Hamiltonian displacement of bidisks inside cylinders

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1 Introduction

The displacement energy of a subset $U \subset \mathbb{R}^{2n}$ is defined by

$$e(U) = \inf\{\|H\| | \phi_H(U) \cap U = \emptyset\}$$

where the infimum is taken over all compactly supported (Hamiltonian) functions $H : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$, and writing $H_t(x) = H(x,t)$ the Hofer norm $\|H\| = \int_0^1 \sup_x H_t(x) - \inf_x H_t(x) dt$. The diffeomorphism ϕ_H is the Hamiltonian diffeomorphism generated by H, that is, the time-1 flow of the time-dependent vectorfield X_t defined by $X_t \rfloor \omega = dH_t$. The form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ is the standard symplectic form on $\mathbb{R}^{2n}$ in coordinates $x_1, y_1, ..., x_n, y_n$. It is sometimes convenient to write $\|\phi\| = \inf \|H\|$ where the infimum is taken over all Hamiltonian functions H satisfying $\phi_H = \phi$.

The notion of displacement energy extends to arbitrary symplectic manifolds, in particular to subsets $M \subset \mathbb{R}^{2n}$. In this case the displacement energy is defined by

$$e^M(U) = \inf\{\|H\| | \phi_H(U) \cap U = \emptyset\}$$

where the infimum is now taken over Hamiltonian functions with compact support in $M \times [0,1]$.

We emphasize that, in cases when U can be displaced within M, for $e^M(U)$ to differ from $e(U)$ it is important that the support of the functions H and not just the image $\phi_H(U)$ lie in M. In fact, suppose that a Hamiltonian diffeomorphism ψ satisfies $\psi(U) = V$ where $U \cap V = \emptyset$. If there exists a $W \subset M$ which is Hamiltonian diffeomorphic to U but with $U \cap W = \emptyset$ then we can find another Hamiltonian diffeomorphism f with support disjoint from U such that $f(V) = W$. So $\phi = f\psi f^{-1}$ satisfies $\phi(U) = W$ but by the invariance of the Hofer norm ϕ can be generated by Hamiltonians of the same norm as ψ.

For convenience of notation we will frequently identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, writing $z_j = x_j + iy_j$ in standard coordinates. In this paper we will focus on the displacement energy of bidisks

$$D(a, b) = \{\pi|z_1|^2 < a, \pi|z_2|^2 < b\} \subset \mathbb{C}^2$$

where $a \leq b$.

In his original work on the subject [7], [8], H. Hofer showed that $e(D(a, b)) = a$. In fact the infimum can be realized by a Hamiltonian function $H(z_1)$ depending only upon z_1. On the other hand, if $D(a, b) \subset M$ and volume$(M) < 2ab$ we clearly have $e^M(D(a, b)) = \infty$.

A natural candidate for M where we might expect strict inequalities

$$e(D(a, b)) < e^M(D(a, b)) < \infty$$

is the cylinder $M = Z(a + \epsilon) = \{\pi|z_1|^2 < a + \epsilon\}$ for ϵ small. If $\epsilon > a$ then again we have $e^{Z(a+\epsilon)}(D(a, b)) = a$. By translating vertically we see that $e^{Z(a+\epsilon)}(D(a, b)) \leq b$ but it is not immediately clear that $e^{Z(a+\epsilon)}(D(a, b)) > e(D(a, b)) = a$ for any $\epsilon > 0$.

Our main theorem gives fairly tight estimates for the displacement energies of bidisks inside cylinders.

Theorem 1.1. *Let* $Z = Z(1 + \epsilon)$. *Then*

$$(\frac{1}{2} - \epsilon)\lfloor S \rfloor + \epsilon \leq e^Z(D(1, S)) \leq \frac{S}{2} + \frac{1}{2}.$$

The upper bound here is established by an explicit construction in section 2. The lower bound relies on some symplectic embedding obstructions.

We recall the main theorem from [6]. Let $B^{2n}(A) \subset \mathbb{R}^{2n}$ denote the round ball with capacity A, that is, of radius r satisfying $\pi r^2 = A$.

Theorem 1.2. *For any* $0 < A < 3$ *there are no symplectic embeddings of* $D^2(1) \times B^{2(n-1)}(S)$ *into* $B^4(A) \times \mathbb{R}^{2(n-2)}$ *when* S *is sufficiently large.*

The paper [6] also stated an analogous theorem for embeddings into bidisks.

Theorem 1.3. *If $a < 2$ then there are no symplectic embeddings of $D^2(1) \times B^{2(n-1)}(S)$ into $D(a,b) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

In order to deduce the lower bound in Theorem 1.1 we will apply a quantitative version of Theorem 1.3.

Let $E(a,b,c) \subset \mathbb{C}^3$ denote the ellipse

$$E(a,b,c) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \,\middle|\, \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} + \frac{\pi|z_3|^2}{c} \leq 1 \right\}.$$

Theorem 1.4. *Let $d \geq 1$ be an integer. If there exist symplectic embeddings*

$$E(1, 2d+1, 2d+1) \hookrightarrow D(1+\epsilon, T) \times \mathbb{C}$$

then $T \geq d(1-\epsilon) + 1$.

In section 3 we show how to derive the lower bound in Theorem 1.1 from Theorem 1.4. This will be done using the technique of symplectic folding.

In section 4 we prove Theorem 1.4. This follows [6] closely, but we need to exercise care with the dimensions of the ellipse. However there are some simplifications resulting from considering embeddings into bidisks rather than balls, as was the focus in [6].

2 Displacing a bidisk

This section is devoted to proving the following. Let $\epsilon > 0$ and as before set $Z = Z(1+\epsilon)$.

Theorem 2.1. $e^Z(D(1,S)) \leq \frac{S}{2} + \frac{1}{2}$.

We fix a $0 < \delta << \epsilon$ and the proof will consist of explicitly constructing a Hamiltonian diffeomorphism ϕ with a generating Hamiltonian of norm less than $\frac{S}{2} + \frac{1}{2} + \delta$. By abuse of notation, occasionally we will also simply write δ or ϵ for quantities differing only by a universal constant.

Changing notation slightly, we will use coordinates (u,v,x,y) on $\mathbb{R}^4$. We define $p : \mathbb{R}^4 \to \mathbb{R}^2$ to be the projection onto the (u,v)-plane and set

$$D(1,S) = \left\{ 0 < u < 1, 0 < v < 1, 0 < x < 1, \frac{-S}{2} < y < \frac{S}{2} \right\} \subset \mathbb{R}^4$$

and

$$Z = p^{-1}\{(u,v) \in D\} \subset \mathbb{R}^4$$

where D is a region of area $1+\epsilon$ containing $D(1) = p(D(1,S))$. Up to Hamiltonian diffeomorphism these domains are equivalent to those previously defined.

The diffeomorphism ϕ will be a composition of four Hamiltonian diffeomorphisms which we define now in the following steps.

Step 1

Let $H_1(y)$ satisfy the following

- $H_1(y) = 0$ whenever $|y| > \frac{S}{2} + \delta$
- $H_1'(y) = 1 + \delta$ if $\frac{-S}{2} < y < -\delta$
- $H_1'(y) = -1 - \delta$ if $\delta < y < \frac{S}{2}$

Such functions H_1 clearly exist with $\|H_1\| \leq (1+\delta)\frac{S}{2}$. We call the resulting Hamiltonian diffeomorphism ψ_1.

Let ξ_1 be a Hamiltonian diffeomorphism also generated by a Hamiltonian function of x, y but supported in $\{x < 0\} \cup \{x > 1\}$. Then $\phi_1 = \xi_1 \circ \psi_1 \circ \xi_1^{-1}$ is also a Hamiltonian diffeomorphism with $\|\phi_1\| = \|\psi_1\| \leq (1+\delta)\frac{S}{2}$ by the biinvariance of the Hofer metric. We choose ξ_1 such that the image of $D(S) = \left\{0 < x < 1, \frac{-S}{2} < y < \frac{S}{2}\right\}$ under ϕ_1, which is just $\xi_1(\psi_1(D(S)))$, is contained in the region described by Figure 1.

Let $F_1 = \phi_1(D(1,S))$.

Step 2

Let $n = \lceil \frac{1}{2\epsilon} \rceil$. Then we can divide $D(1)$ into $2n$ vertical strips R_i of area less than ϵ, see Figure 2.

Let $b(u,v)$ be a function supported in D with level-sets as shown in Figure 2. So for all i, we have that $b = 0$ on $R_i \cap R_{i+1}$, $b < 0$ away from the boundary of R_i if i is odd, and $b > 0$ away from the boundary of R_i if i is even. Define $H_2(u,v,x) = b(u,v)x$.

We can choose b such that the derivative $b' = 0$ on $b^{-1}(0)$, and assume that t is sufficiently small that if $|b| < t$ and $|x| < 2$ then the (u,v)-component, xX_b, of the Hamiltonian flow of H_2, which has modulus xb', is bounded by $\frac{\epsilon}{n}$ and so the flow restricted to $\{|b| < t\} \cap D(1)$ remains in an $\frac{\epsilon}{n}$ neighborhood of $\{|b| < t\} \cap D(1)$. Furthermore, we can suppose that $\{|b| < t\} \cap R_i$ lies in an $\frac{\epsilon}{n}$-neighborhood of ∂R_i.

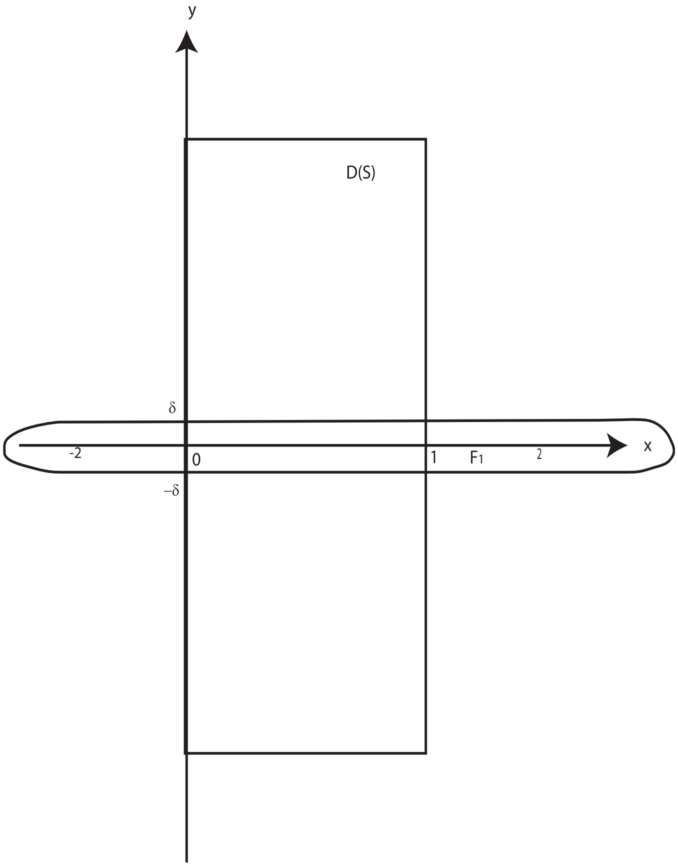

Figure 1: *The region F_1*

The choice of t here is independent of δ and so we may assume that $\delta << t < \epsilon$.

Let $\chi : \mathbb{R} \to [0, 1]$ be a bump function with support in $\{|x| < 3\}$ and with $\chi = 1$ whenever $|x| < 2$. Let ϕ_2 be the diffeomorphism generated by $\chi(x)H_2$. We see that $\|\phi_2\| \leq 3\|b\| < \epsilon$ using our conventions, and also note that ϕ_2 preserves both x and $b \circ p$.

If $(u', v', x, y') = \phi_2(u, v, x, y)$ and $|x| < 2$ then $y' = y + b(u, v)$ and so $|y' - b| < \delta$. (Since $|y| < \delta$, see Figure 1.)

For $1 \leq i \leq n$ we define E_i^+ to be the component of $\{b > \frac{t}{4}\}$ intersecting R_{2i} and E_i^- to be the component of $\{b < \frac{t}{4}\}$ intersecting R_{2i-1}. Set $E^+ = \bigcup_i E_i^+$ and $E^- = \bigcup_i E_i^-$. For $1 \leq i \leq 2n + 1$, let G_i be an $\frac{\epsilon}{n}$-neighborhood of the component of $\{|b| < t\}$ intersecting R_{i-1} and R_i, set $G = \bigcup_i G_i$. We

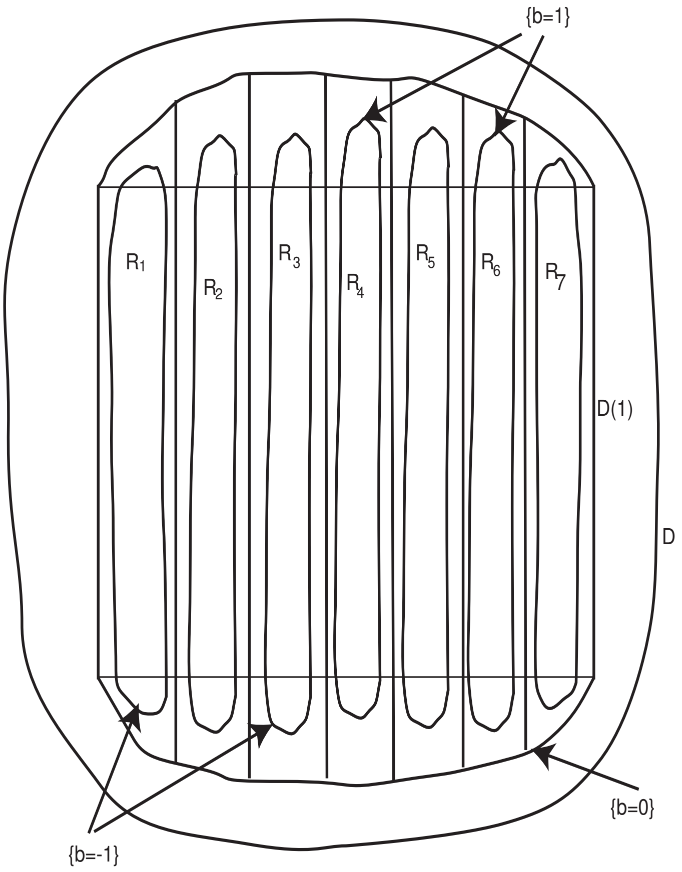

Figure 2: *Level sets of b*

may assume that G is a disjoint union of disks of total area bounded by δ.

Let $F_2 = \phi_2(F_1)$.

Provided that δ is sufficiently small and $|r| < 2$, the level sets $L_{r,s} = F_2 \cap \{x = r, y = s\}$ satisfy the following.

- if $s > \frac{t}{2} - \delta$ then $L_{r,s} \subset E^+$;
- if $s < \frac{-t}{2} + \delta$ then $L_{r,s} \subset E^-$;
- if $-\frac{t}{2} - \delta \leq s \leq \frac{t}{2} + \delta$ then $L_{r,s} \subset G$.

Step 3

We notice that the fibers of $F_2 \cap \{|x| < 2\}$ under p which project to $E^+ \setminus G$ are contained in $\{y > \frac{t}{4} - \delta\}$ and so are distinct from the corresponding fibers over $E^- \setminus G$. We will use this to carry out an analogue of symplectic folding ϕ_3 such that the projection $p(\phi_3(F_2)\cap\{|x|<2\})$ lies in a small neighborhood of $E^- \cup G$.

Up to symplectomorphism we can think of $D(1) \subset D$ as resembling a flower, with $G \subset D(\delta)$ lying in a central bud and the components of E^+ and E^- forming petals surrounding the bud. Hence there exists a Hamiltonian diffeomorphism ψ_3 of D having norm roughly $\frac{1}{2n}$ which rotates an arbitrarily large subset of $E^+ \setminus G$ onto $E^- \setminus G$ and is cut-off to equal the identity on $D(\delta) \supset G$. Denote the generating Hamiltonian by $H(u,v)$.

Thinking now of H as a function on $\mathbb{R}^4$, we see that $H|_{F_2}$ is constant on the region $\{0 < x < 1, -\frac{t}{2} - \delta \leq y \leq \frac{t}{2} + \delta\}$. Thus we can replace $H|_{0<x<1}$ by another function, say $\tilde{H}$, which is equal to H when $y > \frac{t}{2}$ but which is constant on F_2 for $y < \frac{t}{2}$. The Hamiltonian flow generated by $\tilde{H}$ when applied to $F_2 \cap \{0 < x < 1\}$ preserves the coordinates x and y and moves a large compact subset of $F_2 \cap p^{-1}(E^+ \setminus G)$ into $p^{-1}(E^- \setminus G)$. Hence the projection of the image lies in a small neighborhood of $D(\delta) \cup E^-$.

We need to extend the flow of $\tilde{H}$ to a Hamiltonian flow defined on all of F_2. The resulting diffeomorphism will be denoted by ϕ_3 and we will set $F_3 = \phi_3(F_2)$. We will do this in such a way that the flow still preserves x and such that $p(F_3 \cap \{0 < x < 1\}) \subset D(\delta) \cup E^-$.

Let $\chi : \mathbb{R} \to [0,1]$ be a function which is increasing on $[-2,0]$, decreasing on $[1,2]$ and with $\chi(x) = 0$ if $|x| > 2$ and $\chi(x) = 1$ if $0 < x < 1$.

The function $\tilde{H}$ extends trivially as a smooth function on $\{|y| > \frac{t}{2}\} \cup p^{-1}D(\delta)$ (as $H \circ p$ on $\{y > \frac{t}{2}\}$ and as a constant on $p^{-1}D(\delta) \cup \{y < \frac{t}{2}\}$), but even though it contains $F_2 \cap \{|x| < 2\}$ this region does not include all of F_2. Therefore we define $H_3 = \chi(x)\tilde{H}$.

This defines a smooth extension of $\tilde{H}$ to $R = \{|y| > \frac{t}{2}\} \cup p^{-1}D(\delta) \cup \{|x| > 2\}$. Furthermore $\frac{\partial H_3}{\partial y} = 0$ on R and so the corresponding flow preserves the x coordinate as required, and is in fact the identity on $\{y < \frac{t}{2}\}$. To ensure that the flow does indeed exist we require F_2 to remain in R. For this, we compute $\frac{\partial H_3}{\partial x} = \chi'\tilde{H}$. We may assume that $|\chi'| \leq 1$ and so the magnitude of the y component of our Hamiltonian vectorfield is bounded by $|\tilde{H}|$. Hence the flow will exist for time 1 provided that the y coordinate of all points $z \in F_2 \cap \{y > \frac{t}{2}\}$ is at least $\tilde{H}(z)$. But this y coordinate is roughly equal to $b(p(z))$, see Step 2. We can see from Figure 2 that each E_i^+ has boundary components disjoint from G. Therefore we may assume that b is arbitrarily large on arbitrarily large subsets of E_i^+ disjoint from G, and in particular exceeds $\tilde{H}$, which is bounded by $\frac{1}{2n}$.

Hence ϕ_3 is a well defined extension of the flow of $\tilde{H}$, and as it presreves x we do indeed satisfy the condition $p(F_3 \cap \{0 < x < 1\}) \subset D(\delta) \cup E^-$.

Step 4

Again if δ is sufficiently small we may assume that there exists a Hamiltonian diffeomorphism ψ_4 of norm roughly $\frac{1}{2}$ mapping $D(\delta) \cup E^-$ into $T = D \setminus (D(\delta) \cup E^-)$.

Let $c(u, v)$ be a bump function supported in T and equal to 1 in a neighborhood of the image of ψ_4. Let ξ be the Hamiltonian diffeomorphism generated by $Kc(u, v)x$, where K is very large. We set $\phi_4 = \xi \circ \psi_4 \circ \xi^{-1}$ and $F_4 = \phi_4(F_3)$. Then $\|\phi_4\| = \|\psi_4\| \simeq \frac{1}{2}$.

We note that ϕ_4 preserves x and thus $\{0 < x < 1\}$. Therefore as $F_3 \cap \{0 < x < 1\} \subset p^{-1}(D(\delta) \cup E^-)$ we have $F_4 \cap \{0 < x < 1\} \subset \{y > K\}$ which is disjoint from $\{|y| < S\}$ and so we have displaced the bidisk from itself as required.

Up to factors of order ϵ, δ we have $\|\phi_1\| \leq \frac{S}{2}$ and $\|\phi_4\| \leq \frac{1}{2}$. The norms $\|\phi_2\|$ and $\|\phi_3\|$ can be taken to be arbitrarily small.

Thus up to an arbitrarily small error the norm of the composition is bounded by $\frac{S}{2} + \frac{1}{2}$ and we have established Theorem 2.1 as required.

3 Symplectic embeddings and displacement energy

Let e denote the displacement energy $e^Z(D(1, S))$ where as usual Z is the cylinder $Z(1 + \epsilon) \subset \mathbb{C}^2$.

Then for any $\delta > 0$ we can find a Hamiltonian function H of norm $e + \delta$ generating a flow which displaces $D(1, S)$ inside Z.

Let $\chi(x_3)$ be a smooth increasing function equal to 0 when $x_3 \leq 0$ and 1 when $x_3 \geq 1$.

Let V be a δ-neighborhood in a z_3-plane of the union of disks D_1 and D_2 of radius $\sqrt{\frac{S}{2\pi}}$ centered at $(-\sqrt{\frac{S}{2\pi}}, 0)$ and $(\sqrt{\frac{S}{2\pi}} + 1, 0)$ respectively, and the interval $[0, 1]$ on the x_3-axis. Then there exists a symplectic embedding $D(S) \hookrightarrow V$ and by taking a product $f : D(1, S) \times D(S) \hookrightarrow Z \times V$.

We apply the Hamiltonian diffeomorphism ϕ generated by $\chi(x_3)H(z_1, z_2)$ to the image of f. If we define $V' = V \cup [0, 1] \times [0, e + \delta]$ then $U =$

$\phi(\text{image}(f)) \subset Z \times V'$ and the fibers of U over the disk D_1 are disjoint from the fibers over the disk D_2.

Thus if g is a smooth map of V' which preserves ω_0 and sends D_1 onto D_2 then id. $\times\, g|_U$ gives a symplectic embedding of U into a domain symplectomorphic to $Z \times D(\frac{S}{2} + e + \delta)$.

We recall that $E(a, b, c) \subset \mathbb{C}^3$ denotes the ellipse

$$E(a,b,c) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \,\middle|\, \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} + \frac{\pi |z_3|^2}{c} \leq 1 \right\}.$$

The following can be established using the technique of symplectic folding, as we will recall momentarily.

Lemma 3.1. *For any $r > 1$ there exists a symplectic embedding*

$$E(1, 2d+1, 2d+1) \hookrightarrow D(r) \times D(r(d+1)) \times D(r(d+1)).$$

Putting everything together, if $S = (d + 1)$ then we have a symplectic embedding

$$E(1,(2d{+}1),(2d{+}1)) \hookrightarrow rZ \times D(r(\frac{S}{2}{+}e{+}\delta)) = D(r(1{+}\epsilon), r(\frac{S}{2}{+}e{+}\delta)) \times \mathbb{C}.$$

Applying Theorem 1.4 then, with $r \to 1^+$, we find $\frac{S}{2}+e+\delta \geq d(1-\epsilon)+1$. Therefore the displacement energy $e \geq d(\frac{1}{2} - \epsilon) + \frac{1}{2} - \delta$ and letting $\delta \to 0$ this gives our result as required.

Proof of Lemma 3.1

We construct the embedding by performing a sympectic fold twice, for a detailed study of symplectic folding see [17].

Write $E = E(1, 2d+1, 2d+1)$. If we project E to the z_2-plane, the fibers are ellipses. At a point where $\pi |z_2|^2 = k(2d + 1)$ the fiber is $E((1 - k), (1 - k)(2d + 1))$, where similarly to the above we write

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^3 \,\middle|\, \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\}.$$

The projection to the z_2-plane is a disk of area $2d + 1$ and we can identify it with a δ-neighborhood of the union two disks D_1^2 and D_2^2 of area $\frac{2d+1}{2}$ centered at $(-\sqrt{\frac{2d+1}{2\pi}}, 0)$ and $(\sqrt{\frac{2d+1}{2\pi}} + 1, 0)$ respectively, and the interval $[0, 1]$ on the x_2-axis. We may further assume that under this identification points with $\pi|z_2|^2 > \frac{2d+1}{2}$ map into the disk D^2_2. By abuse of notation, we will now denote by z_2 the push-forward of the standard z_2-coordinate under this identification of regions in the plane. We can do the same for the projection to the z_3-plane, so that it lies in a δ neighborhood of the union of the same two disks D^3_1 and D^3_2 thought of now as lying in the z_3-plane and the interval $[0, 1]$ on the x_3-axis.

Now, if $\pi|z_2|^2 > \frac{2d+1}{2}$, the projection to the z_1-plane of the corresponding fiber is contained in the disk centered at the origin of area $\frac{1}{2}$, and can be displaced within the disk of area $r\pi$ by a Hamiltonian diffeomorphism generated by a Hamiltonian of norm $\frac{1}{2}$. In fact, we can arrange this Hamiltonian such that away from a δ-neighborhood of the boundary points with $\pi|z_1|^2 < s$ are mapped to those with $\pi|z_1|^2 > 1 - s$ for all $s < \frac{1}{2}$. Thus the fiber over a point with $\pi|z_2|^2 > k(2d + 1)$ for $k > \frac{1}{2}$ now lies in $\{\pi|z_1|^2 > k\}$.

Similarly to the above, let $\chi(x_2)$ be a smooth increasing function equal to 0 when $x_2 \leq 0$ and 1 when $x_2 \geq 1$ and apply the Hamiltonian flow generated by χH. This is the identity on points projecting onto D^2_1 and the image is such that the fibers over $\{\pi|z_2|^2 = k(2d + 1)\} \subset D^2_2$ are disjoint from those over $\{\pi|z_2|^2 = (1 - k)(2d + 1)\} \subset D^2_1$. The projection to the z_2 plane of the image lies in the union of D^2_1 and D^2_2 and the rectangle $[0, 1] \times [0, \frac{1}{2}]$.

We note that in the ellipse E, if $\pi|z_2|^2 > \frac{2d+1}{2}$ then $\pi|z_3|^2 \leq \frac{2d+1}{2}$ and vice versa. Therefore we can carry out the same construction supported at points where $\pi|z_2|^2 < \frac{2d+1}{2}$ in order to arrange that the fibers of the projection to the z_3-plane over D^3_2 are disjoint from the corresponding fibers over D^3_1. We call the resulting domain E'.

Now we apply symplectic folding. Namely we apply the composition of two maps $\psi_2 = \phi_2 \times \mathrm{id}_{13}$ and $\psi_3 = \phi_3 \times \mathrm{id}_{12}$ where ϕ_2 is a map of the z_2-plane taking D^2_2 onto D^2_1 and id_{13} is the identity on the (z_1, z_3)-planes. Similarly ϕ_3 is a map of the z_3-plane taking D^3_2 onto D^3_1 and id_{12} is the identity on the (z_1, z_2)-planes. The maps ϕ_2 and ϕ_3 are area preserving. For all $s < \frac{1}{2}$ we ensure that ϕ_2 takes points with $\pi|z_2|^2 > (1 - s)\frac{2d+1}{2}$ to points with $\pi|z_2|^2 < s\frac{2d+1}{2}$. Similarly ϕ_3 takes points with $\pi|z_3|^2 > (1 - s)\frac{2d+1}{2}$ to points with $|z_3|^2 < s\frac{2d+1}{2}$.

This composition applied to E', written as described, has image E'' contained in a neighborhood of

$$D(1) \times (D^2_1 \cup [0, 1] \times [0, \frac{1}{2}]) \times (D^3_1 \cup [0, 1] \times [0, \frac{1}{2}]).$$

Thus since $D^2_1 \cup [0, 1] \times [0, \frac{1}{2}]$ and $D^3_1 \cup [0, 1] \times [0, \frac{1}{2}]$ are symplectomorphic to disks of area $\frac{2d+1}{2}+\frac{1}{2}$ the proof of the lemma is complete once we show that the composition is injective.

If a point in E' has coordinates with $(\pi|z_1|^2, \pi|z_2|^2, \pi|z_3|^2) = (r,s,t)$ and $s > \frac{2d+1}{2}$ then $r > \frac{s}{2d+1}$ and the image under the fold ψ_2 is a point with coordinates $(r, 2d+1-s, t)$. This is disjoint from E' because $r + \frac{2d+1-s}{2d+1} + \frac{t}{2d+1} > 1$. Similarly the image of the support of ψ_3 is disjoint from E'.

Finally suppose that a point with coordinates $(|z_1|^2, |z_2|^2, |z_3|^2) = (r,s,t)$ and $s > \frac{2d+1}{2}$ is mapped until ψ_2 to the image under ψ_3 of a point with coordinates (r', s', t') and $t' > \frac{2d+1}{2}$. Then $(r, 2d+1-s, t) = (r', s', 2d+1-t')$. As (r,s,t) are the coordinates of a point in E' we know that $\frac{s+t}{2d+1} < 1$, but this implies that $\frac{2d+1-s'+2d+1-t'}{2d+1} < 1$ or $\frac{s'+t'}{2d+1} > 1$ giving a contradiction.

4 An obstruction to symplectic embeddings

Here we prove the following. The notation is preserved from the previous sections. We also now denote by $S^2(a) \times S^2(b)$ the manifold $S^2 \times S^2$ with the product symplectic form such that the first factor has area a and the second area b.

Theorem 4.1. *Let $S_1, S_2 > 2d+1$ for an integer $d \geq 1$ and suppose that there exists a symplectic embedding*

$$E(1, S_1, S_2) \hookrightarrow S^2(1+\epsilon) \times S^2(T) \times \mathbb{C}.$$

Then $T \geq (1-\epsilon)d + 1$.

As there exists an embedding (by inclusion) $D(1+\epsilon, T) \times \mathbb{C} \hookrightarrow S^2(1+\epsilon) \times S^2(T) \times \mathbb{C}$, Theorem 4.1 implies Theorem 1.4 as required.

Theorem 4.1 is a quantitative version of Theorem 1.3 from [6]. The proof of Theorem 1.3 was omitted in [6] as it is entirely analogous to Theorem 1.2 concerning embeddings into $B^4(R) \times \mathbb{C}$, or more generally $\mathbb{C}P^2 \times \mathbb{C}$ (and in fact it is slightly simpler). We include the proof here for completeness, outlining parts which already appear in [6].

Proof of Theorem 4.1

We may assume that $1, S_1, S_2$ are linearly independent over $\mathbb{Q}$. Then the characteristic line field on $\partial E(1, S_1, S_2) \subset \mathbb{C}^3$ has exactly three closed orbits. These are the circles $\gamma_i = \{z_j = 0; j \neq i\}$ with actions 1, S_1 and S_2

respectively. With respect to the trivialization induced from $\mathbb{C}^3$ the Conley-Zehnder index of the r-fold cover $\gamma_1^{(r)}$ of γ_1 is given by

$$\mu(\gamma_1^{(r)}) = 2r + \left(2\left\lfloor \frac{r}{S_1} \right\rfloor + 1\right) + \left(2\left\lfloor \frac{r}{S_2} \right\rfloor + 1\right).$$

In particular, if $r \leq 2d+1$ the index $\mu(\gamma_1^{(r)}) = 2r+2$.

Now suppose that there exists a symplectic embedding

$$E(t, tS_1, tS_2) \hookrightarrow S^2(1+\epsilon) \times S^2(T) \times \mathbb{C} = X.$$

Then $X \setminus E(t, tS_1, tS_2)$ has the structure of a symplectic manifold with a concave end symplectomorphic to $\partial E \times (-\infty, 0]$ and compatible almost-complex structures can be defined as in [6], section 2.4. Original sources for this are [4] or even [9].

For any such almost-complex structure there are moduli spaces of finite energy J-holomorphic planes mapping into $X \setminus E$ and exponentially asymptotic to multiple covers of $\gamma_1 \times (-\infty, 0]$ outside of a compact set. The reparameterization group $G = \mathrm{Aut}(\mathbb{C})$ acts on such planes.

If such a plane u is asymptotic to $\gamma_1^{(r)}$ then we write $u \sim \gamma_1^{(r)}$. In this case we can add an r-fold cover of the disk $\{z_2 = z_3 = 0\} \subset E$ to the image of u in order to construct a 2-dimensional homology class $[u] \in H_2(X)$. We will write $[u] = (k, l)$ if $[u] \bullet (\text{pt.} \times S^2) = k$ and $[u] \bullet S^2 \times (\text{pt.}) = l$.

Lemma 4.2. *(i) Let*

$$\mathcal{M}(J) = \{u : \mathbb{C} \to X \setminus E | \mid \bar{\partial}_J u = 0, [u] = (d,1), u \sim \gamma_1^{(2d+1)}\}/G.$$

Then for generic J the moduli space $\mathcal{M}(J)$ is a compact 0-dimensional manifold.

(ii) Given a family of embeddings $E(t, tS_1, tS_2) \hookrightarrow X$ for $\epsilon \leq t \leq 1$ and a corresponding family J_t of compatible almost-complex structures we define

$$\mathcal{M}(\{J_t\}) = \{u : \mathbb{C} \to X \setminus E, t \in [0,1] | \mid \bar{\partial}_{J_t} u = 0, [u] = (d,1), u \sim \gamma_1^{(2d+1)}\}/G.$$

Then for generic families $\{J_t\}$ the moduli space $\mathcal{M}(\{J_t\})$ is a compact 1-dimensional manifold, giving a cobordism between $\mathcal{M}(J_\epsilon)$ and $\mathcal{M}(J_1)$.

Proof. The relevent index formula here, see [4], gives the deformation index of such a finite energy plane for fixed J as

$$\text{index} = 2c_1([u]) - \mu(\gamma_1^r) = 2(2d+2) - (2(2d+1)+2) = 0.$$

The various compactness statements follow exactly as in [3], the point being that any bubbling is of codimension at least 2. □

Let $\phi_\epsilon : E(\epsilon, \epsilon S_1, \epsilon S_2) \hookrightarrow X$ be a symplectic embedding which restricts to an embedding $E(\epsilon, \epsilon) \hookrightarrow S^2 \times S^2$ on $\{z_3 = 0\}$ and such that the image $\phi_\epsilon(E(\epsilon, \epsilon S_1, \epsilon S_2))$ is invariant under rotations about the origin in the z_3-plane. For ϵ sufficiently small such symplectic embeddings exist and are isotopic through embeddings of $E(t, tS_1, tS_2)$ for $\epsilon \leq t \leq 1$ to any given embedding of $E(1, S_1, S_2)$.

Lemma 4.3. *Let J_ϵ be a regular compatible almost-complex structure on $X \backslash \phi_\epsilon(E(\epsilon, \epsilon S_1, \epsilon S_2))$ which is invariant under rotations in the z_3-plane. Then $\mathcal{M}(J_\epsilon)$ contains a positive number of equivalence classes of curves, counting with multiplicity.*

It follows as in [6], Lemma 4.4 that such regular J_ϵ do indeed exist. Thus, given Lemma 4.3, $\mathcal{M}(J_\epsilon)$ represents a nontrivial cobordism class. Therefore by Lemma 4.2, if an embedding $\phi_1 : E(1, S_1, S_2) \hookrightarrow X$ exists and J_1 is a compatible almost-complex structure on $X \setminus \phi_1(E(1, S_1, S_2))$ then $\mathcal{M}(J_1)$ is also nonempty.

The compatibility condition for J_1 implies that curves u in $\mathcal{M}(J_1)$ have positive symplectic area. Computing, this area is $d(1 + \epsilon) + T - 2d + 1 \geq 0$, and so we obtain the inequality required for Theorem 4.1.

Proof of Lemma 4.3. Any curves in $\mathcal{M}(J_\epsilon)$ must lie in $\{z_3 = 0\}$ since any other curves would appear in a 1-dimensional family (given by rotating the z_3-plane) and so could not be regular.

Thus we can focus on an embedding $\phi_\epsilon : E(\epsilon) = E(\epsilon, \epsilon S_1, \epsilon S_2) \to S^2 \times S^2$ and look for finite energy curves in $S^2 \times S^2 \setminus \phi_\epsilon(E(\epsilon))$ asymptotic to $\gamma_1^{(2d+1)}$. It follows from work of C. Wendl, [18], see also [6], Lemma 4.4, that any such curves have positive orientation.

Fix points $p_1, ..., p_{2d+1} \in \phi_\epsilon(E(\epsilon)) = E$. Then given a generic tame almost-complex structure J on $S^2 \times S^2$ there exists a unique J-holomorphic sphere v in the homology class $(d, 1)$ and passing through the points. That the oriented count is 1 here is the statement that the corresponding Gromov-Witten invariant is 1. To see this, we can place d points on $0 \times S^2$ and d points on $\infty \times S^2$ and the remaining point p_{2d+1} elsewhere. Then for the standard product complex structure curves through $p_1, ..., p_{2d}$ correspond to meromorphic functions on $\mathbb{C}P^1$ with specified zeros and poles. Such functions are well-defined up to scale and the scale is fixed by p_{2d+1}. That there is in fact a unique sphere for any J now follows from automatic regularity in dimension 4, see [5], [10], which implies that all curves are positively oriented.

Now we 'stretch the neck' along ∂E following [3]. The result is a holomorphic building, see [6], consisting of holomorphic curves in completions of E and $S^2 \times S^2 \setminus E$ and in the symplectization $\mathbb{R} \times \partial E$. Generically all components have deformation index 0.

We focus on the components lying in $S^2 \times S^2 \setminus E$. Suppose that such a component F has s_1^- negative ends asymptotic to multiples of γ_1, and s_2^- negative ends asymptotic to multiples of γ_2. If the i^{th} negative end covering γ_1 does so a_i^- times, and the i^{th} negative end covering γ_2 does so b_i^- times, then the virtual deformation index of the component is

$$\mathrm{index}(F) = (-1)(2 - s_1^- - s_2^-) + 2c_1(F) - \sum_{i=1}^{s_1^-} \mu(\gamma_1^{(a_i^-)}) - \sum_{i=1}^{s_2^-} \mu(\gamma_2^{(b_i^-)}).$$

With our choices of trivialization the Chern class $c_1(F) = 2d + 2$.

The Conley-Zehnder index $\mu(\gamma_2) = 2 + 2(2d+1) + 1$. Thus any component with a negative end asymptotic to a multiple of γ_2 has deformation index at most

$$(-1)(2-1) + 2(2d+2) - 2 - 2(2d+1) - 1 = -2$$

and so such components generically do not exist.

Similarly no component can have a negative end asymptotic to $\gamma_1^{(r)}$ for $r > 2d+1$. Here the relevant Conley-Zehnder index is $\mu(\gamma_1^{(r)}) = 2r + 2\left\lfloor \frac{r}{S_1} \right\rfloor + 1$. So in this case the deformation index is at most

$$(-1)(2-1) + 2(2d+2) - (2r+3) = 4d - 2r < -2.$$

So suppose that we have K components with a total number of s negative ends each asymptotic to $\gamma_1^{(r_i)}$. Then the sum of the indices of these components is

$$\begin{aligned} -2K + s + 2(2d+2) - \sum_{i=1}^{s} \mu(\gamma_1^{r_i}) \\ = -2K + 2(2d+2) - \sum_{i=1}^{s} 2r_i. \end{aligned}$$

Now, as in [6], by monotonicity we may assume that the components inside E have total area at least $2d+1$ (by situating the points at the center of disjoint balls). It follows that $\sum_{i=1}^{s} r_i \geq 2d+1$ and so

$$\text{index} \leq -2K + 2(2d+2) - 2(2d+1) = -2K + 2.$$

Therefore for generic J we must have $K = 1$.

In summary, after stretching the neck we have a single component F in $S^2 \times S^2 \setminus E$ with, say, s negative ends each asymptotic to a multiple of γ_1. The components in $E \cup (\mathbb{R} \times \partial E)$ therefore must fit together to form s disks which can (abstractly at least) be glued to the ends of F to form our original genus 0 curve.

Suppose that $s > 1$. Then we can pick two of the points, for convenience say p_1 and p_2, which lie in different components in $E \cup (\mathbb{R} \times \partial E)$. Consider families of $2d+1$ points $\{p_1(t), ..., p_{2d+1}(t)\}$ in E which switch p_1 and p_2 and leaves the other points fixed. More precisely, suppose that $p_i(0) = p_i$ for all i, $p_1(1) = p_2$, $p_2(1) = p_1$, and $p_i(1) = p_i$ for all $i > 2$. For any tame almost-complex structure J on $S^2 \times S^2$ there exist corresponding families of J-holomorphic spheres C_t in the class $(d,1)$ passing through the points $p_1(t), ..., p_{2d+1}(t)$. By our computation of the Gromov-Witten invariant we observe that $C_0 = C_1$. Set $J = J_N$ where J_N is the result of stretching the neck along ∂E to a length N. Then by Proposition 2.13, [6], (or its exact analogue in our case which we review now) the components of the limits of the C_t in $S^2 \times S^2 \setminus E$ all coincide. The proof of this result proceeded by contradiction. If the components differ then there exists a $t_0 \in [0,1]$ such that the family of J_N-holomorphic spheres C_{t_0} converge to a building having a nonrigid component in $S^2 \times S^2 \setminus E$, or in other words a component of deformation index greater than 0. In fact, since the index formulas are all even, the component has index at least 2. But if the $p_i(t)$ are in sufficiently general position then components in $E \cup (\mathbb{R} \times \partial E)$ must all have index at least -1 for all t, and so in fact nonnegative index. This contradicts the conservation of indices in the limit.

Therefore, for all N sufficiently large, the intersection of our J_N-holomorphic spheres C_t with $S^2 \times S^2 \setminus E$ are all C^∞ close and are embedded spheres with s disks removed. But for C_0 one boundary is connected to a disk in E passing through p_1 whereas for C_1 the same boundary is connected to a disk passing through p_2, contradicting the fact that $C_0 = C_1$. Thus $s = 1$.

In conclusion, we have constructed a holomorphic plane in $S^2 \times S^2 \setminus E$ with a single end asymptotic to $\gamma_1^{(2d+1)}$ as required, and Lemma 4.3 is proved. $\square$

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