

Hamiltonian displacement of bidisks inside cylinders

R. Hind

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1 Introduction

The displacement energy of a subset $U \subset \mathbb{R}^{2n}$ is defined by

$$e(U) = \inf\{\|H\| \mid \phi_H(U) \cap U = \emptyset\}$$

where the infimum is taken over all compactly supported (Hamiltonian) functions $H : \mathbb{R}^{2n} \times [0, 1] \rightarrow \mathbb{R}$, and writing $H_t(x) = H(x, t)$ the Hofer norm $\|H\| = \int_0^1 \sup_x H_t(x) - \inf_x H_t(x) dt$. The diffeomorphism ϕ_H is the Hamiltonian diffeomorphism generated by H , that is, the time-1 flow of the time-dependent vectorfield X_t defined by $X_t \lrcorner \omega = dH_t$. The form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ is the standard symplectic form on \mathbb{R}^{2n} in coordinates $x_1, y_1, \dots, x_n, y_n$. It is sometimes convenient to write $\|\phi\| = \inf \|H\|$ where the infimum is taken over all Hamiltonian functions H satisfying $\phi_H = \phi$.

The notion of displacement energy extends to arbitrary symplectic manifolds, in particular to subsets $M \subset \mathbb{R}^{2n}$. In this case the displacement energy is defined by

$$e^M(U) = \inf\{\|H\| \mid \phi_H(U) \cap U = \emptyset\}$$

where the infimum is now taken over Hamiltonian functions with compact support in $M \times [0, 1]$.

We emphasize that, in cases when U can be displaced within M , for $e^M(U)$ to differ from $e(U)$ it is important that the support of the functions H and not just the image $\phi_H(U)$ lie in M . In fact, suppose that a Hamiltonian diffeomorphism ψ satisfies $\psi(U) = V$ where $U \cap V = \emptyset$. If there exists a $W \subset M$ which is Hamiltonian diffeomorphic to U but with $U \cap W = \emptyset$ then

we can find another Hamiltonian diffeomorphism f with support disjoint from U such that $f(V) = W$. So $\phi = f\psi f^{-1}$ satisfies $\phi(U) = W$ but by the invariance of the Hofer norm ϕ can be generated by Hamiltonians of the same norm as ψ .

For convenience of notation we will frequently identify \mathbb{R}^{2n} with \mathbb{C}^n , writing $z_j = x_j + iy_j$ in standard coordinates. In this paper we will focus on the displacement energy of bidisks

$$D(a, b) = \{\pi|z_1|^2 < a, \pi|z_2|^2 < b\} \subset \mathbb{C}^2$$

where $a \leq b$.

In his original work on the subject [5], [6], H. Hofer showed that $e(D(a, b)) = a$. In fact the infimum can be realized by a Hamiltonian function $H(z_1)$ depending only upon z_1 . On the other hand, if $D(a, b) \subset M$ and $\text{volume}(M) < 2ab$ we clearly have $e^M(D(a, b)) = \infty$.

A natural candidate for M where we might expect strict inequalities

$$e(D(a, b)) < e^M(D(a, b)) < \infty$$

is the cylinder $M = Z(a + \epsilon) = \{\pi|z_1|^2 < a + \epsilon\}$ for ϵ small. If $\epsilon > a$ then again we have $e^{Z(a+\epsilon)}(D(a, b)) = a$. By translating vertically we see that $e^{Z(a+\epsilon)}(D(a, b)) \leq b$ but it is not immediately clear that $e^{Z(a+\epsilon)}(D(a, b)) > e(D(a, b)) = a$ for any $\epsilon > 0$.

Our main theorem gives fairly tight estimates for the displacement energies of bidisks inside cylinders.

Theorem 1.1. *Let $Z = Z(1 + \epsilon)$. Then*

$$\left(\frac{1}{2} - \epsilon\right)[S] + \epsilon \leq e^Z(D(1, S)) \leq \frac{S}{2} + 3.$$

The upper bound here is established by an explicit construction in section 2. The lower bound relies on some symplectic embedding obstructions.

We recall the main theorem from [4]. Let $B^{2n}(A) \subset \mathbb{R}^{2n}$ denote the round ball with capacity A , that is, of radius r satisfying $\pi r^2 = A$.

Theorem 1.2. *For any $0 < A < 3$ there are no symplectic embeddings of $D^2(1) \times B^{2(n-1)}(S)$ into $B^4(A) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

The paper [4] also stated an analogous theorem for embeddings into bidisks.

Theorem 1.3. *If $a < 2$ then there are no symplectic embeddings of $D^2(1) \times B^{2(n-1)}(S)$ into $D(a, b) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

In order to deduce the lower bound in Theorem 1.1 we will apply a quantitative version of Theorem 1.3.

Let $E(a, b, c) \subset \mathbb{C}^3$ denote the ellipse

$$E(a, b, c) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} + \frac{\pi|z_3|^2}{c} < 1 \right\}.$$

Theorem 1.4. *Let $d \geq 1$ be an integer. If there exist symplectic embeddings*

$$E(1, R, R) \hookrightarrow D(1 + \epsilon, T) \times \mathbb{C}$$

for some $R > 2d + 1$ then $T \geq d(1 - \epsilon) + 1$.

In section 3 we show how to derive the lower bound in Theorem 1.1 from Theorem 1.4. This will be done using the technique of symplectic folding.

In section 4 we prove Theorem 1.4. This follows [4] closely, but we need to exercise care with the dimensions of the ellipse. However there are some simplifications resulting from considering embeddings into bidisks rather than balls, as was the focus in [4].

2 Displacing a bidisk

This section is devoted to proving the following. Let $\epsilon > 0$ and as before set $Z = Z(1 + \epsilon)$.

Theorem 2.1. $e^Z(D(1, S)) \leq \frac{S}{2} + 3$.

We fix a $0 < \delta \ll \epsilon$ and the proof will consist of explicitly constructing a Hamiltonian diffeomorphism ϕ with a generating Hamiltonian of norm less than $\frac{S}{2} + 3 + \delta$. By abuse of notation, occasionally we will also simply write δ or ϵ for quantities differing only by a universal constant.

Changing notation slightly, we will use coordinates (u, v, x, y) on \mathbb{R}^4 . We define $p : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ to be the projection onto the (u, v) -plane and set

$$D(1, S) = \left\{ 0 < u < 1, 0 < v < 1, -\frac{1}{2} < x < \frac{1}{2}, \frac{-S}{2} < y < \frac{S}{2} \right\} \subset \mathbb{R}^4$$

and

$$Z = p^{-1} \{(u, v) \in D\} \subset \mathbb{R}^4$$

where D is a region of area $1 + \epsilon$ containing $D(1) = p(D(1, S))$. Up to Hamiltonian diffeomorphism these domains are equivalent to those previously defined.

The diffeomorphism ϕ will be a composition of three Hamiltonian diffeomorphisms $\{\phi_i\}_{i=1}^3$ which we define in the following steps. Defining $F_0 = D(1, S)$ we will set $F_i = \phi_i(F_{i-1})$ and require that $F_3 \cap F_0 = \emptyset$ and $\sum_{i=1}^3 \|\phi_i\| < \frac{S}{2} + 3 + \delta$.

Step 1

Outline. Here we apply a Hamiltonian diffeomorphism ϕ_1 which fixes the (u, v) coordinates and maps the (x, y) coordinates into a region as described by Figure 1. In particular, if $(u, v, x, y) \in F_1 \cap \{-\frac{1}{2} < x < \frac{1}{2}\}$ then $|y| < \delta$.

Construction.

Let $H_1(y)$ satisfy the following

- $H_1(y) = 0$ whenever $|y| > \frac{S}{2} + \delta$
- $H_1'(y) = 1 + \delta$ if $\frac{-S}{2} < y < -\delta$
- $H_1'(y) = -1 - \delta$ if $\delta < y < \frac{S}{2}$

Such functions H_1 clearly exist with $\|H_1\| \leq (1 + \delta)\frac{S}{2}$. We call the resulting Hamiltonian diffeomorphism ϕ_1 . We see that $\phi_1(D(S)) \cap F_0 \subset \{|y| < \delta\}$.

Step 2

Outline.

We apply a Hamiltonian ϕ_2 of norm roughly 3ϵ such that $F_2 \cap \{-\frac{1}{2} < x < \frac{1}{2}\} \subset \{|y| < \delta\}$ and $p(F_2 \cap \{-\frac{1}{2} < x < \frac{1}{2}\})$ lies in a simply connected region of area roughly $1 - \epsilon$.

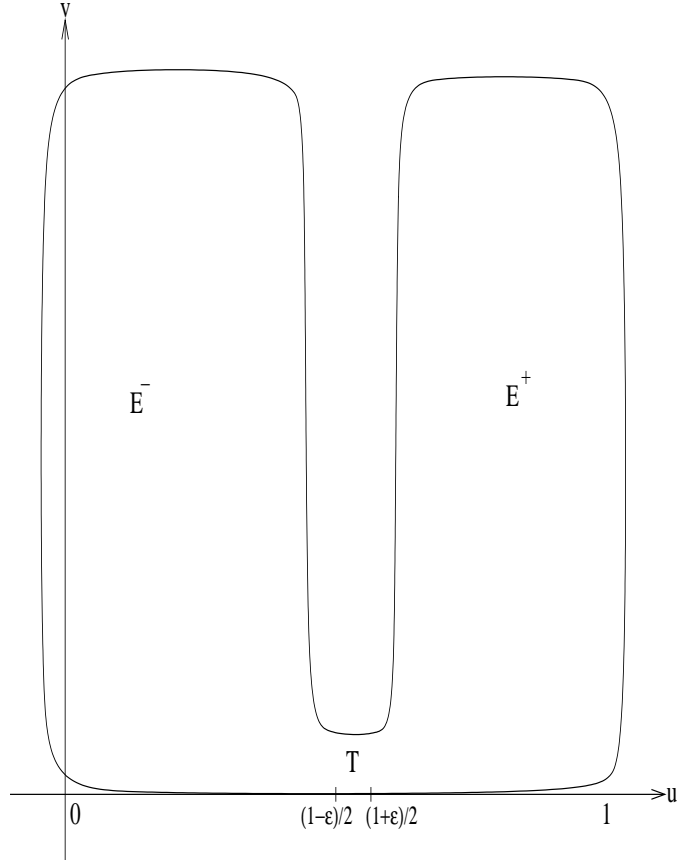
Construction.

Throughout we will be careful to ensure that after each of our diffeomorphisms the preimage of points in $\{-\frac{1}{2} < x < \frac{1}{2}\}$ always lies in $\{-\frac{1}{2} < x < \frac{1}{2}\}$. This will allow us to iterate our procedure in Step 3 below.

We divide $D(1)$ into two regions E^+ and E^- and a thin connecting strip T . Up to symplectomorphism we can, and will, think of these as arranged in Figure 2. The region lying between E^+ and E^- has area roughly ϵ .

Our construction has three stages.

Stage 1

Figure 2: *The division of $D(1) \subset D$*

Hamiltonian flow has a positive $\frac{\partial}{\partial v}$ component and the area of the projection increases roughly by the norm of f , namely ϵ . We can choose β so that the image does not intersect other parts of F_1 , in other words so that $\phi_{2,1}$ is well defined. When restricted to x close to $-\frac{1}{2}$ this increased projection is negligible.

Stage 2

Outline. We apply a Hamiltonian with compact support in $\{-\frac{1}{2} < x < \frac{1}{2} - \delta\}$ and of norm ϵ rotating $p^{-1}(E^+)$ in the direction of $p^{-1}(E^-)$. For x close to $-\frac{1}{2}$ the projection of the image of $p^{-1}(E^+)$ will intersect E^+ in a subset of E^+ of area $\frac{1}{2} - \epsilon$.

Indeed, there exists a positive Hamiltonian function $H_2(u, v)$ of norm ϵ whose time 1 flow moves an arbitrarily large subset of $E^+ \cap \{v < 2\epsilon\}$ onto

$E^- \cap \{v < 2\epsilon\}$. Such a Hamiltonian can be chosen to have support in an arbitrarily small neighborhood of $\{v < 2\epsilon\} \setminus \{\frac{1-\epsilon}{2} < u < \frac{1+\epsilon}{2}, v > 0\}$. Now we apply the Hamiltonian diffeomorphism generated by $H_2(u, v)g(x)$ to $F_{2,1}$, where now the only points which move by definition lie in $p^{-1}(E^+) \cap \{v < 2\epsilon\}$. Here $g : \mathbb{R} \rightarrow [0, 1]$ has compact support in $[-\frac{1}{2}, \frac{1}{2} - \delta]$ and $g(-\frac{1}{2} + \delta) = 1$ and has slope bounded below by approximately -1 . We note that the y component of the resulting flow is then bounded below by $-\epsilon$ and so the corresponding diffeomorphism is again well defined, that is, when points from $p^{-1}(E^+)$ are mapped into $p^{-1}(E^-)$ their y -coordinates are still bounded above 0 and so the points are not mapped onto $p^{-1}(E^-) \cap F_1$. (Really $\epsilon g'(x)$ needs to be bounded below by $-\epsilon + 4\delta$ but we ignore error terms of order δ , which after all is arbitrarily small.) For $-\frac{1}{2} < x < -\frac{1}{2} + \delta$ the y coordinate may increase rapidly.

The diffeomorphism will be denoted $\phi_{2,2}$ and we set $F_{2,2} = \phi_{2,2}(F_{2,1})$. We note that $p(F_{2,2} \cap \{x = -\frac{1}{2} + \delta\})$ lies in a simply connected set of area roughly $1 - \epsilon$, namely $E^- \cup T \cup (E^+ \setminus \{v < 2\epsilon\})$.

Stage 3

Outline. We apply a Hamiltonian diffeomorphism $\phi_{2,3}$ of norm roughly ϵ and depending only on x and y which has the following properties.

- $\phi_{2,3}$ is nondecreasing in the x coordinate and nonincreasing in the y coordinate;
- $\{x = -\frac{1}{2} + \delta\}$ is preserved;
- $\{x > -\frac{1}{2} + 2\delta\} \cap \{y > 0\}$ is mapped to $\{x > \frac{1}{2}\}$;
- $[-\frac{1}{2} + \delta, \frac{1}{2} - \delta] \times [-\delta, \epsilon]$ is mapped into $\{|y| < \delta\}$.

Again we will set $F_{2,3} = \phi_{2,3}(F_{2,2})$. A suitable Hamiltonian satisfying these requirements can be defined by $\epsilon(\frac{1}{2} - \delta + x)\sigma(y)$ where $\sigma(y)$ is a non-increasing function equal to 0 if $y < -\delta$ and -1 if $y > \delta$.

We claim that $F_{2,3}$ satisfies the requirements of Step 2 and hence we can set $\phi_2 = \phi_{2,3} \circ \phi_{2,2} \circ \phi_{2,1}$ and $F_2 = F_{2,3}$. The effect of the final stage is that any point in $F_{2,3} \cap \{-\frac{1}{2} + \delta < x < \frac{1}{2}\}$ is the image under $\phi_{2,3}$ of a point with either x close to $-\frac{1}{2} + \delta$ or $y < 0$ and so its projection under p (which is unchanged under $\phi_{2,3}$) now lies in the union of $E^- \cup T$ with a subset of E^+ of area $\frac{1}{2} - \epsilon$.

Step 3

Here we iterate the construction from Step 2, to be precise replacing $[-\frac{1}{2}, \frac{1}{2}]$ with $[-\frac{1}{2} + \delta, \frac{1}{2} - \delta]$. After each step the area of the projection reduces by ϵ and so we may assume that it lies outside of $D(1)$ after roughly $\frac{1}{\epsilon}$ iterations. The sum of the norms of the Hamiltonians used in Step 2 was 3ϵ . Therefore after all of these iterations the composition will have norm 3, as claimed.

3 Symplectic embeddings and displacement energy

Here we show how to derive the lower bound in Theorem 1.1 from Theorem 1.4.

Let e denote the displacement energy $e^Z(D(1, S))$ where as usual Z is the cylinder $Z(1 + \epsilon) \subset \mathbb{C}^2$.

Then for any $\delta > 0$ we can find a (perhaps time dependent) Hamiltonian function H of norm $e + \delta$ generating a flow which displaces $D(1, S)$ inside Z .

Let $\chi(x_3)$ be a smooth increasing function equal to 0 when $x_3 \leq 0$ and 1 when $x_3 \geq 1$. We may assume $0 \leq \chi' \leq 1 + \delta$.

Let V be a δ -neighborhood in a z_3 -plane of the union of disks D_1 and D_2 of radius $\sqrt{\frac{S}{2\pi}}$ centered at $(-\sqrt{\frac{S}{2\pi}}, 0)$ and $(\sqrt{\frac{S}{2\pi}} + 1, 0)$ respectively, and the interval $[0, 1]$ on the x_3 -axis. Then there exists a symplectic embedding $D(S) \hookrightarrow V$ and by taking a product $f : D(1, S) \times D(S) \hookrightarrow Z \times V$.

We apply the Hamiltonian diffeomorphism ϕ generated by $\chi(x_3)H(z_1, z_2)$ to the image of f . If we define

$$V' = V \cup [0, 1] \times \left[(1 + \delta) \int_0^1 \inf_x H_t(x) dt, (1 + \delta) \int_0^1 \sup_x H_t(x) dt \right]$$

then $U = \phi(\text{image}(f)) \subset Z \times V'$ and the fibers of U over the disk D_1 are disjoint from the fibers over the disk D_2 .

Thus if g is a smooth map of V' which preserves ω_0 and sends D_1 onto D_2 then $\text{id.} \times g|_U$ gives a symplectic embedding of U into a domain symplectomorphic to $Z \times D(\frac{S}{2} + e + \delta)$.

We recall that $E(a, b, c) \subset \mathbb{C}^3$ denotes the ellipse

$$E(a, b, c) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} + \frac{\pi|z_3|^2}{c} < 1 \right\}.$$

The following can be established using the technique of symplectic folding, as we will recall momentarily.

Lemma 3.1. *For any $r > 1$ there exists a symplectic embedding*

$$E(1, 2S - 1, 2S - 1) \hookrightarrow D(r) \times D(rS) \times D(rS).$$

Now, to establish our lower bound it suffices to assume that $S = d + 1$ is an integer, with $d \geq 1$.

Putting everything together, given $r > 1$ we can find an $R > 2S - 1 = 2d + 1$ for which we have a symplectic embedding

$$E(1, R, R) \hookrightarrow rZ \times D\left(r\left(\frac{d+1}{2} + e + \delta\right)\right) = D(r(1 + \epsilon), r\left(\frac{d+1}{2} + e + \delta\right)) \times \mathbb{C}.$$

Applying Theorem 1.4 then, and letting $r \rightarrow 1^+$, we find that $\frac{d+1}{2} + e + \delta \geq d(1 - \epsilon) + 1$. Therefore the displacement energy $e \geq d(\frac{1}{2} - \epsilon) + \frac{1}{2} - \delta$ and letting $\delta \rightarrow 0$ this gives the lower bound of Theorem 1.1 as required.

Proof of Lemma 3.1

We would like to embed $E(1, 2S - 1, 2S - 1)$ symplectically into an arbitrarily small neighborhood of $D(1) \times D(S) \times D(S)$. The embedding is constructed by performing a symplectic fold twice, for a detailed study of symplectic folding see [12].

Write $E = E(1, 2S - 1, 2S - 1)$. If we project E to the z_2 -plane, the fibers are ellipses. At a point where $\pi|z_2|^2 = k(2S - 1)$ the fiber is $E((1 - k), (1 - k)(2S - 1))$, where similarly to the above we write

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^3 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} < 1 \right\}.$$

The projection to the z_2 -plane is a disk of area $2S - 1$ and we can identify it with a δ -neighborhood of the union two disks D_1^2 and D_2^2 of area $\frac{2S-1}{2}$ centered at $(-\sqrt{\frac{2S-1}{2\pi}}, 0)$ and $(\sqrt{\frac{2S-1}{2\pi}} + 1, 0)$ respectively, and the interval $[0, 1]$ on the x_2 -axis. We may further assume that under this identification only points with $\pi|z_2|^2 > \frac{2S-1}{2}$ map into the disk D_2^2 . (Some points with $\pi|z_2|^2 < \frac{2S-1}{2}$ must necessarily map close to the boundary of D_1^2 , but such points lie in the region $\{\pi|z_1|^2 < \frac{1}{2}\}$ and so do not interfere with the folding described below.) By abuse of notation, we will now denote by z_2 the push-forward of the standard z_2 -coordinate under this identification of regions in

the plane. We can do the same for the projection to the z_3 -plane, so that it lies in a δ neighborhood of the union of the same two disks D_1^3 and D_2^3 thought of now as lying in the z_3 -plane and the interval $[0, 1]$ on the x_3 -axis.

Now, if $\pi|z_2|^2 > \frac{2S-1}{2}$, the projection to the z_1 -plane of the corresponding fiber is contained in the disk centered at the origin of area $\frac{1}{2}$, and can be displaced within the disk of area r by a Hamiltonian diffeomorphism of the z_1 -plane generated by a Hamiltonian H of norm $\frac{1}{2}$. In fact, we can arrange this Hamiltonian such that points with $\pi|z_1|^2 < s$ are mapped to those with $\pi|z_1|^2 > 1 - s$ for all $s < \frac{1}{2}$. Thus, applying this diffeomorphism to the fibers over D_2^2 , the new fiber over a point with $\pi|z_2|^2 > k(2S - 1)$ for $k > \frac{1}{2}$ now lies in $\{\pi|z_1|^2 > k\}$.

Similarly to the above, let $\chi(x_2)$ be a smooth increasing function equal to 0 when $x_2 \leq 0$ and 1 when $x_2 \geq 1$ and apply the Hamiltonian flow generated by χH . This is the identity on points projecting onto D_1^2 and the image is such that the fibers over $\{\pi|z_2|^2 = k(2d + 1)\} \subset D_2^2$ are disjoint from those over $\{\pi|z_2|^2 = (1 - k)(2S - 1)\} \subset D_1^2$. The projection to the z_2 plane of the image lies in the union of D_1^2 and D_2^2 and a rectangle symplectomorphic to $[0, 1] \times [0, \frac{1}{2}]$.

We note that in the ellipse E , if $\pi|z_2|^2 > \frac{2S-1}{2}$ then $\pi|z_3|^2 \leq \frac{2S-1}{2}$ and vice versa. Therefore we can carry out the same construction supported at points where $\pi|z_2|^2 < \frac{2S-1}{2}$ in order to arrange that the fibers of the projection to the z_3 -plane over D_2^3 are disjoint from the corresponding fibers over D_1^3 . We call the resulting domain E' .

Now we apply symplectic folding. Namely we apply the composition of two maps $\psi_2 = \phi_2 \times \text{id}_{13}$ and $\psi_3 = \phi_3 \times \text{id}_{12}$ where ϕ_2 is a map of the z_2 -plane taking D_2^2 onto D_1^2 and id_{13} is the identity on the (z_1, z_3) -planes. Similarly ϕ_3 is a map of the z_3 -plane taking D_2^3 onto D_1^3 and id_{12} is the identity on the (z_1, z_2) -planes. The maps ϕ_2 and ϕ_3 are area preserving. For all $s < \frac{1}{2}$ we ensure that ϕ_2 takes points with $\pi|z_2|^2 > (1 - s)(2S - 1)$ to points with $\pi|z_2|^2 < s(2S - 1)$. Similarly ϕ_3 takes points with $\pi|z_3|^2 > (1 - s)(2S - 1)$ to points with $|z_3|^2 < s(2S - 1)$.

This composition applied to E' , written as described, up to symplectomorphism has image E'' contained in a neighborhood of

$$D(1) \times (D_1^2 \cup [0, 1] \times [0, \frac{1}{2}]) \times (D_1^3 \cup [0, 1] \times [0, \frac{1}{2}]).$$

Thus since $D_1^2 \cup [0, 1] \times [0, \frac{1}{2}]$ and $D_1^3 \cup [0, 1] \times [0, \frac{1}{2}]$ are symplectomorphic to disks of area $\frac{2S-1}{2} + \frac{1}{2}$ the proof of the lemma is complete once we show that

the composition is injective.

If a point in E' has coordinates with $(\pi|z_1|^2, \pi|z_2|^2, \pi|z_3|^2) = (r, s, t)$ and $s > \frac{2S-1}{2}$ then $r > \frac{s}{2S-1}$ and the image under the fold ψ_2 is a point with coordinates $(r, 2S-1-s, t)$. This is disjoint from E' because $r + \frac{2S-1-s}{2S-1} + \frac{t}{2S-1} > 1$. Similarly the image of the support of ψ_3 is disjoint from E' .

Finally suppose that a point with coordinates $(|z_1|^2, |z_2|^2, |z_3|^2) = (r, s, t)$ and $s > \frac{2S-1}{2}$ is mapped under ψ_2 to the image under ψ_3 of a point with coordinates (r', s', t') and $t' > \frac{2S-1}{2}$. Then $(r, 2S-1-s, t) = (r', s', 2S-1-t')$. As (r, s, t) are the coordinates of a point in E' we know that $\frac{s+t}{2S-1} < 1$, but this implies that $\frac{2S-1-s'+2S-1-t'}{2S-1} < 1$ or $\frac{s'+t'}{2S-1} > 1$ giving a contradiction.

4 An obstruction to symplectic embeddings

In this section we prove Theorem 1.4. This is a quantitative version of Theorem 1.3 from [4]. The proof of Theorem 1.3 was omitted in [4] as it is entirely analogous to the Theorem 1.2 there concerning embeddings into $B^4(R) \times \mathbb{C}$, or more generally $\mathbb{C}P^2 \times \mathbb{C}$ (and in fact it is slightly simpler). We include the proof here for completeness, outlining parts which already appear in [4].

Proof of Theorem 1.4

The notation is preserved from the previous sections.

Let $1, S_1, S_2$ be linearly independent over \mathbb{Q} with $2d+1 < S_1, S_2 < R$. Then the characteristic line field on $\partial E(1, S_1, S_2) \subset \mathbb{C}^3$ has exactly three closed orbits. These are the circles $\gamma_i = \{z_j = 0; j \neq i\}$ with actions $1, S_1$ and S_2 respectively. With respect to the trivialization induced from \mathbb{C}^3 the Conley-Zehnder index of the r -fold cover $\gamma_1^{(r)}$ of γ_1 is given by

$$\mu(\gamma_1^{(r)}) = 2r + \left(2 \left\lfloor \frac{r}{S_1} \right\rfloor + 1\right) + \left(2 \left\lfloor \frac{r}{S_2} \right\rfloor + 1\right).$$

In particular, if $r \leq 2d+1$ the index $\mu(\gamma_1^{(r)}) = 2r + 2$.

Denote by $S^2(a) \times S^2(b)$ the manifold $S^2 \times S^2$ with the product symplectic form such that the first factor has area a and the second area b . Then there exists an embedding (by inclusion) $D(1+\epsilon, T) \times \mathbb{C} \hookrightarrow S^2(1+\epsilon) \times S^2(T) \times \mathbb{C}$.

Now suppose that there exists a symplectic embedding

$$E(t, tS_1, tS_2) \hookrightarrow S^2(1+\epsilon) \times S^2(T) \times \mathbb{C} = X.$$

Then $X \setminus E(t, tS_1, tS_2)$ has the structure of a symplectic manifold with a concave end symplectomorphic to $\partial E \times (-\infty, 0]$ and compatible almost-complex structures can be defined as in [4], section 2.4. Original sources for this are [2] or even [7]. In our situation the image of $E(t, tS_1, tS_2)$ will always lie in $D(1 + \epsilon, T) \times \mathbb{C} \subset S^2(1 + \epsilon) \times S^2(T) \times \mathbb{C}$ and so we may assume that the almost-complex structures restrict to the standard product structures on the surfaces $S^2(1 + \epsilon) \times \{\infty\} \times \mathbb{C}$ and $\{\infty\} \times S^2(T) \times \mathbb{C}$ which represent the complement of $D(1 + \epsilon, T) \times \mathbb{C}$.

For any such almost-complex structure there are moduli spaces of finite energy J -holomorphic planes mapping into $X \setminus E$ and exponentially asymptotic to multiple covers of $\gamma_1 \times (-\infty, 0]$ outside of a compact set. The reparameterization group $G = \text{Aut}(\mathbb{C})$ acts on such planes.

If such a plane u is asymptotic to $\gamma_1^{(r)}$ then we write $u \sim \gamma_1^{(r)}$. In this case we can add an r -fold cover of the disk $\{z_2 = z_3 = 0\} \subset E$ to the image of u in order to construct a 2-dimensional homology class $[u] \in H_2(X)$. We will write $[u] = (k, l)$ if $[u] \bullet (\text{pt.} \times S^2) = k$ and $[u] \bullet (S^2 \times \text{pt.}) = l$.

Lemma 4.1. (i) *Let*

$$\mathcal{M}(J) = \{u : \mathbb{C} \rightarrow X \setminus E \mid \bar{\partial}_J u = 0, [u] = (d, 1), u \sim \gamma_1^{(2d+1)}\} / G.$$

Then for generic J the moduli space $\mathcal{M}(J)$ is a compact 0-dimensional manifold.

(ii) *Given a family of embeddings $E(t, tS_1, tS_2) \hookrightarrow X$ for $\epsilon \leq t \leq 1$ and a corresponding family J_t of compatible almost-complex structures we define*

$$\mathcal{M}(\{J_t\}) = \{(u, t) \mid u : \mathbb{C} \rightarrow X \setminus E, t \in [0, 1], \bar{\partial}_{J_t} u = 0, [u] = (d, 1), u \sim \gamma_1^{(2d+1)}\} / G.$$

Then for generic families $\{J_t\}$ the moduli space $\mathcal{M}(\{J_t\})$ is a compact 1-dimensional manifold, giving a cobordism between $\mathcal{M}(J_\epsilon)$ and $\mathcal{M}(J_1)$.

Proof. The relevant index formula here, see [2], gives the deformation index of such a finite energy plane for fixed J as

$$\text{index} = 2c_1([u]) - \mu(\gamma_1^r) = 2(2d + 2) - (2(2d + 1) + 2) = 0.$$

The closure of these moduli spaces generically contain no curves with multiply covered components. This follows since any such multiply covered component, say u , must cover, say $r > 1$ times, a somewhere injective curve v . Suppose that $[v] = (k, l)$ and for simplicity assume that v is a plane

with $v \sim \gamma_1^s$ for some s . Analogous statements hold for multiple covers with several negative ends.

First of all we note that $l = 0$. For otherwise, by the positivity of intersection, see [9], the intersection number of u with the surface $S^2(1+\epsilon) \times \{\infty\} \times \mathbb{C}$ will be at least 2. As all components have a nonnegative intersection with this surface, and the sum of these intersections is 1, this gives a contradiction.

So $[v] = (k, 0)$ and $[u] = (rk, 0)$ and by positivity of intersection again $rk \leq d$.

As v is somewhere injective its deformation index is given as above by the formula

$$\text{index}(v) = 4k - (2s + 2\lfloor \frac{s}{S_1} \rfloor + 2\lceil \frac{s}{S_1} \rceil + 2)$$

and generically $\text{index}(v) \geq 0$.

Therefore $s \leq 2k - 1 \leq \frac{2d}{r} - 1$ and so $rs \leq 2d - 1 < S_1, S_2$. Thus

$$\begin{aligned} \text{index}(u) &= 4rk - (2rs + 2\lfloor \frac{rs}{S_1} \rfloor + 2\lceil \frac{rs}{S_1} \rceil + 2) \\ &= 4rk - (2rs + 2) = r(\text{index}(v) + 2) - 2 \geq 2. \end{aligned}$$

In conclusion, any multiply covered components appearing in the limit have strictly positive index. As the total virtual index is preserved in taking a limit, if there are multiply covered components then there must also be (necessarily somewhere injective) components with strictly negative index. In our arrangements all indices are even and so we see components with index at most -2 and such do not generically appear in 1-dimensional families.

The various compactness statements now follow exactly as in [1], the point being that any bubbling is of codimension at least 2. \square

Let $\phi_\epsilon : E(\epsilon, \epsilon S_1, \epsilon S_2) \hookrightarrow X$ be a symplectic embedding which restricts to an embedding $E(\epsilon, \epsilon S_1) \hookrightarrow S^2 \times S^2$ on $\{z_3 = 0\}$ and such that the image $\phi_\epsilon(E(\epsilon, \epsilon S_1, \epsilon S_2))$ is invariant under rotations about the origin in the z_3 -plane. For ϵ sufficiently small such symplectic embeddings exist and are isotopic through embeddings of $E(t, tS_1, tS_2)$ for $\epsilon \leq t \leq 1$ to any given embedding of $E(1, S_1, S_2)$.

Lemma 4.2. *Let J_ϵ be a regular compatible almost-complex structure on $X \setminus \phi_\epsilon(E(\epsilon, \epsilon S_1, \epsilon S_2))$ which is invariant under rotations in the z_3 -plane. Then $\mathcal{M}(J_\epsilon)$ contains a positive number of equivalence classes of curves, counting with multiplicity.*

It follows as in [4], Lemma 4.4 that such regular J_ϵ do indeed exist. Thus, given Lemma 4.2, $\mathcal{M}(J_\epsilon)$ represents a nontrivial cobordism class. Therefore by Lemma 4.1, if an embedding $\phi_1 : E(1, S_1, S_2) \hookrightarrow X$ exists and J_1 is a compatible almost-complex structure on $X \setminus \phi_1(E(1, S_1, S_2))$ then $\mathcal{M}(J_1)$ is also nonempty.

The compatibility condition for J_1 implies that curves u in $\mathcal{M}(J_1)$ have positive symplectic area. Computing, this area is $d(1 + \epsilon) + T - (2d + 1) \geq 0$, and so we obtain the inequality required for Theorem 1.4.

Proof of Lemma 4.2. Any curves in $\mathcal{M}(J_\epsilon)$ must lie in $\{z_3 = 0\}$ since any other curves would appear in a 1-dimensional family (given by rotating the z_3 -plane) and so could not be regular.

Thus we can focus on an embedding $\phi_\epsilon : E(\epsilon) = E(\epsilon, \epsilon S_1) \rightarrow S^2 \times S^2$ and look for finite energy curves in $S^2 \times S^2 \setminus \phi_\epsilon(E(\epsilon))$ asymptotic to $\gamma_1^{(2d+1)}$. It follows from work of C. Wendl, [14], see also [4], Lemma 4.4, that any such curves have positive orientation.

Fix points $p_1, \dots, p_{2d+1} \in \phi_\epsilon(E(\epsilon)) = E$. Then given a generic tame almost-complex structure J on $S^2 \times S^2$ there exists a unique J -holomorphic sphere v in the homology class $(d, 1)$ and passing through the points. That the oriented count is 1 here is the statement that the corresponding Gromov-Witten invariant is 1. To see this, we can place d points on $0 \times S^2$ and d points on $\infty \times S^2$ and the remaining point p_{2d+1} elsewhere. Then for the standard product complex structure curves through p_1, \dots, p_{2d} correspond to meromorphic functions on $\mathbb{C}P^1$ with specified zeros and poles. Such functions are well-defined up to scale and the scale is fixed by p_{2d+1} . That there is in fact a unique sphere for any J now follows from automatic regularity in dimension 4, see [3], [8], which implies that all curves are positively oriented.

Now we ‘stretch the neck’ along ∂E following [1]. The result is a holomorphic building, see [4], consisting of holomorphic curves in completions of E and $S^2 \times S^2 \setminus E$ and in the symplectization $\mathbb{R} \times \partial E$. Generically all components have deformation index 0. Multiply covered components can be excluded as in Lemma 4.1.

We focus on the components lying in $S^2 \times S^2 \setminus E$. Suppose that such a component F has s_1^- negative ends asymptotic to multiples of γ_1 , and s_2^- negative ends asymptotic to multiples of γ_2 . If the i^{th} negative end covering γ_1 does so a_i^- times, and the i^{th} negative end covering γ_2 does so b_i^- times,

then the virtual deformation index of the component is

$$\text{index}(F) = (-1)(2 - s_1^- - s_2^-) + 2c_1(F) - \sum_{i=1}^{s_1^-} \mu(\gamma_1^{(a_i^-)}) - \sum_{i=1}^{s_2^-} \mu(\gamma_2^{(b_i^-)}).$$

With our choices of trivialization the Chern class $c_1(F) = 2d + 2$.

The Conley-Zehnder index $\mu(\gamma_2) = 2 + 2(2d + 1) + 1$. Thus any component with a negative end asymptotic to a multiple of γ_2 has deformation index at most

$$(-1)(2 - 1) + 2(2d + 2) - 2 - 2(2d + 1) - 1 = -2$$

and so such components generically do not exist.

Similarly no component can have a negative end asymptotic to $\gamma_1^{(r)}$ for $r > 2d + 1$. Here the relevant Conley-Zehnder index is $\mu(\gamma_1^{(r)}) = 2r + 2 \left\lfloor \frac{r}{s_1} \right\rfloor + 1$, and hence the deformation index is at most

$$(-1)(2 - 1) + 2(2d + 2) - (2r + 3) = 4d - 2r < -2.$$

So suppose that we have K components with a total number s of negative ends each asymptotic to $\gamma_1^{(r_i)}$. Then the sum of the indices of these components is

$$\begin{aligned} & -2K + s + 2(2d + 2) - \sum_{i=1}^s \mu(\gamma_1^{r_i}) \\ & = -2K + 2(2d + 2) - \sum_{i=1}^s 2r_i. \end{aligned}$$

Now, as in [4], by monotonicity we may assume that the components inside E have total area at least $2d + 1$ (by situating the points at the center of disjoint balls of radius close to 1, see [13] for this). It follows that $\sum_{i=1}^s r_i \geq 2d + 1$ and so

$$\text{index} \leq -2K + 2(2d + 2) - 2(2d + 1) = -2K + 2.$$

Therefore for generic J we must have $K = 1$.

In summary, after stretching the neck we have a single component F in $S^2 \times S^2 \setminus E$ with, say, s negative ends each asymptotic to a multiple of γ_1 . The components in $E \cup (\mathbb{R} \times \partial E)$ therefore must fit together to form s disks

which can (abstractly at least) be glued to the ends of F to form our original genus 0 curve.

Suppose that $s > 1$. Then we can pick two of the points, for convenience say p_1 and p_2 , which lie in different components in $E \cup (\mathbb{R} \times \partial E)$. Consider families of $2d + 1$ points $\{p_1(t), \dots, p_{2d+1}(t)\}$ in E which switch p_1 and p_2 and leaves the other points fixed. More precisely, suppose that $p_i(0) = p_i$ for all i , $p_1(1) = p_2$, $p_2(1) = p_1$, and $p_i(1) = p_i$ for all $i > 2$. For any tame almost-complex structure J on $S^2 \times S^2$ there exist corresponding families of J -holomorphic spheres C_t in the class $(d, 1)$ passing through the points $p_1(t), \dots, p_{2d+1}(t)$. By our computation of the Gromov-Witten invariant we observe that $C_0 = C_1$. Set $J = J_N$ where J_N is the result of stretching the neck along ∂E to a length N . Then by Proposition 2.13, [4], (or its exact analogue in our case which we review now) the components of the limits of the C_t in $S^2 \times S^2 \setminus E$ all coincide. The proof of this result proceeded by contradiction. If the components differ then, since curves of index 0 in $S^2 \times S^2 \setminus E$ are isolated, there exists a $t_0 \in [0, 1]$ such that the family of J_N -holomorphic spheres C_{t_0} converge as $N \rightarrow \infty$ to a building having a nonrigid component in $S^2 \times S^2 \setminus E$, or in other words a component of deformation index greater than 0. In fact, since the index formulas are all even, the component has index at least 2. But if the $p_i(t)$ are in sufficiently general position then components in $E \cup (\mathbb{R} \times \partial E)$ must all have index at least -1 for all t , and so in fact nonnegative index. This contradicts the conservation of indices in the limit.

Therefore, for all N sufficiently large, the intersection of our J_N -holomorphic spheres C_t with $S^2 \times S^2 \setminus E$ are all C^∞ close and are embedded spheres with s disks removed. But for C_0 one boundary is connected to a disk in E passing through p_1 whereas for C_1 the same boundary is connected to a disk passing through p_2 , contradicting the fact that $C_0 = C_1$. Thus $s = 1$.

In conclusion, we have constructed a holomorphic plane in $S^2 \times S^2 \setminus E$ with a single end asymptotic to $\gamma_1^{(2d+1)}$ as required, and Lemma 4.2 is proved. \square

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