M$ .

Following the work of H. Hofer, K. Wysocki and E. Zehnder, see [21] (and see below), and as in [16], after taking suitable subsequences of  $N = (N_1, N_2)$  in which both entries tend towards infinity, for  $i, j = 0, 1$  families of  $J_{N,i}$ -holomorphic curves in  $\mathcal{F}_j$  will converge to unions of finite energy planes as  $N \rightarrow \infty$ . The limiting finite energy planes can be chosen to extend to foliations of three symplectic manifolds with cylindrical ends, namely the completion  $W$  of the complement of  $M$  in  $S^2 \times S^2$  with an end symplectomorphic to the negative symplectization of  $\Sigma$ , that is  $((-\infty, 0) \times \Sigma, d(e^t \alpha))$ , the completion of  $U_i$ , which will be a copy of  $T^*S^2$ , and the completion of  $M \setminus U_i$  with two ends symplectomorphic to the positive symplectization of  $\Sigma$  and the negative symplectization of the boundary of  $U_i$ . A priori these foliations will depend upon the subsequence  $(N_1, N_2) \rightarrow \infty$ . For the relevant compactness result here see [2]. Taking a diagonal subsequence this result produces finite energy curves through a dense set of points in each of the three symplectic manifolds, and another application of the compactness result finds finite energy curves through every point. These curves either coincide or are distinct by positivity of intersection, see [22], since we are taking limits of foliations, and hence the limit is also a foliation. Further facts about finite energy curves, such as definitions, asymptotic convergence to Reeb orbits and Fredholm properties can be found in the series of papers [18], [19], [20]. A brief summary containing the facts we need here appears in section 2 of [16].

The foliations of the completion of  $U_i$  were determined in [16], Lemma 10. For  $U_i$  and its almost-complex structure suitably chosen, the Reeb flow on  $\partial U_i$

is foliated by closed orbits, and exactly one curve in each foliation is asymptotic to each closed orbit. Also, each curve in the foliation from  $\mathcal{F}_0$  intersects in a single point each curve in the foliation from  $\mathcal{F}_1$  provided that the curves have different asymptotic limits. Another result coming from the analysis in [16], see Lemmas 8 and 9, is that the curves in both foliations are transverse to the zero-section, and it follows that the curves in the foliations of  $S^2 \times S^2$  are transverse to  $L_i$  for  $N$  sufficiently large.

By the positivity of intersections, any intersections of limiting finite energy planes must also be seen as intersections of holomorphic spheres in the foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$ . It follows that finite energy curves in the two limiting foliations of  $W$  and the completion of  $M \setminus U_i$  either have the same image or are disjoint. For, if any of these curves were to intersect in an isolated point we could find  $J_{N,i}$ -holomorphic curves in  $\mathcal{F}_0$  and  $\mathcal{F}_1$ , for  $N$  sufficiently large, with intersection number at least two. There would be one intersection point near our point in  $W$  or  $M \setminus U_i$  and another inside  $U_i$ . This gives a contradiction. Thus in particular the two foliations of  $W$  must coincide. Two curves in different homotopy classes which are disjoint in the completion of  $M \setminus U_i$  and  $U_i$  must be asymptotic to the same periodic orbit in  $\partial U_i$  and come from taking a limit of  $J_{N,i}$ -holomorphic curves through the same point of  $\Delta$ . The resulting finite energy curves in  $W$  coincide.

We can also obtain finite energy foliations of  $W$  and the completion of  $M$  by stretching the neck only along  $\Sigma$  and studying limits of  $J_{N,t}$ -holomorphic curves. For  $t = i = 0$  or  $1$ , for each fixed  $N_2$  there exists a subsequence  $N_1 \rightarrow \infty$  such that in the limit we obtain foliations of  $W$  and the completion of  $M$ . We define a function  $F$  such that the  $J_{N,i}$ -holomorphic curves restricted to  $U_i$  and  $W$  are uniformly  $\epsilon$ -close to their limit as  $N_1 \rightarrow \infty$  for  $N_1 \geq F(N_2)$  and  $N_1$  in a diagonal subsequence. After taking another subsequence, for  $N_2$  sufficiently large we may assume that these foliations of the completion of  $M$  will consist of curves transverse to  $L_i$  (since near  $L_i$  the curves converge to the limiting curves in the completion of  $U_i$ ). We fix this  $N_2$  and look at the families of almost-complex structures  $J_{N,t}$  connecting  $J_{N,i}$  where now  $N = (N, N_2)$ . For each  $N$ ,

$t$  we have corresponding foliations of  $S^2 \times S^2$  and for subsequences  $N \rightarrow \infty$  obtain foliations of  $W$  and the completion of  $M$ .

Regarding the resulting foliations of  $W$ , we will use the following lemma, which will be proven at the end of the section. Suppose that the contact form  $\alpha$  on  $\Sigma$  and the almost-complex structure on  $W$  are chosen generically so that periodic orbits of the Reeb flow on  $\Sigma$  are isolated and embedded finite energy curves in  $W$  appear in families whose dimension is as predicted by the index theorem, see [8] and [20].

**Lemma 14** *The deformation index of generic finite energy curves  $C$  in the foliation of  $W$  satisfies  $\text{index}(C) \leq 2$ .*

Since 2-dimensional families of curves are needed to foliate  $W$ , this lemma implies that all finite energy planes sufficiently close to a curve  $C$  in our foliation actually appear in the foliation. In particular, if two finite energy foliations of  $W$  have a generic curve in common then they must coincide.

The following is the key proposition.

**Proposition 15** *There exists an  $N$  and diffeomorphisms  $f_{N,t}$  of  $\Delta$  such that the spheres  $L_t$  given by intersecting the  $J_{N,t}$ -holomorphic sphere in  $\mathcal{F}_0$  through points  $x \in \Delta$  with the  $J_{N,t}$ -holomorphic sphere in  $\mathcal{F}_1$  through  $f_{N,t}(x) \in \Delta$  lie in  $M$ . Also,  $L_0 = \overline{\Delta}$  and  $L_1 = L$ .*

**Proof** Fix now a subsequence  $N \rightarrow \infty$  such that the  $J_{N,t_i}$ -holomorphic curves  $C_i$  through points  $x_i$  (in  $\mathcal{F}_0$  or  $\mathcal{F}_1$ ) all converge, where  $t_i = \frac{i}{n}$ ,  $1 \leq i \leq n$  and  $x_i \in \Delta$ . Choosing  $n$  sufficiently large and points  $x_i$  such that the curves  $C_i$  and  $C_{i+1}$  are disjoint in  $M$  for all  $i$ , we can redefine the  $J_{N,t}$  on  $M$  such that for  $t_i \leq t \leq t_{i+1}$  either  $C_i$  or  $C_{i+1}$  is  $J_{N,t}$  holomorphic for all  $N$ . Suppose that for some  $t$  the curves  $C_i$  are  $J_{N,t}$ -holomorphic for all  $N$ . Then, taking subsequences and limits as  $N \rightarrow \infty$ , the foliations of  $W$  corresponding to  $t$  and  $t_i$  have a curve in common, and therefore must coincide by Lemma 14. More generally, we see that for any  $t$  and convergent sequence of  $J_{N,t}$ -holomorphic

spheres, the reparameterized curves in  $W$  must converge to the same foliations as for  $t = 0, 1$ . In particular the foliations coming from  $\mathcal{F}_0$  and  $\mathcal{F}_1$  coincide.

We now study the foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  corresponding to the family of almost-complex structures  $J_{N,t}$  on  $S^2 \times S^2$  for large  $N$ . As in section 3 we can find corresponding diffeomorphisms  $f_{N,t} : \Delta \rightarrow \Delta$  such that  $\bar{\Delta} = L_0$  consists of the intersections of  $J_{N,0}$ -holomorphic spheres in  $\mathcal{F}_0$  through points  $x \in \Delta$  with the spheres in  $\mathcal{F}_1$  through  $f_{N,0}(x)$ , and  $L = L_1$  consists of the intersections of  $J_{N,1}$ -holomorphic curves in  $\mathcal{F}_0$  through points  $x \in \Delta$  with the spheres in  $\mathcal{F}_1$  through  $f_{N,1}(x)$ . By the result in section 2 we may assume that  $f_{N,t}(x) \neq x$  for all  $x, t$ . As  $N \rightarrow \infty$ , the convergence of  $J_{N,0}$  and  $J_{N,1}$ -holomorphic spheres implies that we may assume that the  $f_{N,0}$  and  $f_{N,1}$  converge to continuous maps  $f_{\infty,0}$  and  $f_{\infty,1}$  and hence that the  $f_{N,t}$  converge to maps  $f_{\infty,t}$  satisfying  $f_{\infty,t}(x) \neq x$  for all  $t$  and  $x$ .

Suppose that the proposition is false. Then for each  $N$  there exists a  $t_N$  and  $x_N \in \Delta$  such that the  $J_{N,t_N}$ -holomorphic sphere in  $\mathcal{F}_0$  through  $x_N$  intersects the  $J_{N,t_N}$ -holomorphic sphere in  $\mathcal{F}_1$  through  $f_{N,t_N}(x_N)$  inside  $W$ . Taking a subsequence of  $N \rightarrow \infty$  suppose that  $t_N \rightarrow t$  and  $x_N \rightarrow x$  so  $f_{N,t_N}(x_N) \rightarrow f_{\infty,t}(x) \neq x$ . Also, a limiting finite energy curve through the point  $x$  will intersect a limiting curve through  $f_{\infty,t}(x)$  inside  $W$ . But limits of the  $J_{N,t_N}$ -holomorphic spheres again give foliations of  $W$ , and since for each  $N$  sufficiently large the  $J_{N,t_N}$ -holomorphic foliations contain a  $J_{N,t}$ -holomorphic sphere these foliations are again standard, giving a contradiction as required and proving Proposition 15.

We now complete the proof of Theorem 1. Following the method of section 3, we can find a family of symplectic forms  $\omega_t$  on  $S^2 \times S^2$  such that  $L_t$  is Lagrangian with respect to  $\omega_t$ . The  $\omega_t$  restrict to exact symplectic forms on  $M$ , say  $\omega_t = d\alpha_t$ , which tame  $J_t$ . In a tubular neighborhood  $V = (-\epsilon, 0) \times \Sigma$  of the boundary  $\Sigma = \{0\} \times \Sigma$  of  $M$ , define a function  $\chi : V \rightarrow [0, 1)$  such that  $\chi(r, y)$  is an increasing function of  $r$ ,  $\chi(r, y) = 0$  for  $r$  close to  $-\epsilon$  and  $\chi(r, y) = 1$  for  $r$  close to 0. Then, first scaling  $\alpha_t$  if necessary, we can replace it by  $\beta_t = (1-\chi)\alpha_t + \chi e^r \alpha$  in  $V$ . The new form  $\omega_t = d\beta_t$  will still be symplectic and

tamed by  $J_t$  (for  $\alpha_t$  suitably scaled) but now agrees with  $\omega$  near  $\Sigma$ . Assuming  $V$  to be disjoint from all  $L_t$ , the submanifolds  $L_t$  will still be Lagrangian with respect to  $\omega_t$ .

We now apply Moser's method as in section 3 to find a symplectomorphism between  $(M, \omega_t)$  and  $(M, \omega)$  and thereby isotope the  $L_t$  into Lagrangian submanifolds of  $(M, \omega)$ . As before, this can be arranged to fix  $L_0$  and  $L_1$  and now also the neighborhood  $V$ . Thus it gives our Lagrangian isotopy as required.

**Proof of Lemma 14**

Let  $C$  be a finite energy curve in the foliation of  $W$ . The curve  $C$  will be one component of a limit of holomorphic spheres. The other components can be assumed to be curves  $D_i$  in the symplectization of  $\Sigma$  and curves  $E_j$  in the completion of  $M$ .

An analysis of Chern classes as in [16] shows that only one component of a limit can intersect  $\Delta$ . Since  $W \setminus \Delta$  admits a plurisubharmonic function approaching  $+\infty$  towards  $\Delta$  and a constant towards the negative end, by the maximum principle there are no finite energy curves lying entirely in  $W \setminus \Delta$ . Therefore the limiting curve has only one component in  $W$ , and, since the finite energy curves are limits of curves of genus 0, the  $D_i$  and  $E_j$  have only one positive asymptotic limit.

We recall that the index formula for a finite energy curve  $F$  depends upon a trivialization of the contact planes along its asymptotic limits (which are Reeb orbits in certain contact manifolds). We can then define a Conley-Zehnder index for each asymptotic limit and a Chern class  $c_1(F)$  relative to these trivializations. Suppose that the positive asymptotic limits have Conley-Zehnder indices  $\mu_k^+$  for  $1 \leq k \leq m$  and the negative asymptotic limits have index  $\mu_l^-$  for  $1 \leq l \leq n$ . If the asymptotic limits are nondegenerate the formula for the deformation index of  $F$  modulo reparameterizations is

$$\text{index}(F) = -(2 - m - n) + 2c_1(F) + \sum_{k=1}^m \mu_k^+ - \sum_{l=1}^n \mu_l^-.$$

In our case, a global trivialization of the contact planes in  $T^1S^2$  extends over any 1-handles to a trivialization of  $\xi = \{\alpha = 0\}$  on  $\Sigma$  and we compute our

indices relative to this. Then the curves in the symplectization of  $\Sigma$  and the completion of  $M$  have Chern class 0 and our curve  $C$  has  $c_1(C) = 2$ .

A curve  $E_j$  has a single positive asymptotic limit. If this has index  $\mu_j^+$  then we obtain

$$\text{index}(E_j) = -1 + \mu_j^+.$$

But for a generic choice of almost-complex structure this index must be non-negative and so  $\mu_j^+ \geq 1$  for all  $j$ .

Suppose that a curve  $D_i$  has a positive asymptotic limit with index  $\mu_i^+$  and  $m_i$  negative asymptotic limits with index  $\mu_{ik}^-$  for  $1 \leq k \leq m_i$ . Then the index formula becomes

$$\text{index}(D_i) = -1 + m_i + \mu_i^+ - \sum_{k=1}^{m_i} \mu_{ik}^-.$$

Again generically this index must be nonnegative. Therefore if all of the negative asymptotic limits are positive asymptotic limits of curves  $E_j$  we obtain  $\mu_i^+ \geq 1$ . By an induction on the number of levels of the limiting finite energy curve we deduce that in fact  $\mu_i^+ \geq 1$  for all curves  $D_i$ .

Finally we look at  $C$ . It has only negative asymptotic limits and if these have index  $\mu_l^-$  for  $1 \leq l \leq n$  then

$$\text{index}(C) = 2 + n - \sum_{l=1}^n \mu_l^-.$$

But all of these negative asymptotic limits are positive limits of curves  $D_i$  or  $E_j$ . Hence  $\mu_l^- \geq 1$  for all  $l$  and so  $\text{index}(C) \leq 2$  as required.

## 5 Proof of Theorem 4

In this section we study the symplectic manifold  $W$ , which is a plumbing of two copies of  $T^*S^2$ . Namely we take two copies of  $T^*S^2$  and identify the cotangent fibers projecting to a disk  $D$  in  $S^2$  with a product  $D \times E$  in each copy. We then identify the two copies of  $D \times E$ , preserving the product structure but reversing the factors. Alternatively  $W$  can be realized as a Stein manifold by adding a

2-handle to a disk bundle  $T^1S^2$  along the boundary of one fiber, a Legendrian curve for the natural choice of contact structure.

In any case,  $W$  is naturally a symplectic manifold with symplectic form  $\omega_0$  and contains two Lagrangian spheres  $L_1$  and  $L_2$  corresponding to the two zero-sections. We will think of its non-compact end as a copy of  $[0, \infty) \times M$  where  $M$  carries a contact structure with contact form  $\alpha$  and the symplectic structure on the end is given by  $\omega = d(e^t\alpha)$ .

The manifold  $M$  is a Lens space  $L(3, 2)$ . The contact form can be described as follows.

Let  $S$  be the 3-sphere given by

$$S = \{(z_1, z_2) \in \mathbb{C}^2 \mid H(x) = 1\}$$

where

$$H(x) = |z_1|^2 + \frac{1}{r^2}|z_2|^2$$

and equipped with the contact form  $\lambda|_S$  where

$$\lambda = \frac{i}{4} \sum_{j=1}^2 (z_j d\bar{z}_j - \bar{z}_j dz_j).$$

Let  $r^2 > 1$  and irrational, and periodic orbits  $p_0 = \{z_2 = 0\} \cap S$  and  $p_1 = \{z_1 = 0\} \cap S$ .

**Lemma**(see [17] Lemma 1.6) *The associated Reeb vectorfield possesses precisely two periodic orbits  $p_0$  and  $p_1$ . They are nondegenerate and have Conley-Zehnder indices  $\mu(p_0) = 3$  and  $\mu(p_1) = 2n + 1$  where  $n < r^2 + 1 < n + 1$ .*

Now we observe that  $S$  and  $\lambda|_S$  are invariant under the map  $\sigma : (z_1, z_2) \mapsto (e^{\frac{2\pi i}{3}} z_1, e^{\frac{4\pi i}{3}} z_2)$  and so project to  $L(3, 2)$  to give the contact form  $\alpha$ . The orbits  $p_0$  and  $p_1$  triple cover periodic orbits  $x_0$  and  $x_1$  on our  $L(3, 2)$ . Let  $X$  be the corresponding Reeb vectorfield.

Our proof will proceed as follows. On  $[0, \infty) \times M$  we choose a tame almost-complex structure  $J$  which is translation invariant, preserves the contact planes on  $M$  and satisfies  $J(\frac{\partial}{\partial t}) = X$ . Throughout the proof we will fix this almost-complex structure. It can be extended to a tame almost-complex structure  $J$

on  $W$  and for each extension we will describe a foliation of  $W$  by finite energy planes asymptotic to multiple covers of  $x_0$ . Let  $L \subset W$  be a Lagrangian sphere homotopic to  $L_1$ . Then we pay specific attention to the pattern of the foliation relative to  $L$  when we change  $J$  by stretching the neck near  $L$ . This is all done in section 5.1.

In section 5.2, using our holomorphic foliations we can construct plurisubharmonic exhaustion functions on  $W$ . These functions will have exactly one minimum and two critical points of index 2. It will turn out that after stretching the neck along  $L$ , the unstable manifold of one critical point will be disjoint from  $L$ .

All such plurisubharmonic exhaustions give isotopic symplectic structures on  $W$ . The final part of the proof, in section 5.3, will use these isotopies to construct the symplectomorphism needed for our theorem. Of course Theorem 1 will also be used, in a form which says that a Lagrangian sphere disjoint from the unstable manifold of one critical point is Hamiltonian isotopic to the stable manifold of the other critical point.

## 5.1 Finite energy holomorphic curves in $W$

### 5.1.1 Finite energy foliations

As stated above,  $W$  admits a foliation by finite energy planes. More specifically the following is true.

**Theorem 16** *For any tame extension  $J$ , the almost-complex manifold  $(W, J)$  can be foliated by finite energy planes. Exactly three planes in the foliation,  $E_0, E_1, E_2$ , are asymptotic to  $x_0$ . The other finite energy planes are all asymptotic to  $3x_0$ . After choosing orientations for  $L_1$  and  $L_2$  we may assume that  $E_i \bullet L_j = -\delta_{ij}$  and  $E_0 \bullet L_j = 1$  for  $i, j = 1, 2$ .*

#### **Proof**

This is very similar to the proof in [14], (which of course is heavily reliant on the series of papers [18], [19], [20]) but the arrangement of finite energy

planes is different to the situation covered there. In fact, [14] described finite energy foliations of Stein manifolds diffeomorphic to disk bundles over  $S^2$  whose boundaries are the Lens spaces  $L(p, 1)$ . The basic case of the foliation of  $T^*S^2$  with boundary  $\mathbb{R}P^3$  was worked out earlier in [13]. The proofs, and this one, follow the same path in that they start with the finite energy planes in  $\mathbb{R} \times S^3$  constructed in [17] (using the method of filling by holomorphic disks) and project these to get finite energy planes in  $W$  which are topologically trivial relative to the boundary but appear in a 2-dimensional family. A process of elimination using index and area inequalities then determines the behaviour of the family of curves as they propagate into  $W$ .

More precisely, this reasoning, originating in the works of H. Hofer, K. Wysocki and E. Zehnder, [17], Theorem 5.1, implies that there is a 2-dimensional moduli space of unparameterized disjoint embedded finite energy planes asymptotic to  $3x_0$ . An  $S^1$  family of these planes lie in  $[0, \infty) \times M$  and each plane in the family touches  $\{1\} \times M$  in a single point. Choosing  $R$  large, this  $S^1$  family will intersect  $\{R\} \times M$  in an  $S^1$  family of circles, a 2-torus, which bounds a solid torus  $U$  containing the periodic orbit  $x_0$ . Let  $B$  be the intersection of the  $S^1$  family of finite energy planes with  $[0, R] \times M$ . Then a small perturbation of  $B \cup U$  is a pseudoconvex hypersurface bounding a domain  $V$  Stein homotopic to  $W$ . In fact the perturbation of  $B \cup U$  can be isotoped into  $\{R\} \times M$  through a family of pseudoconvex hypersurfaces.

We are interested in an extension of our moduli space to a family of finite energy planes foliating  $V$ . After the perturbation of  $B$  we may suppose that our  $S^1$  family intersects  $B$  in a circle  $\gamma$  of complex tangencies. Other finite energy planes in our moduli space will intersect  $B$  in circles linking  $\gamma$ , see for example Figure 2 in [14]. Since  $M$  is an  $L(3, 2)$ , after choosing coordinates on  $U$  we may assume that the finite energy planes intersecting  $\partial U$  do so in  $(3, 1)$  curves, where the first component represents the class of a longitude homotopic to  $x_0$ .

The planes in the moduli space intersecting  $U$  do not form a compact set. In fact, as in [14], Lemma 3.2, bubbling occurs and sequences of finite energy planes asymptotic to  $3x_0$  will converge to three finite energy planes  $E_0, E_1,$

$E_2$  asymptotic to  $x_0$ . (The topology of  $V$  implies that we now get bubbling into three planes, energy considerations imply that they are all asymptotic to  $x_0$ .) We call these rigid planes since the moduli space of finite energy planes asymptotic to  $x_0$  modulo reparameterization has dimension 0. Together with the finite energy planes asymptotic to  $3x_0$  the rigid planes complete our foliation.

We notice as in [14] that  $V$  is homotopic to the intersections of the three rigid planes with  $V$ , after identifying their boundaries in  $U$ . (This implies that there is no further bubbling.) To check the intersection numbers, we can choose a convenient almost-complex structure  $J$  since the numbers are independent of the choice. In fact there is an  $S^1$  subgroup of symplectomorphisms of  $W$  which on each cotangent bundle corresponds to the extension via differentials of the rotations of  $L_j$  about the axis through the intersection point  $q \in L_1 \cap L_2$ . Let  $q_1$  and  $q_2$  be the antipodal point of  $q$  in  $L_1$  and  $L_2$  respectively. If the almost-complex structure is invariant under these symplectomorphisms, then so are the rigid planes (as they appear only in dimension 0). Stokes' Theorem implies that holomorphic planes cannot intersect our Lagrangians in circles (since they are symplectic and the symplectic form on  $W$  is exact) and so the rigid planes must intersect the two Lagrangians in fixed points of the  $S^1$ -action. A plane disjoint from the Lagrangians is homotopic to a plane in  $[0, \infty) \times M$  where the asymptotic limit  $x_0$  is not contractible. Therefore each rigid plane does indeed intersect a Lagrangian and we can order our planes so that  $E_0 \cap L_j = \{q\}$ ,  $E_1 \cap L_1 = \{q_1\}$  and  $E_2 \cap L_2 = \{q_2\}$ . Choosing orientations for  $L_1$  and  $L_2$  gives the theorem as required.

Topologically the intersections of our finite energy planes with  $U$  can be visualized as follows. We note however that this is an idealized picture. In practice holomorphic curves can have quite complicated tangencies with pseudoconvex hypersurfaces. In the next section we will use the technique of filling by holomorphic disks to ensure that the pattern we describe here does indeed occur.

Figure 1: flowlines in the cross-section  $A$

We look at a cross-section  $A$  of  $U$ . The interior of  $A$  has three special points corresponding to the intersection of  $A$  with the rigid planes. By taking  $R$  sufficiently large, the rigid planes can be assumed to intersect  $U \subset \{R\} \times M$  transversally. A finite energy plane intersecting  $\partial U$  hits  $\partial A$  in three points. Choosing a path from one of these points to one of the special points determines a 1-parameter family of finite energy planes intersecting the path. The intersections of these planes with  $A$  generate two more paths from our points in  $\partial A$  to the remaining special points. Conversely a path in our moduli space starting from a plane intersecting  $\partial U$  and converging to the bubbled planes generates three paths in  $A$ . Starting with other planes intersecting  $\partial U$  we can generate a vector field on  $A$  with elliptic points corresponding to the rigid planes. The vector field will necessarily have hyperbolic points corresponding to tangencies of finite energy planes with  $U$ . Assuming that there are no more elliptic points (which could occur if a finite energy plane became tangent to  $U$  from the outside) there must be two hyperbolic points and the various integral curves are illustrated in Figure 1. The three marked points on the boundary are the intersections of a typical finite energy plane with  $\partial A$ . With our choice of subscripts the central special elliptic point in Figure 1 corresponds to  $E_0$ . Notice that the same picture is obtained in each cross-section  $A_\theta$  of  $U$  for  $\theta \in S^1$  and we can continuously choose coordinates in each  $A_\theta$  so that the elliptic points lie in the same position. But then the points on  $\partial A_\theta$  corresponding to a fixed finite energy plane will rotate through  $\frac{2\pi}{3}$  in these coordinates as  $\theta$  moves once around. The integral curves leaving our points on  $A_\theta$  can be chosen so that they correspond to the same family of finite energy planes for each  $\theta$ . These integral curves will encounter a hyperbolic point for two values of  $\theta$ , corresponding to a 1-parameter family in the moduli space becoming tangent to  $U$  twice before bubbling. Topologically this means that two curves in the plane must contract to the boundary and that the plane will bubble into three components.

### 5.1.2 Stretching the neck

In this subsection we consider which finite energy planes in the foliation will intersect  $L$  if we perform a stretching-the-neck operation to deform  $J$  along the boundary of a tubular neighborhood of  $L$ .

The almost-complex structure is replaced by other almost-complex structures  $J_N$  as in section 4 where  $\Sigma$  is now the boundary of a tubular neighborhood  $Z$  of our Lagrangian  $L$ , which of course is diffeomorphic to  $\mathbb{R}P^3 = L(2, 1)$ . We fix a contact form on  $\Sigma$  as above, now quotienting  $S^3$  by the map  $\sigma : (z_1, z_2) \mapsto (-z_1, -z_2)$ . Denote by  $y_0$  and  $y_1$  the corresponding Reeb periodic orbits on  $\Sigma$ . Note that this form and the corresponding Reeb vector field are nondegenerate unlike the Morse-Bott type form on  $\partial U_i$  used in section 4.

The stretching-the-neck procedure in section 4 applies again here to produce a finite energy foliation of the completed tubular neighborhood of  $L$  which is now identified with  $T^*L = T^*S^2$ . Since  $y_1$  has large Conley-Zehnder index the finite energy curves must be asymptotic to  $y_0$ . The resulting foliation was first described in [13], (see Theorem 2.1, or alternatively, for a description entirely in terms of finite energy planes, rather than disks, Theorem 2.3 in [14]). There are two finite energy planes asymptotic to  $y_0$  and the remaining planes are asymptotic to  $2y_0$ . The planes asymptotic to  $y_0$  have intersection number  $\pm 1$  with  $L$ . They are rigid in the sense that the corresponding moduli space has dimension 0.

Taking limits of finite energy planes in the holomorphic foliations of  $(W, J_N)$  also results in a collection of finite energy curves lying in a completion of  $W \setminus Z$  and the symplectization of  $\Sigma$  equipped with suitable almost-complex structures. After taking subsequences and additional limits we also obtain finite energy foliations of  $W \setminus Z$ .

Suppose that an embedded finite energy curve  $u$  in  $W \setminus Z$  has one positive asymptotic limit  $mx_0$  and  $k$  negative asymptotic limits asymptotic to  $n_i y_0$ ,  $1 \leq i \leq k$ . The virtual dimension of the moduli space of finite energy curves

containing  $u$  modulo reparameterization is given by

$$\text{index}(u) = -(2 - 1 - k) + \mu(mx_0) - \sum_{i=1}^k \mu(n_i y_0)$$

where the  $\mu$  are Conley-Zehnder indices with respect to a suitable trivialization giving  $c_1(TW) = 0$ . For  $m, n_i$  not too large  $\mu(mx_0) = m$  and  $\mu(n_i y_0) = \lfloor \frac{n_i}{2} \rfloor + n_i$  where  $\lfloor z \rfloor$  denotes the greatest integer less than or equal to  $z$ . Hence

$$\text{index}(u) = m - 1 - \sum_{i=1}^k (\lfloor \frac{n_i}{2} \rfloor + n_i - 1).$$

Our assumption is that  $L$  is homotopic to  $L_1$ . Thus the limits of the  $J_N$  holomorphic rigid planes  $E_0$  and  $E_1$  must contain rigid planes in  $T^*L$ . The components of the limits in  $W \setminus Z$  must have positive and negative asymptotic limits asymptotic to  $x_0$  and  $y_0$  respectively. It will turn out that these curves coincide for  $E_0$  and  $E_1$ . Meanwhile, we next observe that the limiting curves corresponding to  $E_2$  have no component in  $T^*L$ . To see this, we note that for the component of the limit in  $W \setminus Z$  to have nonnegative index, since  $m = 1$  its negative asymptotic limit can cover  $y_0$  only once. Therefore any components in  $T^*L$  are rigid curves. But then the component of the limit of  $E_2$  must coincide with a component of the limit of  $E_0$  or  $E_1$ . In either case this implies that for  $N$  sufficiently large  $E_2$  will intersect non-rigid  $J_N$ -holomorphic planes which intersect  $L$  close to  $E_0$  or  $E_1$ , a contradiction. Therefore, crucially for us, the  $J_N$ -holomorphic rigid planes  $E_2$  are disjoint from  $L$  for  $N$  sufficiently large.

We now look at a limit of  $J_N$ -holomorphic finite energy planes passing through a point  $p \in L$  disjoint from the rigid planes. We claim that the limiting curve has a single component in  $T^*L$  asymptotic to  $2y_0$  and two components in  $W \setminus Z$ , one a double cover of the component of the limits of  $E_0$  or  $E_1$  and the other the limit of the rigid planes  $E_2$ .

To justify the claim, we note that one component of the limit in  $T^*L$  must be a plane asymptotic to  $2y_0$ . The sum of the positive asymptotic limits of the limiting components in  $W \setminus Z$  is  $3x_0$  and as in section 4, the negative asymptotic limits of components of a limiting curve in  $W \setminus Z$  cover  $y_0$  at least as many

times as the positive asymptotic limits of the corresponding limit curve in  $T^*L$ . Suppose that a component of the limit in  $W \setminus Z$  with negative asymptotic limit  $2y_0$  is embedded and the almost-complex structure is generic so the curve has nonnegative index. Then by the above formula the positive asymptotic limit must cover  $x_0$   $m = 3$  times and index = 0. Thus such curves are isolated. But taking a limit as our initial point  $p \in L$  approaches a rigid plane, such limits must approach the limits of the  $E_i$ , which is impossible. Therefore the limiting components in  $W \setminus Z$  with negative asymptotic limit  $2y_0$  are multiple covers of the limits of  $E_0$  and  $E_1$ , which must coincide. (We recall that by positivity of intersections [22] a limit of embedded holomorphic curves is either embedded or a multiple cover.) There is another component of the limit in  $W \setminus Z$  with positive asymptotic limit  $x_0$ . Since such curves are again isolated we see that this must coincide with the limit of the  $E_2$  as required.

The claim implies that for  $N$  sufficiently large the nonrigid planes intersecting  $L$  will intersect  $U$  in two disjoint circles, one close to the intersection of  $U$  with  $E_0$  and  $E_1$  and homotopic to  $2x_0$ , the other close to the intersection with  $E_2$ .

## 5.2 Plurisubharmonic exhaustion functions

In this section we produce a filling (or foliation) of  $V$  by holomorphic disks with boundary on the perturbation of  $B \cup U$  and use it to construct a plurisubharmonic exhaustion for  $V$ . The key property is that  $L$  will be disjoint from the unstable manifold of one of the two index 2 critical points.

**Theorem 17** *For any extension  $J$ , the almost-complex manifold  $(V, J)$  admits a plurisubharmonic exhaustion function with three critical points, one a minimum and the others of index 2. The Lagrangian  $L$  is disjoint from the unstable manifold of one of the index 2 critical points.*

It would be convenient simply to use the intersections of  $V$  with finite energy planes as our filling. Unfortunately it seems hard to control the tangencies of

Figure 2: holomorphic disks filling a surface

such planes with  $U$ . Therefore we singularly foliate  $U$  with surfaces, each of which in turn can be singularly foliated by the boundaries of holomorphic disks. Together with the finite energy planes intersecting  $B$  these will complete the filling.

**Proof** There exist  $S^1$  families of  $J_N$ -holomorphic finite energy planes which divide the finite energy planes in the moduli space intersecting  $L$  from those lying entirely in some  $[R, \infty] \times M$ . As  $N$  approaches infinity the planes in this family can each be chosen to converge to a union of curves having a component in  $T^*L$ . Thus the planes in the family will converge to the curve in  $W \setminus Z$  which is a double-cover of the limit of the  $J_N$ -holomorphic  $E_0$  and  $E_1$  and to the limit of  $E_2$ . Therefore, since the family is compact, for  $N$  sufficiently large the curves in the family will intersect  $U$  transversally in two families of circles, one homotopic to  $2x_0$  and the other to  $x_0$ . The first family will foliate a torus  $I$  enclosing  $(E_0 \cup E_1) \cap U$  and the second will foliate a torus enclosing  $E_2 \cap U$ .

We define vector fields on each  $A_\theta \subset U$  looking exactly as described in the previous section, but not necessarily corresponding to the intersections of the  $A_\theta$  with finite energy planes. The integral curves of our vector field converging to the intersection of a particular curve  $C$  with  $\partial U$  will form a surface diffeomorphic to a sphere with four disks removed. The four boundary components are the intersections of  $U$  with  $C$ ,  $E_0$ ,  $E_1$  and  $E_2$ . Now, it is easy to adjust our vector field such that each of these surfaces intersect the torus  $I$  in the boundary of one of the finite energy planes in our  $S^1$  family.

Next we use the theory of filling by holomorphic disks, see [5], [1], [7], [12] (Theorem 1 in [12] unifies alot of the previous work) to singularly foliate each of the surfaces above by boundaries of holomorphic disks in  $V$ . The foliation extends the boundaries of  $C$  and the  $E_i$  and is unique, therefore it includes the intersection of the surface with  $I$ . The filling looks approximately as in Figure 2. In particular since it includes the disk through  $I$  the arrangement of the

singular (hyperbolic) points  $p$  and  $q$  is as shown.

We construct a plurisubharmonic function by following [7], see also [13], [14]. We start by defining a function  $g$  which is constant on the holomorphic disks in our filling. We now fix  $J = J_N$  for  $N$  suitably large. Recall that  $\gamma$  is the circle of complex tangencies in  $B \subset \partial V$  and let  $T_1, T_2$  be tori in  $U$  formed by the boundaries of holomorphic disks passing through the hyperbolic points  $p$  and let  $S_1, S_2$  be tori in  $U$  formed by the boundaries of holomorphic disks passing through the points  $q$ . We label things so that the inside of  $T_1$  in  $U$  encloses  $S_1$  and  $S_2$ . We define  $g$  to be a Morse function on  $\gamma$  with a single minimum at 0 and a single maximum at 1. As in [13] we define  $g$  to be constant on families of holomorphic disks converging to points on  $\gamma$ . These families of disks can be chosen to be parameterized either by an interval with the disks converging to the points  $g^{-1}(t) \in \gamma$  for  $t \leq \frac{1}{4}$  or  $t \geq \frac{3}{4}$  or alternatively by an interval with one end converging to a point  $g^{-1}(t) \in \gamma$  for  $\frac{1}{4} < t < \frac{3}{4}$  and the other to a cusp-disk with boundary on  $T_1 \cup T_2$ . This defines  $g$  on the disks passing through the complement of the insides of  $T_1$  and  $T_2$ .

Inside  $T_2$  we simply define  $g$  to be constant on 1-parameter families of disks connecting the disks on which  $g = t$ . Inside  $T_1$  we again define  $g$  to be constant on families of disks connecting the disks on which  $g = t$  and  $\frac{1}{4} < t < \frac{3}{8}$  or  $\frac{5}{8} < t < \frac{3}{4}$ . We also let  $g = t$  on intervals of disks connecting disks with  $g = t \in [\frac{3}{8}, \frac{5}{8}]$  on one side and cusp-disks with boundary on  $S_1 \cup S_2$  on the other. Inside  $S_1$  and  $S_2$  we extend  $g$  to be constant on the 1-parameter families of disks connecting the disks on which  $g = t$  as before. Altogether this defines a function  $g$  whose level-sets are foliated by holomorphic disks.

Now, as in [7], see also [13], [14], the level-sets of  $g$  are Levi flat (foliated by holomorphic curves) but we can perturb  $g$  such that they become pseudoconvex. One way to do this is to choose a function  $\psi$  on  $W$  which satisfies  $dd^c\psi(X, JX) \gg |d\psi|$  for any unit vector  $X$  tangent to the foliation (with respect to any fixed metric). Then we can replace  $g$  by  $g + \psi$ . Recall that the level-sets of a function  $f$  are pseudoconvex if  $-dd^c f$  is positive on the complex tangencies  $d^c f = 0$ . (Before the perturbation  $-dd^c g$  vanishes on this subspace.)

We then have that  $dd^c(g + \psi)$  is positive on the complex tangencies of the level-sets of  $g$ , but the tangencies to the level-sets of  $g + \psi$  differ only by order  $|d\psi|$  and so the same is true for these subspaces. Next, composing  $g$  with a sufficiently convex function  $\phi$  on  $\mathbb{R}$  with  $\phi'' \gg \phi' > 0$  it then becomes strictly plurisubharmonic. (To see this, we compute  $-dd^c(\phi \circ g) = -d(\phi' d^c g) = -\phi' dd^c g + \phi'' d^c g \wedge dg$  and observe that the first term is positive on complex tangencies to the level sets of  $g$  while the second term vanishes, but on sufficiently transverse complex planes the second term is positive and overwhelms the first.) Next set  $f = \max(g, h)$  where  $h$  is a function increasing rapidly towards  $\partial V$ . The function  $f$  can be smoothed to give a plurisubharmonic exhaustion. Investigating the pattern of holomorphic disks as in [13], section 3, we see that it has three critical points. There is an index 0 critical point near the minimum of  $g$  on  $\gamma$  and there are index 2 critical points near the maxima of  $g$  on  $S_1 \cap S_2$  and  $T_1 \cap T_2$ . We call these points  $a$  and  $b$  respectively. The construction ensures that  $f(b) > f(a)$ . Furthermore  $f < f(b)$  on all disks lying inside the hypersurface formed by the holomorphic disks intersecting  $I$ . Therefore  $L$  is disjoint from the unstable manifold of  $b$  as required, as it lies inside this hypersurface.

We close this section by remarking that since  $f$  provides a plurisubharmonic exhaustion of  $W$  (or more precisely an almost-complex manifold  $V$  Stein homotopic to  $W$ ), we can adjust  $f$  near its minimum such that the stable manifolds of the two critical points are embedded spheres intersecting transversally at the minimum.

### 5.3 Symplectomorphisms

The plurisubharmonic function  $f$  from the previous section gives a symplectic form  $\omega = -dd^c f$  on  $W$  where  $d^c f = df \circ J$ . This in turn gives us a vector field  $v = \text{grad} f$  defined by  $v \lrcorner \omega = d^c f$ . By a suitable choice of  $f = h$  near  $\partial V$  we may assume that  $v$  is complete in the sense that its positive integral flow exists for all time.

The stable manifolds of the two critical points are Lagrangian spheres with

respect to the form  $\omega$ . By Weinstein's Lagrangian neighborhood theorem applied to a pair of transversally intersecting Lagrangians, a neighborhood of these two stable manifolds is symplectomorphic to a neighborhood of  $L_1 \cup L_2 \subset (W, \omega_0)$ . We then use [9] to imply the following.

**Lemma 18**  *$(W, \omega)$  and  $(W, \omega_0)$  are symplectomorphic via a symplectomorphism  $\psi$  taking the stable manifolds of the critical points  $a$  and  $b$  of  $f$  onto  $L_1$  and  $L_2$  respectively.*

After perhaps adjusting  $f$  the following is also true. As above  $L$  denotes the Lagrangian sphere homotopic to  $L_1$ .

**Lemma 19** *There exists a symplectomorphism  $\phi$  from  $(W, \omega_0)$  to  $(W, \omega)$  taking  $L$  onto a Lagrangian sphere disjoint from the unstable manifold of the critical point  $b$  of  $f$ .*

Lemmas 18 and 19 together imply our Theorem 4. For, the one-parameter group of diffeomorphisms generated by  $-v = -\text{grad}f$  give an isotopy (which is necessarily Hamiltonian) of  $\phi(L)$  to a Lagrangian sphere in a tubular neighborhood of the stable manifold of the critical point  $a$ . Since this tubular neighborhood can be taken to be symplectomorphic to a unit cotangent bundle of  $S^2$ , Theorem 1 implies that a further Hamiltonian diffeomorphism maps the image of  $\phi(L)$  onto the stable manifold of  $a$  itself. We denote the Hamiltonian diffeomorphism mapping  $\phi(L)$  onto the stable manifold of  $a$  by  $\chi$ . Then  $\psi \circ \chi \circ \phi$  is the symplectomorphism required by Theorem 4.

**Proof of Lemma 19**

By choosing  $f = h$  carefully near  $\partial V$ , now identified with the noncompact end of  $W$ , we may assume that  $\omega = \omega_0$  outside of a compact subset of  $W$ . In fact, both forms are exact and we can write  $\omega - \omega_0 = d\alpha$  where the 1-form  $\alpha$  is identically zero outside of a compact set. Furthermore, since  $\omega$  and  $\omega_0$  tame the same almost-complex structure,  $\omega_t = (1-t)\omega_0 + t\omega$  is a symplectic form on  $W$  for all  $t$ .

Using Moser's method, we observe that the compactly supported time-dependent vector field  $X_t$  defined by  $X_t \lrcorner \omega_t = \alpha$  satisfies  $\mathcal{L}_{X_t} \omega_t = \frac{d}{dt} \omega_t$  and so its flow generates a symplectomorphism from  $(W, \omega_0)$  to  $(W, \omega)$ .

We are interested in the image of  $L$  under such a symplectomorphism, we recall that  $L$  is initially disjoint from the unstable manifold of  $b$  and we want to ensure that this remains the case under the flow of  $X_t$ . Assume that a fixed tubular neighborhood  $Z$  of  $L$  is disjoint from the unstable manifold of  $b$ . It is straightforward to adjust  $f$  to arrange that  $f \geq 0$  and  $Z \subset f^{-1}([0, r])$  for some  $r < f(b)$ . For the manipulation of plurisubharmonic functions see [7] or [6]. In this case, aside from flowing along  $\text{grad} f$  the specific lemma we need is a rearrangement result for critical points of plurisubharmonic functions to ensure that  $f(a) < f(b)$ , analogous to the corresponding result in Morse theory [23]. Lemma 2.1 in [15] explains how to derive this from the analysis in [6].

The composition of  $f$  with an increasing function  $s : [0, \infty) \rightarrow [0, \infty)$  remains plurisubharmonic provided that  $\frac{s''}{s'} \gg 1$ . We choose  $s$  (and its derivatives) to be very small on  $[0, r]$  but then to increase rapidly on  $(r, \infty)$ . Thus we can replace  $f$  by another nonnegative plurisubharmonic exhaustion, still denoted by  $f$ , and having the property that  $f|_Z < 1$ . Further we arrange that  $\omega(X, JX) \ll \omega_0(X, JX)$  on  $f^{-1}([0, 1])$  for all tangent vectors  $X$ , while  $\omega(X, JX) \gg \omega_0(X, JX)$  on  $f^{-1}([2, 3])$  for all  $X$  and now  $f(b) > 3$ .

On the tubular neighborhood  $Z$  we have that  $\omega$  and  $d^c f$  are now uniformly small. Thus the length of  $X_t$  (relative to the Riemannian metric defined by  $\omega_0$  and  $J$ ) remains bounded on this neighborhood for  $t < \frac{1}{2}$  say. Therefore there exists a uniform  $\epsilon$  (depending only upon  $\omega_0$ ,  $J$  and  $Z$ ) such that the flow of  $L$  remains in  $Z$  for  $t < \epsilon$ . But for  $t > \epsilon$  we can suppose that on  $f^{-1}([2, 3])$  the vector field  $X_t$  is closely approximated by  $-\frac{1}{t} \text{grad} f$ . Hence the flow of  $L$  remains in  $f^{-1}([0, 3])$  for all  $0 \leq t \leq 1$  and so the symplectomorphism generated by  $X_t$  can indeed be arranged to leave  $L$  disjoint from the unstable manifold of  $b$  as required.

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