New obstructions to symplectic embeddings

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June 30, 2009

Abstract

In this paper we establish new restrictions on symplectic embeddings of certain convex domains into symplectic vector spaces. These restrictions are stronger than those implied by the Ekeland-Hofer capacities. By refining an embedding technique due to Guth, we also show that they are sharp.

1 Introduction and main results

Consider $\mathbb{R}^{2n}$ equipped with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and its standard symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Let $B^{2n}(R)$ denote the ball of radius R in $\mathbb{R}^{2n}$. Gromov's Nonsqueezing Theorem states that there is no symplectic embedding of $B^{2n}(1)$ into $B^2(R) \times \mathbb{R}^{2(n-1)}$ for $R < 1$, [6]. The following analogous problem remains open.

Question 1. *What, if any, is the smallest value of $R > 0$ such that there exists a symplectic embedding of $B^2(1) \times \mathbb{R}^{2(n-1)}$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$?*

Prior to the current paper, the most that could be said is that if such an emebedding exists, then R must be at least $\sqrt{2}$. This bound is implied by the second Ekeland-Hofer capacity from [4].

In [7], L. Guth constructs new symplectic embeddings of polydiscs which represent a major breakthrough in our understanding of the following *bounded* version of Question 1.

Question 2. *What, if any, is the smallest value of $R > 0$ such that there exists a symplectic embedding of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$ for arbitrarily large $S > 0$?*

Among other things, Guth's work settles the existence issue for this question. [1)] In this setting, the second Ekeland-Hofer capacity again implies that R must be at least $\sqrt{2}$. The following improvement of this bound is the main result of this paper.

Theorem 1.1. *For any $0 < R < \sqrt{3}$ there are no symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

In fact we prove a slightly stronger result. For convenience, identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ using the complex coordinates $z_j = x_j + iy_j$. Let

$$E(1, S, \ldots S) = \left\{ (z_1, \ldots z_n) \in \mathbb{C}^n \;\middle|\; |z_1|^2 + \frac{|z_2|^2}{S^2} + \ldots + \frac{|z_n|^2}{S^2} \leq 1 \right\}$$

and denote by $\mathbb{C}P^2(R)$ the complex projective plane equipped with the symplectic form $R^2\omega_{FS}$ where ω_{FS} is the standard Fubini-Study symplectic form.

Theorem 1.2. *For any $0 < R < \sqrt{3}$ there are no symplectic embeddings of $E(1, S, \ldots S)$ into $\mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

A brief outline of the proof. As in [6], Theorem 1.2 is proved via an existence theorem for certain holomorphic curves. Suppose that for any $S > 0$ there is a symplectic embedding $\phi(S)$ of $E(1, S, \ldots, S)$ into $\mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}$. It suffices to show that for every positive integer $d < S^2$, there exists a holomorphic plane of degree d in the (negative) symplectic completion of $(\mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}) \smallsetminus \phi(E(1, S, \ldots, S))$ whose negative end covers the shortest simple Reeb orbit on the boundary of $\phi(E(1, S, \ldots, S))$ a total of $3d - 1$ times. In particular, the symplectic area of such a curve is both positive and equal to $d\pi R^2 - (3d - 1)\pi$ and so the existence of these curves implies that $R^2 > (3d - 1)/d$ for all $d > 0$.

To prove this existence result we use a cobordism argument to reduce it to an equivalent problem for curves in a four dimensional symplectic manifold. Starting with well known results concerning holomorphic spheres of degree d in $\mathbb{C}P^2$, we then settle this alternative existence problem using automatic regularity theorems, the compactness theorem for splittings from [2], and several new techniques, many of which involve families of holomorphic curves

[1)] Before Guth's work it was commonly thought that the answer to Questions 1 and 2 was that no such R exists.

with varying point constraints. (In fact, we prove the existence problem in dimension four first, and then we apply it in the manner described above.)

Sharpness. We also prove that Theorem 1.1 is sharp in the following sense.

Theorem 1.3. *For any $R > \sqrt{3}$ there exist symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$ for all $S > 0$.*

The proof of Theorem 1.3 involves a refinement of Guth's embedding procedure from [7].

Remark 1.4. We do not know whether there exists a symplectic embedding of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(\sqrt{3}) \times \mathbb{R}^{2(n-2)}$ for all large S.

1.1 A related result

Our approach also allows us to settle the following related problem.

Question 3. *What are the smallest value of $R_1 \leq R_2$ for which there are symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^2(R_1) \times B^2(R_2) \times \mathbb{R}^{2(n-2)}$ for all $S > 0$?*

More precisely, by compactifyng $B^2(R_1) \times B^2(R_2)$ to $\mathbb{C}P^1(R_1) \times \mathbb{C}P^1(R_2)$, and replacing the study of holomorphic curves of high degree, say d, in $\mathbb{C}P^2$ by curves of degree $(d, 1)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$, an identical argument to the one used to prove Theorem 1.1 yields the following result.

Theorem 1.5. *If $R_1 < \sqrt{2}$ then there are no symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^2(R_1) \times B^2(R_2) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

This restriction is again stronger than those imposed by the Ekeland-Hofer capacities which imply that no such embeddings exist, for large enough $S > 0$, when $R_1 < 1$. Moreover, Guth's embedding results again imply (this time directly) that Theorem 1.5 is sharp in the following sense.

Theorem 1.6. *([7]) For any $R > \sqrt{2}$ there exist symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^2(R) \times B^2(R) \times \mathbb{R}^{2(n-2)}$ for all $S > 0$.*

1.2 Organization

The next section contains some background material and the proof of the crucial existence result, Theorem 2.12, which concerns special holomorphic curves in certain four dimensional symplectic manifolds. Theorem 1.2 is then proved in Section 3. Our proof of Theorem 1.5 is completely similar and is thus omitted. In Section 4, we construct the embeddings of Theorems 1.3 and 1.6.

2 Splitting holomorphic curves along the boundary of thin ellipsoids in $\mathbb{C}P^2$

Throughout this section we fix a real number $R > 1$ and a positive integer d.

2.1 A moduli space of curves

Let $\mathcal{J}$ be the space of smooth almost-complex structures on $\mathbb{C}P^2$ which are tame with respect to the symplectic form ω_{FS} (and hence $R^2\omega_{FS}$). For a fixed J in $\mathcal{J}$ and an ordered collection of points $p_1, \dots, p_M$ in $\mathbb{C}P^2$, we consider the moduli space $\mathcal{M}_d(J, p_1, \dots, p_M)$ defined as

$$\big\{(f,(y_1,\dots,y_M)) \in C^\infty(S^2,\mathbb{C}P^2)\times(S^2)^M \mid \bar\partial_J f = 0,\ f(y_i) = p_i,\ [f] = dL\big\}/G,$$

where $(y_, \dots y_M)$ is an M-tuple of pairwise distinct points in S^2, L is the class of a complex line in $\mathbb{C}P^2$ and G is the reparameterization group $\mathrm{PSL}(2,\mathbb{C})$. As is well known, see for example Section 7.4 of [13], for suitably generic J the space $\mathcal{M}_d(J, p_1, \dots, p_M)$ is a smooth orientable manifold of dimension $2(3d - 1 - M)$. We will also make use of the following refinement of this statement.

Proposition 2.1. *For generic J in $\mathcal{J}$, the space $\mathcal{M}_d(J, p_1, \dots, p_{3d-1})$ is a compact, zero-dimensional manifold which contains a fixed number, n_d, of points, independently of J.*

Proof. For suitably generic J, the moduli space $\mathcal{M}_d(J, p_1, \dots, p_{3d-1})$ consists of immersed curves, that is, there are no singular curves or cusp curves. Thus the relevant Gromov-Witten invariant counting curves through the $3d-1$ points is computed by looking at immersed curves, and if these curves are counted with multiplicity then the total is independent of J. On the other hand, the automatic regularity phenomena for holomorphic spheres in dimension four (see Lemma 3.3.3 of [13]) implies that all such curves appear with the same multiplicity 1, and our conclusion follows.

□

For a generic choice of J, $\mathcal{M}_d(J, p_1, \dots, p_{3d-2})$ is a two-dimensional manifold which can be compactified by adding finitely many cusp-curves that consist of the union of two holomorphic spheres each passing through a subset of the $3d-2$ points. In particular, if the points $p_1, \dots, p_{3d-2}$ are in generic position, then one does not need to add cusp-curves with more than two components, nor does one need to add multiply-covered curves. Thus, $\mathcal{M}_d(J, p_1, \dots, p_{3d-2})$ can be considered to be a compact two-dimensional manifold.

Proposition 2.2. *The moduli space $\mathcal{M}_d(J, p_1, \dots, p_{3d-2})$ is connected.*

Proof. Lemma 2.3.3 of [16] yields a version of automatic regularity for holomorphic spheres of nonnegative index in a 4-manifold, which includes the possibility of singularities. In particular, the manifold $\mathcal{M}_d(J, p_1, \dots, p_{3d-2})$ is diffeomorphic to $\mathcal{M}_d(i, p_1, \dots, p_{3d-2})$, where i is the standard complex structure on $\mathbb{C}P^2$. Now, the space of all rational algebraic curves of degree d in $\mathbb{C}P^2$ is dominated by an open set of triples of homogeneous polynomials on $\mathbb{C}P^1$ and so is irreducible. Sets of curves through $3d-1$ points are given by intersecting this variety with an appropriate plane. It then follows from the Uniform Position Principle that any two such curves can be connected with curves passing through any subset of $3d-2$ points, see Appendix E of [8]. But any two curves of degree $d > 1$ have $d^2 > 3d-1$ points in common and so the conclusion follows.

□

2.2 A thin ellipsoid

For $a < b$, let $E(a, b)$ denote the ellipsoid

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2} \leq 1 \right\}.$$

If $\frac{a^2}{b^2} \in \mathbb{R} \smallsetminus \mathbb{Q}$, then the standard Liouville form on $\mathbb{C}^2$ restricts to $\partial E(a, b)$ as a contact form $\alpha_{E(a,b)}$ which has exactly two simple Reeb orbits, γ_1 and

γ_2, which lie in the planes $\{z_2 = 0\}$ and $\{z_1 = 0\}$, respectively. The action of γ_1 is πa^2 and the action of γ_2 is πb^2. The Conley-Zehnder indices of these orbits, with respect to the natural trivialization of the z_1 and z_2 coordinate planes, are given by $\mu(\gamma_1) = 3$ and $\mu(\gamma_2) = 2 + 2\left\lfloor \frac{b^2}{a^2} \right\rfloor + 1$. More generally, if $\gamma_j^{(r)}$ is the r-fold cover of γ_j, then we have

$$\mu(\gamma_1^{(r)}) = 2r + 2\left\lfloor \frac{ra^2}{b^2} \right\rfloor + 1 \tag{1}$$

and

$$\mu(\gamma_2^{(r)}) = 2r + 2\left\lfloor \frac{rb^2}{a^2} \right\rfloor + 1. \tag{2}$$

It follows from Theorem 2 of [15] that for $t < 1/\sqrt{2}$ the ellipsoid $E(t, 1)$ can be symplectically embedded into $B^4(1)$ and hence $\mathbb{C}P^2(R)$. We now fix such an embedding of $E = E(t, 1)$ for a t satisfying

$$t^2 < \frac{1}{d(3d-1)}. \tag{3}$$

In what follows, we will identify E with its image in $\mathbb{C}P^2(R)$.

2.3 Splitting curves in $\mathbb{C}P^2$ along ∂E

In this section we recall from [2] the structure of the limits of holomorphic spheres in $\mathbb{C}P^2$ as one splits this manifold (*stretches the neck*) along the boundary of the embedded ellipsoid E.

Let X be the vector field defined near ∂E as the symplectic dual of the Liouville form. An almost complex structure J on $\mathbb{C}P^2$ is said to be compatible with E if the contact structure $\{\alpha_E = 0\}$ on ∂E is equal to $T(\partial E) \bigcap JT(\partial E)$, and JX is equal to the Reeb vector field of α_E. Denote by $\mathcal{J}_E$ the set of $J \in \mathcal{J}$ which are compatible with E.

The union of the three pieces

$$(E, e^{\tau}\omega_0)\,, \ \big(\partial E \times [-\tau, \tau], d(e^t \alpha_E)\big)\,, \ \text{ and } \ \big(\mathbb{C}P^2 \smallsetminus E, e^{-\tau}\omega_{FS}\big)$$

is a symplectic manifold $((\mathbb{C}P^2)^{\tau}, \omega_{FS}^{\tau})$. For a J in $\mathcal{J}_E$, let J^{τ} be the almost complex structure on $(\mathbb{C}P^2)^{\tau}$ which equals J on the disjoint union of $(\mathbb{C}P^2 \smallsetminus E)$ and E, and is translation invariant on $\partial E \times [-\tau, \tau]$. Fix points $p_1, \dots, p_M$ in $E \subset \mathbb{C}P^2$ and consider a sequence C_N in $\mathcal{M}_d(J^N, p_1, \dots, p_M)$. Fix a representative curve f_N for each C_N. The following compactness theorem is proved by Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder in [2].

Theorem 2.3. *(Theorem 10.6 of [2]) There exists a subsequence of the f_N which converges to a holomorphic building,* $\mathbf{F}$.

We now describe the relevant aspects of this theorem starting with the structure of the limiting building $\mathbf{F}$. The reader is referred to [2] for the precise definitions, statements and proofs of the general compactness theorem for holomorphic curves under splittings.

2.3.1 The limits

We begin by describing a holomorphic building $\mathbf{F}$ which arises as the limit of a subsequence of the curves f_N. The domain of $\mathbf{F}$ is a nodal Riemann sphere $(\mathcal{S}, j)$ with punctures. The building $\mathbf{F}$ then consists of a collection of finite energy holomorphic maps from the collection of punctured spheres of $\mathcal{S} \smallsetminus \{\text{nodes}\}$ to one of the following three symplectic manifolds:

- $\left(E_+^\infty, \omega_+^\infty\right) = (E, \omega_0) \cup (\partial E \times [0, \infty), d(e^t \alpha_E))$,
- $(SE, \omega_{SE}) = (\partial E \times \mathbb{R}, d(e^t \alpha_E))$,
- $\left((\mathbb{C}P^2 \smallsetminus E)_-^\infty, \omega_-^\infty\right) = (\mathbb{C}P^2 \smallsetminus E, \omega_{FS}) \cup (\partial E \times (-\infty, 0], d(e^t \alpha_E))$.

More precisely, these target manifolds are equipped with the almost-complex structures which are induced by J and are translation invariant on the subsets which are cylinders over ∂E. The curves of $\mathbf{F}$ are then holomorphic with respect to these almost-complex structures and the complex structures on the punctured spheres induced by the structure j on $\mathcal{S}$. Since each curve of $\mathbf{F}$ has finite energy, they are all asymptotically cylindrical near each puncture, to some multiple of either γ_1 or γ_2.

Let F be a curve of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$. The punctures of the domain of F are all negative, i.e as $z \in S^2$ approaches a puncture on the domain of F, $F(z)$ takes values in $\partial E \times (-\infty, 0]$ and its $(-\infty, 0]$-component converges to $-\infty$. One can also associate to F a unique smooth map $\overline{F}$ which takes the oriented blow-up of the domain of F to the closure of $\mathbb{C}P^2 \smallsetminus E$ such that the circle corresponding to each puncture is mapped to the closed Reeb orbit on ∂E which determines the asymptotic behavior of F near that puncture. Following [2] we will refer to $\overline{F}$ as the compactification of F. We also note that the images of F and $\overline{F}$ coincide away from an arbitrarily small neighborhood of the boundary of $\mathbb{C}P^2 \smallsetminus E$.

A curve of $\mathbf{F}$ whose image lies in SE can have both negative and positive punctures. It also admits a compactification which maps the oriented blow-up of its domain to ∂E, taking each boundary circle to the appropriate closed Reeb orbit. Similarly, each curve of $\mathbf{F}$ with image in E_+^∞ has only positive ends, and admits a compactification which maps the oriented blow-up of its domain to the closure of E and again takes each boundary circle to the corresponding closed Reeb orbit.

The limiting holomorphic building $\mathbf{F}$ is also equipped with a *level structure.* If $\mathbf{F}$ is a building of level k, this structure is encoded by a labeling of the punctured Riemann spheres of $\mathcal{S} \smallsetminus \{\text{nodes}\}$ by integers from 0 to $k+1$, called levels, such that the levels of two components which share a node differ at most by 1. Let $\mathcal{S}_r$ denote the union of components of level r and denote by v_r the holomorphic curve of $\mathbf{F}$ with (possibly disconnected) domain $\mathcal{S}_r$. Then $v_0 : \mathcal{S}_0 \to E_+^\infty$, $v_r : \mathcal{S}_r \to SE$, for $1 \leq r \leq k$, and $v_{k+1} : \mathcal{S}_{k+1} \to (\mathbb{C}P^2 \smallsetminus E)_-^\infty$. Moreover, each node shared by $\mathcal{S}_r$ and $\mathcal{S}_{r+1}$ is a positive puncture for v_r and a negative puncture for v_{r+1}, each asymptotic to the same Reeb orbit. As well, v_r extends continuously across each node within $\mathcal{S}_r$.

Lastly, we recall that the curves of $\mathbf{F}$ have the following two collective properties. The sum over components of their virtual indices (see below) is equal to $2(3d-1-M)$, the deformation index of the curves f_N. As well, their compactifications fit together to form a continuous map $\overline{\mathbf{F}} \colon S^2 \to \mathbb{C}P^2$.

2.3.2 The convergence

Let $\mathbf{F}$ be a holomorphic building of level k, as above, whose domain is the Riemann surface with nodes $(\mathcal{S}, j)$. If $\mathbf{F}$ is the limit of the holomorphic spheres f_N in the sense of [2], then there exist maps $\sigma_N : S^2 \to \mathcal{S}$ and sequences $s_N^r \in \mathbb{R}$, $r = 1, \ldots, k$, such that:

(i). The σ_N are diffeomorphisms except that they may collapse a finite collection of circles in S^2 to nodes in $\mathcal{S}$. Moreover, $\sigma_{N*} i$ converges to j away from the nodes of $\mathcal{S}$, where i is the standard complex structure on S^2.

(ii). The sequences of maps $f_N \circ \sigma_N^{-1} : \mathcal{S}_0 \to E_+^\infty$ and $f_N \circ \sigma_N^{-1} : \mathcal{S}_{k+1} \to (\mathbb{C}P^2 \smallsetminus E)_-^\infty$ converge in the C^∞_{loc}-topology to the maps v_0 and v_{k+1}, respectively. For $1 \leq r \leq k$ the maps $\psi^{s_N^r} \circ f_N \circ \sigma_N^{-1} : \mathcal{S}_r \to SE$ converge to v_r in the C^∞_{loc}-topology where $\psi^{s_N^r}$ is the diffeomorphism of $SE = \partial E \times \mathbb{R}$ which translates the $\mathbb{R}$-component by s_N^r.

Here, as is necessary, we are identifying $\partial E \times (-N, N) \subset (\mathbb{C}P^2)^N$ with an increasing sequence of domains in SE, $E \cup \partial E \times (-N, N) \subset (\mathbb{C}P^2)^N$ with an increasing sequence of domains in E_+^∞, and $\mathbb{C}P^2 \setminus E \cup \partial E \times (-N, N) \subset (\mathbb{C}P^2)^N$ with an increasing sequence of domains in $(\mathbb{C}P^2 \setminus E)_-^\infty$.

2.4 Finer restrictions on limiting holomorphic buildings

When the number of point constraints M is equal to $3d-1$, one can use the special nature of the Reeb flow of α_E on ∂E, together with the regularity results of Dragnev from [3], to obtain more restrictions on the curves comprising our limiting holomorphic buildings. Throughout this section we will consider curves of a limiting building which have connected domains.

Proposition 2.4. *Let $\mathbf{F}$ be a holomorphic building which is the limit of a sequence of curves f_N representing classes in $\mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$. For a generic choice of J in $\mathcal{J}_E$:*

(i). The curves in $\mathbf{F}$ whose image lies in $(\mathbb{C}P^2 \setminus E)_-^\infty$ are somewhere injective, regular, and have deformation index equal to zero. Their (negative) ends are all asymptotic to multiples of γ_1.

(ii). The curves in $\mathbf{F}$ whose image lies in SE are all multiple covers of holomorphic cylinders over γ_1, and they have virtual index zero.

(iii). The curves in $\mathbf{F}$ whose image lies in E_+^∞ are somewhere injective, regular, and have deformation index equal to zero (with the point constraints). Their (positive) ends are all asymptotic to multiples of γ_1.

Proof. By [3], we may choose J in $\mathcal{J}_E$ so that the corresponding almost-complex structures on $(\mathbb{C}P^2 \setminus E)_-^\infty$, SE, and E_+^∞ are regular in the sense that for all somewhere injective curves which have finite energy and genus zero, the deformation index of the curve is equal to its virtual index.

Consider a curve F of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \setminus E)_-^\infty$. Suppose that F has s_1^- negative ends asymptotic to multiples of γ_1, and s_2^- negative ends asymptotic to multiples of γ_2. If the i^{th} negative end covering γ_1 does so a_i^- times, and the i^{th} negative end covering γ_2 does so b_i^- times, then the virtual index of F is

$$\operatorname{index}(F) = (-1)(2 - s_1^- - s_2^-) + 2c_1(F) - \sum_{i=1}^{s_1^-} \mu(\gamma_1^{(a_i^-)}) - 2\sum_{i=1}^{s_2^-} \mu(\gamma_2^{(b_i^-)}).$$

By the iteration formulas (1) and (2) , this simplifies to

$$\mathrm{index}(F) = -2 + 2c_1(F) - 2\sum_{i=1}^{s_1^-}(a_i^- + \lfloor a_i^- t^2 \rfloor) - 2\sum_{i=1}^{s_2^-}(b_i^- + \lfloor b_i^-/t^2 \rfloor). \quad (4)$$

The Chern class $c_1(F)$ is equal to that of the closed curve in $\mathbb{C}P^2$ defined by capping the compactification $\overline{F}$ with a with a disc contained in E. Hence, $c_1(F)$ is at most $3d$, the Chern class of degree d curves in $\mathbb{C}P^2$. As well, the bound on t from (3) implies that $\lfloor b_i^-/t^2 \rfloor \geq b_i^- 3d$, and so

$$\mathrm{index}(F) \leq -2 + 6d - 2\sum_{i=1}^{s_1^-} a_i^- - (2+6d)\sum_{i=1}^{s_2^-} b_i^-.$$

Together with (4), this expression leads immediately to the following result.

Lemma 2.5. *If a holomorphic curve F of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ has nonnegative virtual index, then $c_1(F) > 0$, $s_2^- = 0$ and $\sum_{i=1}^{s_1^-} a_i^- \leq 3d - 1$. Moreover,*

$$\mathrm{index}(F) = -2 + 2c_1(F) - 2\sum_{i=1}^{s_1^-} a_i^-.$$

Any multiply covered holomorphic curve in a symplectic cobordism must cover a somewhere injective curve, [3, 11]. We may therefore suppose that F is an r-fold cover of a simple curve $\widetilde{F}$. It follows from the regularity of J, that $\mathrm{index}(\widetilde{F}) \geq 0$. Lemma 2.5 then implies that $c_1(\widetilde{F}) > 0$, $\tilde{s}_2^- = 0$, $\sum_{i=1}^{\tilde{s}_1^-} a_i^- \leq 3d - 1$, and

$$\mathrm{index}(\widetilde{F}) = -2 + 2c_1(\widetilde{F}) - 2\sum_{i=1}^{\tilde{s}_1^-} \tilde{a}_i^-.$$

Hence, for F we have $s_2^- = 0$ and

$$\mathrm{index}(F) \geq -2 + r(\mathrm{index}(\widetilde{F}) + 2) \geq 0. \quad (5)$$

Thus, the hypothesis of Lemma 2.5 also holds for each curve F of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ and we have

Lemma 2.6. *Let F be a curve of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$. Then $c_1(F) > 0$ and the ends of F are all asymptotic to some multiple of γ_1 and cover γ_1 at most $3d-1$ times. The virtual index of F is nonnegative and is strictly positive when F is a multiple cover.*

Now consider a curve G of $\mathbf{F}$ whose image lies in the symplectization SE. Since none of the curves of $\mathbf{F}$ in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ have negative ends asymptotic to multiples of γ_2, it follows from the existence of the map $\overline{\mathbf{F}}$ that, at least if G is of the top level k, the positive ends of G can only be asymptotic to multiples of γ_1. Suppose that G has s_1^+ such ends, and that the i^{th} one covers γ_1 a total of a_i^+ times. As established above, each curve of $\mathbf{F}$ in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ has at most $3d-1$ negative ends when counted with multiplicity, and there are at most d such curves since each one has a positive Chern number. Hence

$$\sum_{i=1}^{s_1^+} a_i^+ \leq d(3d-1) < 1/t^2. \tag{6}$$

Suppose that G has s_1^- negative ends asymptotic to multiples of γ_1 with the i^{th} such end covering this orbit b_i^- times, and G has s_2^- negative ends asymptotic to multiples of γ_2 with the i^{th} such end covering γ_2 a total of c_i^- times. Then, by Stokes' Theorem we have

$$\begin{aligned} 0 &\leq \int_G d\alpha_E \\ &= \left(\sum_{i=1}^{s_1^+} a_i^+ - \sum_{i=1}^{s_1^-} b_i^-\right)\pi t^2 - \left(\sum_{i=1}^{s_2^-} c_i^-\right)\pi \\ &\leq \pi t^2 \sum_{i=1}^{s_1^+} a_i^+ - \pi \sum_{i=1}^{s_2^-} c_i^- \\ &\leq \pi t^2 d(3d-1) - \pi \sum_{i=1}^{s_2^-} c_i^-. \end{aligned}$$

Our choice of t satisfying (3) implies that

$$\sum_{i=1}^{s_2^-} c_i^- \leq t^2 d(3d-1) < 1,$$

and so $s_2^- = 0$. Integrating $d\alpha_E$ over G once again, we now have

$$\sum_{i=1}^{s_1^+} a_i^+ - \sum_{i=1}^{s_1^-} b_i^- \geq 0. \tag{7}$$

Hence the total number of positive ends of G is no less than the total number of its negative ends and by (6) and (7) we have

$$\begin{aligned} \text{index}(G) &= -2 + s_1^+ - s_1^- + \sum_{i=1}^{s_1^+} \mu(\gamma_1^{(a_i^+)}) - \sum_{i=1}^{s_1^-} \mu(\gamma_1^{(b_i^-)}) \\ &= -2 + 2s_1^+ + 2\sum_{i=1}^{s_1^+}(a_i^+ + \lfloor a_i^+ t^2 \rfloor) - 2\sum_{i=1}^{s_1^-}(b_i^- + \lfloor b_i^- t^2 \rfloor) \\ &= 2(s_1^+ - 1) + 2\left(\sum_{i=1}^{s_1^+} a_i^+ - \sum_{i=1}^{s_1^-} b_i^-\right) \\ &\geq 0. \end{aligned}$$

Hence, the virtual index of G is strictly positive unless $s_1^+ = 1$ and $\sum_{i=1}^{s_1^+} a_i^+ = \sum_{i=1}^{s_1^-} b_i^-$. This condition is equivalent to the curve being a multiple cover of a cylinder over γ_1. As G has no negative ends asymptotic to γ_2 then same conclusions apply by induction to lower level curves mapping to SE. To summarize, we have

Lemma 2.7. *Let G be a curve of $\mathbf{F}$ with image in the symplectization SE. The positive and negative ends of G are all asymptotic to some multiple of γ_1 and the positive ends cover γ_1 at least as many times as the negative ends. The virtual index of G is nonnegative and is strictly positive unless G has one positive end and is a multiple cover of a cylinder over γ_1.*

Finally, we consider a curve H of $\mathbf{F}$ whose image is in E_+^∞. Our analysis of the curves G above implies that none of the positive ends of H are asymptotic to multiples of γ_2. Suppose that H has s_1^+ positive ends asymptotic to multiples of γ_1 with the i^{th} such end covering this orbit b_i^+ times. Then

$$\begin{aligned}\operatorname{index}(H) &= (-1)(2-s_1^+) + \sum_{i=1}^{s_1^+} \mu(\gamma_1^{(b_i^+)}) + -2M \\ &= -2 + 2s_1^+ - 2M + 2\sum_{i=1}^{s_1^+}(b_i^+ + \lfloor b_i^+ t^2 \rfloor)\end{aligned}$$

where $M \leq 3d-1$ is the number of point constraints achieved by the curve H. The fact that the negative ends of the collection of curves of $\mathbf{F}$ in $(\mathbb{C}P^2 \setminus E)_-^\infty$ cover γ_1 at most $d(3d-1)$ times, together with the fact that positive ends of the curves in SE cover γ_1 at least as many times as their negative ends, implies that

$$\sum_{i=1}^{s_1^+} b_i^+ \leq d(3d-1). \tag{8}$$

Hence

$$\operatorname{index}(H) = -2 + 2s_1^+ - 2M + 2\sum_{i=1}^{s_1^+} b_i^+. \tag{9}$$

Arguing as above, the curve H can be realized as an r-fold cover of a somewhere injective curve $\widetilde{H}$ with the same point constraints. By regularity

$$\operatorname{index}(\widetilde{H}) = -2 + 2\tilde{s}_1^+ - 2M + 2\sum_{i=1}^{\tilde{s}_1^+} \tilde{b}_i^+ \geq 0. \tag{10}$$

Since $\sum_{i=1}^{s_1^+} b_i^+ = r\sum_{i=1}^{\tilde{s}_1^+} \tilde{b}_i^+$ and $s_1^+ \geq \tilde{s}_1^+$, we then have

$$\operatorname{index}(H) \geq \operatorname{index}(\widetilde{H}) + (r-1)2\sum_{i=1}^{\tilde{s}_1^+} \tilde{b}_i^+ \geq 0. \tag{11}$$

Lemma 2.8. *Let H be a curve of $\mathbf{F}$ with image in E_+^∞. The positive ends of H are asymptotic to some multiple of γ_1. The virtual index of H is non-negative and is strictly positive if H is multiply covered.*

Note that the deformation index of the curves in $\mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$ is zero, and so the sum of the virtual indices of the curves of $\mathbf{F}$ is also zero.

It follows from Lemmas 2.6, 2.7, and 2.8, that every curve of $\mathbf{F}$ must have virtual index zero. The same three lemmas then yield the three statements of Proposition 2.4.

$\square$

For any integer $l > 0$, one can symplectically embed l disjoint balls of radius one into the ellipsoid $E(1, \sqrt{l})$, [17]. Hence, for t satisfting (3), we may assume that the points $p_1, \dots, p_{3d-1}$ lie at the center of disjoint balls in $E = E(t, 1)$ which have radii slightly less than t and whose closures lie in the interior of E. Let $J \in \mathcal{J}_E$ be an almost complex structure which agrees with the standard complex structure in the balls around the points p_j, and induces almost-complex structures on $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$, SE, and E^{∞}_{+} which are regular for somewhere injective, finite energy curves of genus zero. Consider again a sequence of curves f_N which represent classes in $\mathcal{M}_d(J^N, p_1, \dots, p_{3d-1})$ and converge, in the sense of [2], to a holomorphic building $\mathbf{F}$. For this new choice of J we get the following refinements of Proposition 2.4.

Lemma 2.9. *There is exactly one curve of* $\mathbf{F}$ *whose image lies in* $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$. *It has exactly* $3d - 1$ *negative ends when counted with multiplicity. Moreover, the curves of* $\mathbf{F}$ *with image in* SE *each have exactly one positive puncture and the curves of* $\mathbf{F}$ *with image in* E^{∞}_{+} *are all holomorphic planes.*

Proof. We first show that the negative ends of the curves of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$ cover γ_1 at least $3d-1$ times. Let G_E be the collection of curves of $\mathbf{F}$ with images in either SE or E^{∞}_{+}. Denote by $\overline{G}_E$ the map to E formed by fitting together the compactifications of the curves of G_E. By the existence of the map $\overline{\mathbf{F}} \colon S^2 \to \mathbb{C}P^2$, the ends of $\overline{G}_E$ cover γ_1 the same number of times, say r, as the curves of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$. Now, the symplectic area of $\overline{G}_E$ is $r\pi t^2$. On the other hand, since the compactifications of curves E^{∞}_{+} are holomorphic away from ∂E, the monotonicity theorem and our choice of J imply that the intersections of $\overline{G}_E$ with the balls centered at the points p_j have symplectic area arbitrarily close to $\pi(3d-1)t^2$. Hence, $r \geq 3d - 1$.

So, the curves of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$ have a total Chern number of $3d$ and, collectively, their negative ends cover γ_1 at least $3d-1$ times. As shown in the proof of Proposition 2.4, the deformation index of a curve F in $\mathbf{F}$ whose image lies in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$ is given by

$$\text{index}(F) = -2 + 2c_1(F) - 2 \sum_{i=1}^{s_1^-} a_i^- .$$

where s_1^- represents the number of negative ends and a_i^- is the number of times the i^{th} negative end covers γ_1. If there are K such curves, then their total deformation index is at most

$$-2K + 2(3d) - 2(3d-1) = -2K + 2.$$

Since the total index must be nonnegative we have $K = 1$. Hence, there is exactly one curve, say F, of $\mathbf{F}$ in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$, and F has index zero and exactly $3d-1$ negative ends, when counted with multiplicity.

The remaining statements of Lemma 2.9 follows easily from the first as F is a limit of genus 0 curves.

□

2.5 Holomorphic curves with varying point constaints

In this section we prove two results involving families of (moduli spaces of) curves with varying point constraints. Let $p_i(t)$ be a set of paths in E for $1 \leq i \leq 3d-1$ and $0 \leq t \leq 1$. For each $N \in \mathbb{N}$ set

$$\mathcal{N}_N = \{C | C \in \mathcal{M}_d(J^N, p_1(t), \dots, p_{3d-1}(t)) \text{ for some } t \in [0,1]\},$$

where J^N is the almost complex structure on $((\mathbb{C}P^2)^N, \omega_{FS}^N)$ defined by $J \in \mathcal{J}_E$ in the manner described in §2.3.

Proposition 2.10. *For generic choices of the paths $p_i(t)$ and the almost complex structure $J \in \mathcal{J}_E$, the moduli spaces $\mathcal{N}_N$ are compact 1-dimensional manifolds and the natural projection onto $[0,1]$ is a covering map of degree n_d.*

Proof. For generic paths we may assume that our curves are immersed. Automatic regularity for holomorphic spheres in dimension four, see Lemma 3.3.3 of [13], implies that immersed curves with nonnegative virtual deformation index are in fact regular. The same is true for moduli spaces of curves satisfying point constraints, as can be seen by identifying curves passing through the points with unconstrained curves in a suitable blow-up at the points. Alternatively, and more directly, we can apply Theorem 4.5.3 of [16] which gives regularity for all curves in a generic path.

□

Proposition 2.11. *Under the hypotheses of Proposition 2.10, let $f_N(t)$ for $t \in [0,1]$ be a 1-parameter family of curves representing a component of $\mathcal{N}_N$. If the subsequences $f_{N_j}(0)$ and $f_{N_j}(1)$ both converge, in the sense of [2], then the curves of the limiting holomorphic buildings with images in $(\mathbb{C}P^2 \setminus E)^\infty_-$ are identical.*

Proof. Assume that the limits of $f_{N_j}(0)$ and $f_{N_j}(1)$ in $(\mathbb{C}P^2 \setminus E)^\infty_-$ are distinct. We may again assume that the limiting almost complex structures on $(\mathbb{C}P^2 \setminus E)^\infty_-$ and E^∞_- are regular for somewhere injective finite energy curves of genus zero. The first statement of Proposition 2.4 implies that the limits of $f_{N_j}(0)$ and $f_{N_j}(1)$ in $(\mathbb{C}P^2 \setminus E)^\infty_-$ must be rigid finite energy curves, that is, curves of deformation index zero. Since there exist only finitely many such curves, for each N_j we can choose a t_{N_j} such that $f_{N_j}(t_{N_j})(S^2) \cap (\mathbb{C}P^2 \setminus E)$ is a fixed distance from the compactification of any rigid curve in $(\mathbb{C}P^2 \setminus E)^\infty_-$. Passing to a subsequence, if necessary, we may then assume that the t_{N_j} converge to some $t_\infty \in (0,1)$ and that $f_{N_j}(t_{N_j})$ converges to a holomorphic building whose curve in $(\mathbb{C}P^2 \setminus E)^\infty_-$ lies a fixed distance from any rigid curve. This component of the limit is therefore not rigid, i.e., it has deformation index at least 2 since our indices are even. If the points $p_i(t_\infty)$ are in general position, the curves in the limit of $f_{N_j}(t_{N_j})$ with image in $SE \cup E^\infty_+$ must have a negative total deformation index, and so cannot pass through all of the required $3d-1$ fixed points. This gives a contradiction as required. □

2.6 Holomorphic planes away from E

The key to the proof of Theorem 1.1 is the following existence result.

Theorem 2.12. *For any integer $d \geq 1$ and a suitable choice of almost-complex structure $J \in \mathcal{J}_E$, there exists a regular, finite energy holomorphic plane of degree d in $(\mathbb{C}P^2 \setminus E)^\infty_-$ whose negative end covers the periodic orbit γ_1 precisely $3d-1$ times.*

The rest of this section is devoted to the proof of Theorem 2.12. We begin by fixing points $p_1, \ldots, p_{3d-1}$ in $E \subset \mathbb{C}P^2$ and an almost complex structure J as in Lemma 2.9. Consider the set of holomorphic buildings which occur as the limit of a sequence of curves representing classes in the spaces $\mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$. By Lemma 2.9 these limit buildings each have a single curve with image in $(\mathbb{C}P^2 \setminus E)^\infty_-$ and the negative ends of this curve cover γ_1 a total of $3d-1$ times. Let m be the maximum number of times a single negative end of one of these limit curves in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$ covers γ_1. To prove Theorem 2.12 it suffices to show that $m = 3d-1$.

Proposition 2.13. $m > 1$.

Proof. Arguing by contradiction, suppose that in any limit $\mathbf{F}$, as above, the unique curve in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$ has precisely $3d-1$ negative ends each of which covers γ_1 once. For each N we take a suitably generic family of paths $p_i(t)$ in E with $p_i(0) = p_i$ and $p_i(1) = p$ for all i, where p is in E. By Proposition 2.10 there exist corresponding families of curves $f_N(t)$ for each N which pass through the points $p_i(t)$ and satisfy $f_N(0) = f_N$. By Proposition 2.11, for all sufficiently large N the curves $f_N(t)$ intersect compact subsets of $\mathbb{C}P^2 \smallsetminus E$ in curves lying arbitrarily close to a rigid curve with $3d-1$ negative ends. Thus, for $t = 1$ and N large there will be $3d-1$ distinct branches of the curves $f_N(1)$ in E which pass through the point p. On the other hand, for each N there exists a (unique) J^N-holomorphic sphere D in $(\mathbb{C}P^2)^N$ which has degree one and passes through p and another fixed point $q \in (\mathbb{C}P^2)^N$. By positivity of intersection we then have $d = D \cdot (f_N(1)(S^2)) \geq 3d-1$ for all large N, a contradiction.

□

Arguing again by contradiction, let us assume that m is less than $3d-1$, i.e., $m \in (1, 3d-1)$. Let $\mathbf{F}$ be a limiting holomorphic building for a subsequence of the curves f_N with $[f_N] \in \mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$, such that the unique curve of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$ has a negative end which covers γ_1 exactly m times. For simplicity, we will suppress the choice of a subsequence, and assume that the f_N converge to $\mathbf{F}$ in the sense of [2]. Denote the level of $\mathbf{F}$ by k and its domain by $(\mathcal{S}, j)$.

Partitions. We now consider a family of partitions for the set of point constraints $\{p_1, \ldots p_{3d-1}\}$, which is determined by $\mathbf{F}$. Let G_E be the collection of curves of $\mathbf{F}$ with images in either SE or E_+^∞ and denote by $\overline{G}_E$ the map to E formed by fitting together the compactifications of the curves of G_E, as described in §2.3.1. It follows from Lemma 2.9, that the domain of $\overline{G}_E$ is a union of discs. The intersections of the images of these discs with the point constraints determines a partition P of $\{p_1, \ldots p_{3d-1}\}$. In particular, we say that two points lie in the same block of P if they lie in the image of the same disc of the domain of $\overline{G}_E$.

Note that our assumption that $m < 3d-1$ implies that there are at least two discs in the domain of $\overline{G}_E$. So, we may assume that p_1 lies in the image of a disc whose end covers γ_1 a total of m times, and that p_2 lies in a different disc. Since $m > 1$, it follows from monotonicity and our choice of J that the disc through p_1 intersects at least one other point constraint, say p_3. In other words we can label the point constraints so that p_1 and p_3 lie in the same block of P and this block doesn't contain p_2.

Similarly, for any integer $r \in [1, k-1]$ we can also define a partition P_r of the set $\{p_1, \dots p_{3d-1}\}$ by considering the maps to E constructed from the compactifications of the curves of $\mathbf{F}$ with level at most r. It follows from Lemma 2.9 that for $r > r'$ the partition $P_{r'}$ is a (possibly trivial) refinement of P_r. Hence p_1 and p_2 lie in different blocks of each P_r.

A distinguished level and curve of F. Let r_0 be the maximum integer in $[1, k-1]$ such that the block of P_r which contains p_1 is the same for all $r \geq r_0$. Equivalently, r_0 can be defined as the lowest level such that for all $r > r_0$ the curve of G_E at level r which corresponds to the block of P_r containing p_1 has exactly one positive and one negative end, each of which cover γ_1 exactly m times.

Relabeling the constraint points, if necessary, we may choose the point p_3 as above so that it also lies in a different block to p_1 for all partitions P_r with $r < r_0$. We now show that for this choice of p_3, we can recognize r_0 as the level of a special curve of $\mathbf{F}$. For each N, let us choose the curve f_N such that $f_N(0) = p_1$, $f_N(1) = p_3$ and $f_N(\infty) \in \Sigma_\infty$ where Σ_∞ is the line at infinity in $\mathbb{C}P^2$. This choice is not unique (since our curves intersect the line at infinity in d points) but we note that (as required below) a continuous choice can be made over generic 1-parameter families of curves. Now, the sequence of maps $\mathrm{id} \circ \sigma_N^{-1}$ converges to a continuous map $z \colon \mathcal{S} \to S^2$ which has removable singularities at the nodes of $\mathcal{S}$. In fact, the maps $\mathrm{id} \times f_N : S^2 \to S^2 \times (\mathbb{C}P^2)^N$ converge as before to a holomorphic building whose curves map to one of $S^2 \times (\mathbb{C}P^2 \smallsetminus E)^\infty_-$, $S^2 \times SE$ or $S^2 \times E^\infty_+$, and z is the projection of this building onto the S^2 factor. This picture shows slightly more. The projection from $\mathcal{S}$ to S^2 is holomorphic of degree 1 (since it is a limit of such maps) and so can be nonconstant on exactly one component of $\mathcal{S} \smallsetminus \{\text{nodes}\}$. We call this component $\mathcal{S}_f$, and denote the curve of F with this domain as f and its level by l.

Lemma 2.14. *The curve f maps into either SE or E_+^∞, that is, $l \leq k$. In fact, with p_3 chosen as above, we have $l = r_0$.*

Proof. First suppose that $r_0 > 0$ and consider the curve of level r_0 corresponding to block of P_{r_0} containing p_1. This curve has a positive puncture asymptotic to $\gamma_1^{(m)}$ and at least two negative punctures, one connected through a chain of curves to a curve through p_1, and another connected similarly to a curve through p_3 (these punctures are different by the choice of p_3). The positive puncture is connected through a chain of curves to a curve through Σ_∞. Now, the z coordinate on the curves which pass through p_1, p_3 and Σ_∞ includes the values 0, 1 and ∞ respectively. Therefore, as the function z can be nonconstant on at most one of these three distinct chains of spheres, the value of z at two of the punctures on our curve at level r_0 are different and so z is nonconstant on this curve which must therefore be f.

Finally, if $r_0 = 0$ then there is a finite energy plane in E_+^∞ passing through p_1 and p_3 and the z coordinate is nonconstant on this curve as it includes the values 0 and 1.

□

Partitions revisited. As we now describe, the partition P_{r_0} (in fact any of the P_r) can also be defined in terms of the curves f_N converging to $\mathbf{F}$. Consider a sequence of neighborhoods of the form

$$A_N = \partial E \times (\mathbf{s} - s_N^{r_0} - 1, \mathbf{s} - s_N^{r_0} + 1),$$

where $\mathbf{s} \in \mathbb{R}$ and the $s_N^{r_0}$ are the shifts that determine the convergence of the f_N at level r_0 (see §2.3.2). (Of course, if $r_0 = 0$ then these shifts are not necessary.) For large enough N, each A_N lies in the cylindrical portion $\partial E \times (-N, N)$ of $(\mathbb{C}P^2)^N$. The sets

$$\psi^{s_N^{r_0}} \left(f_N(S^2) \cap A_N \right)$$

converge to a subset of $v_{r_0}(\mathcal{S}_{r_0})$, the image of the level r_0 curves of $\mathbf{F}$. In particular, modulo translations, the sets A_N can be viewed to be converging to the set $A = \partial E \times (\mathbf{s} - 1, \mathbf{s} + 1)$ in the copy of SE which acts as the target of the level r_0 curves of $\mathbf{F}$ (or, if $r_0 = 0$, in the cylindrical portion of E_+^∞). Choosing $\mathbf{s}$ to be sufficiently large, we may assume that the sets $\sigma_N(f_N^{-1}(A_N))$ converge to a collection of annuli, one in each component of

$\mathcal{S}_{r_0}$, which are arbitrarily close to the positive asymptotic puncture in each component. (That is, the components of the complements of these annuli that contain the positive punctures are precisely punctured disks). For sufficiently large N, we can then redefine the partition P_{r_0} by stating that two points p_i and p_j lie in the same block if for all large N there exists a continuous path in $S^2 \setminus f_N^{-1}(A_N)$ which starts at y_i and ends at y_j. Here the $y_k \in f_N^{-1}(p_k)$ are as in the definition of section 2.1. The convergence of f_N to $\mathbf{F}$ implies this definition gives the same partition as above.

Another building realizing m. We now observe that there exists another limiting building $\mathbf{F}'$ whose curve in $(\mathbb{C}P^2 \setminus E)^\infty_-$ agrees with that of $\mathbf{F}$, but which determines a different family of partitions of $\{p_1, \dots p_{3d-1}\}$. To see this, recalling section §2.5, consider the space $\mathcal{N}_N$ determined by a family of paths $p_i(t)$ in E which switches p_1 and p_2 and leaves the other points fixed. More precisely, suppose that $p_i(0) = p_i$ for all i, $p_1(1) = p_2$, $p_2(1) = p_1$, and $p_i(1) = p_i$ for all $i > 2$. Let $f_N(t)$ be a 1-parameter family of curves representing a component of $\mathcal{N}_N$ such that $[f_N(0)] = [f_N]$. Set $f'_N = f_N(1)$. Passing to subsequences we may assume that the f'_N also converge to some $\mathbf{F}'$. By Proposition 2.11, the curves of $\mathbf{F}$ and $\mathbf{F}'$ with image in $(\mathbb{C}P^2 \setminus E)^\infty_-$ are the same. On the other hand, by our choice of the paths $p_i(t)$, it follows from Proposition 2.10 that the points p_1 and p_3 are in different blocks of the partition P' of $\{p_1, \dots p_{3d-1}\}$ which corresponds to $\mathbf{F}'$. Hence $P \neq P'$.

More importantly, note that for large N and the sets A_N above, the components of $(f'_N)^{-1}(A_N)$ also determine a partition of the point constraints which is again defined by stating that two points p_i and p_j are in the same block if for all large N there exists a path in $S^2 \setminus (f'_N)^{-1}(A_N)$ which starts at y_i and ends at y_j. We denote this partition by $P'(r_0)$. (It need not equal the partition determined by the curves of $\mathbf{F}'$ with level at most r_0).

Lemma 2.15. *The partition $P'(r_0)$ is not equal to P_{r_0}. In particular, the points p_1 and p_3 lie in different blocks of $P'(r_0)$.*

Proof. To see this, we argue by contradiction. Assume there exists a path $\gamma = \gamma(1)$ in $S^2 \setminus (f'_N)^{-1}(A_N)$, and in particular in $S^2 \setminus ((f'_N)^{-1}(\mathbb{C}P^2 \setminus E))$, which connects ${f'_N}^{-1}(p_1)$ and ${f'_N}^{-1}(p_3)$. Let $f_N(t)$ be the family of curves used to define $\mathbf{F}'$ above. By Proposition 2.11, the sets $f_N(t)^{-1}(\mathbb{C}P^2 \setminus E)$ have the same topology relative to the marked points. Therefore the path $\gamma(1)$ can be included in a homotopy of paths $\gamma(t)$ connecting $f_N^{-1}(p_2(t))$ to

$f_N^{-1}(p_3)$ within $S^2 \setminus (f_N(t)^{-1}(\mathbb{C}P^2 \setminus E))$. But then $\gamma(0)$ connects $f_N^{-1}(p_2)$ and $f_N^{-1}(p_3)$ in $S^2 \setminus f_N^{-1}(\mathbb{C}P^2 \setminus E)$, which contradicts the fact that p_2 and p_3 are in different blocks of P.

$\square$

A technical aside. To complete the proof of Theorem 2.12 we will need the distinguished curve f to be rigid. This is automatic if f maps to E_+^∞ and the constraint points are in generic position. Let us assume then that f maps to SE, in which case f is a multiply covered cylinder with one positive and at least two negative punctures. To achieve rigidity for f now we will perturb it so that the resulting curve $\tilde{f}$ is somewhere injective, and is a regular solution of a Cauchy-Riemann type equation with some domain dependence. This process is carried out carefully in the following paragraphs, and is possible precisely because f is the unique curve of $\mathbf{F}$ on which the coordinate function z is nonconstant. The curve $\tilde{f}$ will have the same asymptotic behavior as f, and the other curves of $\mathbf{F}$ will be unchanged by this procedure. In particular, the new limiting building will still be the limit of holmorphic spheres (holomorphic with respect to a domain dependent almost-complex structure) and will have the same curve of level $k+1$ with a negative end covering the orbit γ_1 precisely m times.

Let $A = \partial E \times (\mathbf{s}-1, \mathbf{s}+1) \subset SE$. As above, we view A as the limit of the sets $\psi^{s_N^k}(A_N)$ and as a subset of the copy of SE that corresponds to the target of the level r_0 curves of $\mathbf{F}$. For large $\mathbf{s}$ the set $f^{-1}(A)$ is arbitrarily close to the positive puncture $z = \infty$ in the domain of f. By Lemma 2.14, f is asymptotically cylindrical to $\gamma_1^{(m)}$ over this puncture. As the convergence is exponential ([10]), for large $\mathbf{s}$ we may then assume that:

- the argument function increases from 0 to 2π on $f^{-1}(A)$;
- $f(z) = f(w) \in A$ implies that $\arg z - \arg w \simeq \frac{2\pi n}{m}$ for some $n \in \mathbb{Z}$, and $n = m\mathbb{Z}$ if and only if $z = w$.

In particular, intersecting branches of the multiply covered curve f have different arguments in $f^{-1}(A)$.

Let $\tilde{f}$ be a generic perturbation of f on the preimage of A such that any self-intersections of $\tilde{f}$ on A are isolated. For all $z \in \tilde{f}^{-1}(A)$ we can then define a domain dependent almost complex structure $\tilde{J}(z, \tilde{f}(z)) = \tilde{J}(\arg z, \tilde{f}(z))$

such that $\tilde{f}$ becomes $\tilde{J}$-holomorphic. In other words, for local coordinates $z = x + iy$ on $\tilde{f}^{-1}(A) \subset S^2$ we have

$$\frac{\partial \tilde{f}}{\partial y} = \tilde{J}(\arg z, \tilde{f}(z)) \frac{\partial \tilde{f}}{\partial x}.$$

Define $\tilde{J}$ as an extension of J to $\mathbb{R}/2\pi\mathbb{Z} \times A$ such that $\tilde{J}(\theta, p) \neq \tilde{J}(\theta', p)$ for all $\theta \neq \theta'$ and $p \in A$. This is possible since the intersection points of $\tilde{f}$ have different arguments. Since $\arg z$ is constant on other level r_0 curves passing through A, we may assume that they still satisfy the original holomorphic curve equation. The $\tilde{J}$-holomorphic curve $\tilde{f}$ has the same asymptotic limits as f and thus still has virtual deformation index 0 (now in the space of parameterized spheres with the set of punctures including the points 0, 1 and ∞). However, it is no longer multiply covered and so, for a generic choice of $\tilde{J}$, it will be rigid and hence isolated away from other maps from the sphere to SE which satisfy the new Cauchy-Riemann equation, and have bounded energy, deformation index equal to zero and punctures at 0, 1 and ∞. Let $\tilde{\mathbf{F}}$ be the holomorphic building obtained from $\mathbf{F}$ by replacing f by $\tilde{f}$.

By the nature of their convergence to the original f and J, both f_N and J^N can be perturbed analogously to obtain a sequence of curves $\tilde{f}_N$ which converge to $\tilde{f}$ and are holomorphic with respect to a sequence of domain dependent almost-complex structures $\tilde{J}^N$ which converge to $\tilde{J}$ on A. More precisely, for $f_N(z) \notin \psi^{-s_N^{r_0}}(A)$ we have $\tilde{f}_N(z) = f_N(z)$ and $\tilde{J}^N(z, f_N(z)) = J^N(f_N(z))$, and for $f_N(z) \in \psi^{-s_N^{r_0}}(A)$ we have $\tilde{J}^N(z, \tilde{f}_N(z)) = \tilde{J}^N(\arg z, \tilde{f}_N(z))$. Note that this gives smooth (domain dependent) almost-complex structures $\tilde{J}^N$.

The curves f'_N which converge to the limiting building $\mathbf{F}'$ can now be obtained from the $\tilde{f}_N$ as before (by following paths of points $p_i(t)$) but now using the domain dependent almost-complex structures $\tilde{J}^N$. By Proposition 2.11 the limit $\mathbf{F}'$ of the f'_N will still have the same curve as $\tilde{\mathbf{F}}$ in $(\mathbb{C}P^2 \setminus E)_-^{\infty}$.

Back to the proof of Theorem 2.12. For the sake of simplicity we will now drop the tildes from the notation of our technical aside. We just retain the fact that the level r_0 curve f of $\mathbf{F}$ which corresponds to the block of P_{r_0} containing p_1, is now rigid.

As described previously, for sufficiently large N the components of $f_N^{-1}(A_N)$ and $(f'_N)^{-1}(A_N)$ determine partitions P_{r_0} and $P'(r_0)$ of $\{p_1, p_2, \ldots, p_{3d-1}\}$. We also recall that these partitions are distinct since p_1 and p_3 are in the same block of P_{r_0} but are in different blocks of $P'(r_0)$. Hence, if we forget the point p_2, then P_{r_0} and $P'(r_0)$ still determine distinct partitions of $\{p_1, p_3, \dots, p_{3d-1}\}$.

By Proposition 2.2, we can connect $[f_N]$ and $[f'_N]$ by a continuous path in the moduli space $\mathcal{M}_d(\tilde{J}^N, p_1, p_3, \dots, p_{3d-1})$. Let $f_N(t)$ be a corresponding path of curves such that $f_N(0) = f_N$, $f_N(1) = f'_N$, and the class $[f_N(t)]$ belongs to $\mathcal{M}_d(\tilde{J}^N, p_1, p_3, \dots, p_{3d-1})$ for all $t \in [0,1]$. We may assume that the $f_N(t)$ are all parameterized such that 0, 1 and ∞ map to p_1, p_3 and Σ_∞ as before. When N is large enough, the components of $(f_N(t))^{-1}(A_N)$ determine a family of partitions $P(t)$ of $\{p_1, p_3, \dots, p_{3d-1}\}$ such that $P(0) = P_{r_0}$ and $P(1) = P'(r_0)$. Let $c_N(t)$ be the restriction of $f_N(t)$ to the component of $(f_N(t))^{-1}(A_N)$ which corresponds to the block of $P(t)$ that contains p_1. Since the partitions $P(0)$ and $P(1)$ have different blocks containing p_1, the $c_N(t)$ must vary nontrivially with t. In fact, the topology of the domain of the $c_N(t)$ must change for some t. We now use this transition to obtain a contradiction to the assumption that $m < 3d - 1$.

Henceforth, we only consider values of N which are large enough for the maps $c_N(t)$ to be defined as above. For values of t near 0, each $c_N(t)$ is defined on an annulus whose boundary components are mapped to different components of ∂A_N. If we make an identification between their domains and the standard annulus $[-1,1] \times S^1$, then these $c_N(t)$ define elements in the space

$$\left\{ a \in C^\infty([-1,1] \times S^1, A_N) \mid a(\{\pm 1\} \times S^1) \subset \{s_N \pm 1\} \times \partial E \right\}.$$

Consider the topology on this space determined by the distance function

$$\operatorname{dist}(a_1, a_2) = \inf \|a_1 \circ h - a_2\|_{C^\infty},$$

where the infimum is over smooth diffeomorphisms h of $[-1,1] \times S^1$. Note that this also determines a well-defined distance function on the maps $c_N(t)$, for values of t near 0, since it is independent of how we identify their domains with the standard annulus.

As described above, for some t the domain of $c_N(t)$ must no longer be an annulus. Hence, for all sufficiently small $\epsilon \in (0,1]$ there is a minimal $t_N(\epsilon) \in (0,1)$ such that the distance between $c_N(t_N)$ and $c_N(0)$ is at least ϵ. (The derivative of $c_N(t)$ at the boundary must approach a tangency with ∂A_N as t increases away from 0.) Let us fix such an $\epsilon \in (0,1]$ and set $t_N = t_N(\epsilon)$.

Taking a convergent subsequence of the $f_N(t_N)$ as $N \to \infty$, we obtain a holomorphic building $\mathbf{F}''$. The indices of all component curves of $\mathbf{F}''$ must sum to 2, since this is the index of curves in the original moduli space. (We have forgotten one point constraint.) As we are in a 1-parameter family, all curves generically have index ≥ -1. However, our indices are always even, so all curves have index ≥ 0. Hence, at most one curve of $\mathbf{F}''$ has index 2. By our choice of t_N, the curves of $\mathbf{F}''$ in $(\mathbb{C}P^2 \setminus E)^{\infty}_{-}$ have a negative end which connects to a component in $E^{\infty}_{+} \cup SE$ that passes through m fixed points. (Since for $t \leq t_N$ the domain of $c_N(t)$ is still an annulus and therefore still bounds the same m marked points.)

Let f'' be the curve of $\mathbf{F}''$ with image in SE that includes the limits of the $c_N(t_N)$. (As the complex structure on this copy of SE is not translation invariant, f'' is in fact well defined as a map to SE, not just up to translation.) As $\epsilon > 0$, we have $f'' \neq f$. Nevertheless, we can assume that f'' is close enough to f that it's tangent planes cannot be holomorphic for a domain independent almost-complex structure (for example, because there may be self-intersection points of negative index), in other words f'' will be the unique component of $\mathbf{F}''$ on which the coordinate function z is nonconstant. Consider the set of holomorphic curves from S^2 to SE (with respect to the domain dependent almost complex structure) which have punctures at 0, 1 and ∞, bounded energy, deformation index equal to zero, and whose restriction to A admits a parameterization by an annulus whose distance to the $c_N(0)$, as defined above, is uniformly bounded. This set of curves is finite since it is compact and has dimension zero. Therefore we may assume that the restriction to A of any such curve which does not coincide with that of f is more than some fixed distance away from the restriction of f. Choosing ϵ above to be smaller than this distance we conclude that f'' is not rigid and hence $\mathrm{index}(f'') = 2$.

We now introduce a point $p_2^N(0)$ to the image of $c_N(t_N)$. In the limit we may assume that these points converge to a point p'' in the image of f''. Then in the moduli space of curves passing through $3d-1$ points (when we include $p_2^N(0)$ or p'') the curves $f_N(t_N)$ and the limiting curve f'' can be assumed to be rigid. There is now a component of each $f_N(t_N)$ restricted to $(f_N(t_N))^{-1}(E \cup (\partial E \times [-N, N]))$ passing through $m+1$ points.

We extend $p_2^N(0)$ to a path $p_2(t)$ such that $p_2(1) = p_2$. This can be done generically for each N and to ensure that in any limit the $p_2(t)$ remain a bounded distance from all rigid curves in $E \cup SE$ passing through $p_1, p_3, \ldots, p_{3d-1}$. There are corresponding families of curves connecting $f_N(t_N)$

to another curve h_N that passes through $p_1, p_2, \ldots, p_{3d-1}$. Then, as the limits in $E \cup SE$ must be rigid, by Proposition 2.11 the limit of the h_N lying in $\mathbb{C}P^2 \smallsetminus E$ is the same as the limit of the $f_N(t_N)$. In particular, for the 1-parameter family of curves connecting $f_N(t_N)$ and h_N, the intersections of the images of this family with $\mathbb{C}P^2 \smallsetminus E$ are approximately constant when N is large. Hence, the components of the intersection of these curves with the complement of $\mathbb{C}P^2 \smallsetminus E$ determine identical partitions of the sets $\{p_1, p_2(t), \ldots, p_{3d-1}\}$. For N sufficiently large, it follows that the image of the curve h_N has a component in $E \cup (\partial E \times [-N, N])$ passing through $m+1$ points. Looking at the limit of the h_N, we find a component in $E_+^\infty \cup SE$ that passes through $m+1$ disjoint balls, implying that it must cover γ_1 at least $m+1$ times. This contradicts the definition of m as the maximum such covering number.

3 The proof of Theorem 1.2

Let (M, ω) be the product manifold $\mathbb{C}P^2 \times \mathbb{C}^{n-2}$ equipped with the split symplectic form $R^2 \omega_{FS} \oplus \omega_0$. When convenient, we will equip the $\mathbb{C}^{n-2}$–factor of M with complex coordinates $z_3, \ldots, z_n$. We will also consider the $(n-2)$-dimensional torus, $\mathbb{T}^{n-2}$, acting on $\mathbb{C}^{n-2}$ in the standard way by rotations in the z_j–directions.

Let $E(a_1, a_2, \ldots, a_n)$ be the ellipsoid

$$\left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=2}^{n} \frac{|z_i|^2}{a_i^2} \leq 1 \right\}$$

Suppose that for any $S > 0$ there exists a symplectic embedding

$$\phi(S) : E(1, S, \ldots, S) \hookrightarrow M.$$

To prove Theorem 1.2 we must show that this implies that $R \geq \sqrt{3}$.

Fix an integer $d \geq 1$, and a positive real number S such that S^2 is irrational and $S^2 > d(3d-1)$. Set $\phi = \phi(S)$. For $t \in [1/S, 1]$, let ϕ_t be a smooth family of symplectic embeddings

$$\phi_t : E_t = E(t, tS, \ldots, tS) \hookrightarrow M$$

such that:

- for t in some neighborhood of 1 the embeddings ϕ_t are just the restrictions of ϕ to E_t;
- $\phi_{1/S}$ coincides with the inclusion of $E(1/S, 1, \ldots, 1)$ into $E(1,1) \times \mathbb{C}^{n-2} \subset \mathbb{C}P^2(R) \times \mathbb{C}^{n-2} = M$.

In particular, the image of $\phi_{1/S}$ is chosen to be invariant under the action of $\mathbb{T}^{n-2}$ on M. In what follows, we will identify E_t with its image $\phi_t(E_t)$.

The standard contact form on ∂E_t has n simple closed Reeb orbits, whose images correspond to the intersections $\partial E_t \cap \{z_j = 0; j \neq i\}$ for $i = 1, \ldots, n$. For simplicity, we will denote the closed Reeb orbit on ∂E_t with the smallest action by γ_1, i.e., we suppress the dependence on t. The action of γ_1 is πt^2 and the Conley-Zehnder index of its r-fold cover, $\gamma_1^{(r)}$, is given by

$$\mu(\gamma_1^{(r)}) = 2r + (n-1)\left(2\left\lfloor \frac{r}{S^2} \right\rfloor + 1\right).$$

Let $\mathcal{J}_t = \mathcal{J}_{E_t}$ be the space of smooth almost complex structures on M which are compatible with the symplectic form on M and are also compatible with E_t in the sense of §2.3. As described in that same section, $J_t \in \mathcal{J}_t$ determines an almost complex structure on $(M \smallsetminus E_t)^\infty_-$ which we will denote by the same symbol. Consider a J_t–holomorphic curve in $(M \smallsetminus E_t)^\infty_-$ which has genus zero, degree d, and s^- negative ends asymptotic to multiples of γ_1 such that the i^{th} such end covers γ_1 a total of b_i times. The virtual dimension of the moduli space represented by this curve is

$$(n-3)(2-s^-) + 6d - \sum_{i=1}^{s^-} \left(2b_i + (n-1)\left(2\left\lfloor \frac{r}{S^2} \right\rfloor + 1\right)\right). \qquad (12)$$

Now, let J_t be a smooth family of almost complex structures such that J_t belongs to $\mathcal{J}_t$ for all $t \in [1/S, 1]$. We define $\mathcal{K}_t$ to be the moduli space of somewhere injective J_t–holomorphic planes in $(M \smallsetminus E_t)^\infty_-$ which have finite energy, degree d, and whose negative end is asymptotic to $\gamma_1^{(3d-1)}$. Since $S > \sqrt{3d-1}$, the index formula above implies that each $\mathcal{K}_t$ has virtual dimension

$$(n-3) + 6d - (2(3d-1) + (n-1)) = 0.$$

3.1 The space $\mathcal{K}_{1/S}$.

Here we prove the following result.

Proposition 3.1. *For every regular almost complex structure $J_{1/S}$ in $\mathcal{J}_{1/S}$, the corresponding moduli space $\mathcal{K}_{1/S}$ is an oriented, compact, zero-dimensional manifold whose oriented cobordism class is nontrivial.*

It suffices to prove that the conditions on $\mathcal{K}_{1/S}$ are satisfied for any regular almost complex structure $J_{1/S}$ in $\mathcal{J}_{1/S}$. We will restrict our attention to the subset $\bar{\mathcal{J}}_{1/S}$ of $\mathcal{J}_{1/S}$ consisting of almost-complex structures which are invariant under the $\mathbb{T}^{n-2}$–action on M. Note that for $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$ the submanifold $(\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\} \subset M$ is $J_{1/S}$-holomorphic. Hence, Theorem 2.12 implies that, for a suitable choice of the restriction of $J_{1/S}$ to $(\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\}$, the space $\mathcal{K}_{1/S}$ is nonempty. In particular, the theorem yields a curve with image in

$$((\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\})^{\infty}_{-} \subset (M \smallsetminus E_{1/S})^{\infty}_{-}$$

that represents a class in $\mathcal{K}_{1/S}$.

Proposition 3.2. *For $J_{1/S}$ in $\bar{\mathcal{J}}_{1/S}$, let F be an immersed $J_{1/S}$-holomorphic curve in $(M \smallsetminus E_{1/S})^{\infty}_{-}$ whose image is contained in the subset $((\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\})^{\infty}_{-}$. The linearized normal bundle ν for the image of F splits as a sum of holomorphic line bundles. In particular, $\nu = H \oplus V_3 \oplus \ldots \oplus V_n$, where H is the subbundle of vectors parallel to $\{z_j = 0 \mid j = 3, \ldots n\}$ and V_j is the subbundle of vectors along F parallel to the z_j factor.*

Proof. Let π denote the natural projection $M \to \mathbb{C}P^2$. This is of course holomorphic with respect to the standard integrable complex structure i. Let $J_{1/S}$ be our (stretched) almost-complex structure on $(M \smallsetminus E_{1/S})^{\infty}_{-}$ and also the induced complex structure on ν.

It is clear that H is a holomorphic subbundle. Since the derivative of the projection is holomorphic along F the fibers of the V_j are complex. But we can choose a connection on V_j whose horizontal subspaces are all invariant under rotations in the z_j plane. Then if v is the lift of a vector from the base F, the projection of $J_{1/S}v$ to the fibers is also invariant under rotation. But the component of this projection in the $H \oplus_{i \neq j} V_i$ factors varies linearly with $z_j \in V_j$ and is fixed under rotation, and so must be identically zero. It follows that V_j is a holomorphic subbundle as required.

□

By Proposition 3.2, the linearized operator splits and we can apply the (four-dimensional) automatic transversality results of [18] to conclude that for $J_{1/S}$ in $\bar{\mathcal{J}}_{1/S}$ the curves which represent classes in $\mathcal{K}_{1/S}$ and whose images lie in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$ are in fact regular in $(M \setminus E_{1/S})^\infty_-$ and have the same orientation; genericity implies that such curves will be immersed.

Observe that if $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$ is regular for all curves representing classes in $\mathcal{K}_{1/S}$, then the images of these curves are *all* contained in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$. This is because any such curve whose image is not contained in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$ would have deformation index zero but, by invariance, would appear in a family of curves of dimension at least one. Hence, it follows from Theorem 2.12 and Proposition 3.2 that to prove Proposition 3.1 it suffices for us to establish the existence of a $J_{1/S}$ in $\bar{\mathcal{J}}_{1/S}$ which is regular for $\mathcal{K}_{1/S}$.

A curve which represents a class in $\mathcal{K}_{1/S}$ will be called *orbitally simple* if it intersects at least one orbit of the $\mathbb{T}^{n-2}$–action exactly once. There is an open and dense subset of $\bar{\mathcal{J}}_{1/S}$ for which all orbitally simple curves for the corresponding spaces $\mathcal{K}_{1/S}$ are regular. This follows from the standard methods, exactly as in, say, [13] section 3.2, Here, the condition of orbital simplicity replaces the assumption in [13] that all curves are simple. In particular, the analogue of Proposition 3.2.1 of [13] allows one to construct sections of bundles over $(M \setminus E_{1/S})^\infty_-$ which are both $\mathbb{T}^{n-2}$–invariant and, when restricted to the image of a curve, have support contained in the neighborhood of a single point.

Thus, to detect the desired almost complex structure $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$ it will suffice to find an open subset of $\bar{\mathcal{J}}_{1/S}$ such that every curve for the corresponding spaces $\mathcal{K}_{1/S}$ is either contained in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$ or is orbitally simple. In fact, as orbital simplicity is an open property (by compactness), it will suffice to construct a single such almost-complex structure.

Proposition 3.3. *For a suitable $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$, all curves which represent classes in $\mathcal{K}_{1/S}$ and which do not lie entirely in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$ must intersect some orbit of the $\mathbb{T}^{n-2}$–action exactly once.*

Proof. For a real number a slightly larger than 1, let Σ_a denote the hypersurface $\{S|z_1|^2 + |z_2|^2 = a^2\} \subset (M \setminus E_{1/S})^\infty_-$ which divides $(M \setminus E_{1/S})^\infty_-$ into two regions; V which contains the cylindrical concave end corresponding to $\partial E_{1/S}$, and $W = \{S|z_1|^2 + |z_2|^2 > a^2\}$. We will work with $\mathbb{T}^{n-2}$–invariant almost-complex structures on $(M \setminus E_{1/S})^\infty_-$ for which the projection onto $\{z_j = 0 \mid j = 3, \dots, n\}$ is holomorphic on W. Let J_N denote a sequence of such almost-complex structures which are stretched to a length N along

Σ_a. We observe that Σ_a contains a $2(n-2)$ parameter family of Reeb orbits corresponding to translations of $a\gamma_1$.

Arguing by contradiction, suppose that for all such almost-complex structures J_N there exist curves f_N which represent classes in $\mathcal{K}_{1/S}$ and intersect $\mathbb{T}^{n-2}$–orbits in multiple points, or not at all. Then on W the curves f_N project with a degree $r_N > 1$ onto curves h_N lying in $\{z_j = 0 \mid j = 3, \ldots, n\}$. The sequence of r_N is bounded by d, as the f_N intersect the line at infinity at least r_N times. So, taking a subsequence, we may assume that $r = r_N$ is independent of N.

Now, taking a limit as $N \to \infty$ our curves f_N will converge to a holomorphic building whose component in W has negative ends asymptotic to translations of $a\gamma_1$. Since the component in V has a negative end asymptotic to $\gamma_1^{(3d-1)}$ we may assume, since our curves have positive area, that for a sufficiently close to 1 the negative ends in W cover the orbits a total of at least $3d-1$ times.

The projected curves h_N have bounded area and so will also converge to a holomorphic building which includes a curve h with image in

$$(\{z_j = 0 \mid j = 3, \ldots, n\} \cap \{S|z_1|^2 + |z_2|^2 > a^2\})^\infty_-.$$

Working with $\mathbb{T}^{n-2}$–invariant structures we may still assume that the limiting almost-complex structure on $(\{z_j = 0 \mid j = 3, \ldots, n\} \cap \{S|z_1|^2 + |z_2|^2 > a^2\})^\infty_-$ is regular (indeed, any almost-complex structure preserving $\{z_j = 0 \mid j = 3, \ldots, n\}$ can be extended to W as a $\mathbb{T}^{n-2}$–invariant one).

Suppose that h has degree e and s^- negative ends with the i^{th} such end covering $a\gamma_1$ a total of c_i times. Using formula (4), the deformation index of h is

$$\text{index}(h) \quad = \quad -2 + 6e - 2\sum_{i=1}^{s^-} c_i$$

which is nonnegative only if $\sum_{i=1}^{s^-} c_i \le 3e-1$. Now, the degree e of h is equal to $\frac{d}{r}$, so the negative ends of the limits of our curves f_N cover the translates of $a\gamma_1$ a total of $r\sum_{i=1}^{s^-} c_i \le r(3e-1) = 3d - r$ times. Thus, if $r > 1$ then we have a contradiction as required.

□

3.2 A compact cobordism

For a generic choice of the family J_t, the set $\mathcal{K} = \{\mathcal{K}_t \mid t \in [1/S, 1]\}$ is an oriented 1-dimensional manifold with boundary. By [2], $\mathcal{K}$ is compact modulo convergence to equivalence classes of holomorphic buildings in the $(M \smallsetminus E_t)_-^\infty$. A priori, such buildings could include multiply covered curves, that is, curves not included in the definition of $\mathcal{K}$. In this section we prove

Proposition 3.4. *The family $\mathcal{K}$ is compact.*

Proof. Let $\mathbf{F}$ be a holomorphic building in $(M \smallsetminus E_t)_-^\infty$ representing a limit point of $\mathcal{K}$. Then $\mathbf{F}$ consists of a finite collection of holomorphic curves with connected domains and images in either $(M \smallsetminus E_t)_-^\infty$ or SE_t. Each of these curves has virtual index zero and their compactifications fit together to form a continuous map $\overline{\mathbf{F}}$ from the unit disc to $M \smallsetminus E_t$ which takes the boundary circle to $\gamma_1^{(3d-1)}$. Arguing as in §2.4, one can also show that the curves of $\mathbf{F}$ with image in $(M \smallsetminus E_t)_-^\infty$ have negative ends which are asymptotic only to multiples of γ_1, and the curves with image in SE_t are multiply covered holomorphic cylinders over γ_1. Moreover, as inherited by the curves in $\mathcal{K}$, there is a unique curve of $\mathbf{F}$ of level one, and it has a single negative end which covers γ_1 precisely $3d-1$ times. This is because any curves without negative ends necessarily have positive deformation index.

Our first claim is that there is a single curve of $\mathbf{F}$ with image in $(M \smallsetminus E_t)_-^\infty$, and that it is somewhere injective. Arguing by contradiction, suppose that there are $K > 1$ such curves. Then they have degrees $d_1, \dots, d_K$ with $d_1 + \ldots + d_K = d$ and we may assume that the ends of the i^{th} component cover γ_1 a total number t_i times. Here we allow for some $t_i = 0$, which corresponds to rational curves without (nonremovable) punctures which bubble off in $(M \smallsetminus E_t)_-^\infty$. Since the curves of $\mathbf{F}$ with image in SE_t have at least as many positive ends as negative ends, when counted with multiplicity, and the curve of lowest level has a single negative end which covers γ_1 a total of $3d-1$ times, we have $t_1 + \cdots + t_K \geq 3d-1$. The assumption that $K > 1$ then implies that there must exist a $j \in [1, K]$ for which $t_j \geq 3d_j$. Suppose the corresponding curve F of $\mathbf{F}$ is an r-fold cover of a somewhere injective curve $\widetilde{F}$ (if F is not a multiple cover then we allow $r = 1$ and $\widetilde{F} = F$ here). Then $\widetilde{F}$ has degree $\tilde{d} = \frac{d_j}{r}$ and has some number $\tilde{s}^-$ of negative ends, the i^{th} of which covers γ_1 a total of $\tilde{c}_i$ times. Hence, $\sum_{i=1}^{\tilde{s}^-} \tilde{c}_i = \frac{t_j}{r}$ and the index formula (12) for $\widetilde{F}$

becomes

$$\begin{aligned}\operatorname{index}(\widetilde{F}) &= (n-3)(2-\tilde{s}^-) + 6\tilde{d} - \sum_{i=1}^{\tilde{s}^-}(2\tilde{c}_i + (n-1)) \\ &= (2n-4)(1-\tilde{s}^-) - 2 + \frac{2}{r}(3d_j - t_j) \\ &\leq -2.\end{aligned}$$

For generic choices of almost-complex structure we then have a contradiction.

Thus, there is a single curve G of $\mathbf{F}$ with image in $(M \setminus E_t)^\infty_-$. Suppose that this curve has a total of s^- negative ends, each covering γ_1 a number c_i times. If G is an r-fold cover of a somewhere injective curve $\widetilde{G}$ with $\tilde{s}^-$ negative ends then, as above, the index formula (12) gives

$$\operatorname{index}(\widetilde{G}) = (2n-4)(1-\tilde{s}^-) - 2 + \frac{2}{r}(3d - \sum_{i=1}^{s^-} c_i).$$

But as $\sum_{i=1}^{s^-} c_i \geq 3d-1$, we notice that the index can exceed -2 only if $\tilde{s}^- = 1$ and $r = 1$ and $\sum_{i=1}^{s^-} c_i = 3d-1$. In other words, the unique curve G of $\mathbf{F}$ with image in $(M \setminus E_t)^\infty_-$ is somewhere injective and does indeed lie in $\mathcal{K}$.

Now assume that $\mathcal{K}$ is not compact and that $\mathbf{F}$ represents a limit point not contained in $\mathcal{K}$. Then $\mathbf{F}$ is a holomorphic building of height $k > 1$. As described above, the unique curve of $\mathbf{F}$ of level one has a single negative end which covers γ_1 precisely $3d-1$ times. Since $k > 1$, this curve, say H, has image in SE_t. The compactifications of G, H, and the other curves of $\mathbf{F}$ with image in SE_t fit together to form a curve of genus zero. Hence, the curves with level in $[1, k)$ must each have one positive and one negative end. If follows that the curves of $\mathbf{F}$ with image in SE_t are all cylindrical, and so $\mathbf{F}$ and G represents the same element of $\mathcal{K}$. This contradiction affirms that $\mathcal{K}$ is compact.

□

3.3 The completion of the proof

Theorem 1.2 now follows almost immediately from Propositions 3.1 and 3.4. For any $S > 0$ with $S^2 \in \mathbb{R} \setminus \mathbb{Q}$, and any positive integer d satisfying

$d(3d-1) < S^2$, it follows from Proposition 3.4 that for a generic family J_t the space $\mathcal{K} = \{\mathcal{K}_t \mid t \in [1/S, 1]\}$ is a compact, oriented, 1-dimensional manifold whose boundary is $\mathcal{K}_{1/S} \cup \mathcal{K}_1$. Proposition 3.1 then implies that the moduli space $\mathcal{K}_1$ is nonempty. Hence, there exists a holomorphic plane in $(M \smallsetminus E_1)^{\infty}_{-}$ whose negative end is asymptotic to $\gamma_1^{(3d-1)}$. The symplectic area of this curve is positive and equal to $d\pi R^2 - (3d-1)\pi$. Thus, $R^2 > \frac{3d-1}{d}$, and taking the limit as d (and hence S) goes to ∞ we have $R^2 \geq 3$.

4 Symplectic embeddings

In this section we prove Theorems 1.3 and 1.6.

4.1 The proof of Theorem 1.6

Let $\Sigma(\epsilon)$ be a punctured torus, i.e., a surface of genus one with one boundary component, equipped with a symplectic form of total area ϵ. The Main Lemma of [7] implies the following.

Proposition 4.1. *For any $S, \epsilon > 0$, there exists a symplectic embedding of $B^{2(n-1)}(S)$ into $\Sigma(\epsilon) \times \mathbb{R}^{2(n-2)}$.*

To establish Theorem 1.6 it suffices to find an embedding of $\Sigma(\epsilon) \times B^2(1)$ into $B^2(\sqrt{R}) \times B^2(\sqrt{R})$, where $R > 2$ is fixed and ϵ can be arbitrarily small. The existence of such an embedding is essentially contained in Lemma 3.1 of [7]. We review the construction here for the sake of completeness and because it will play an important role in the proof of Theorem 1.3.

Fix $R = 2 + 2\delta$ for $\delta > 0$. We begin with the following elementary result.

Lemma 4.2. *For every $\delta > 0$ there is a nonnegative function H whose support is contained in $B^2(\sqrt{2+2\delta})$, whose maximum value is less than $\pi + \delta$, and whose time-t Hamiltonian flow, ϕ_H^t, satisfies*

$$\phi_H^t(B^2(1)) \subset B^2(\sqrt{(1+t)(1+\delta/\pi)})$$

and

$$\phi_H^1(B^2(1)) \cap B^2(1) = \emptyset.$$

Proof. Let U be any set whose closure is contained in the interior of the square $[0,\sqrt{\pi+\delta}]\times[0,\sqrt{\pi+\delta}]\subset\mathbb{R}^2$. The time-$t$ Hamiltonian flow of the function $K(x,y)=(\sqrt{\pi+\delta})x$ on $\mathbb{R}^2$ is given by

$$\phi^t_K(x_1,y_1)=(x_1,y_1+t\sqrt{\pi+\delta}).$$

(We use the convention that the Hamiltonian vectorfield, X_K, of K is defined by the equation $i_{X_K}\omega_0=-dK$.) Hence, for all $t>0$ the set $\phi^t_K(U)$ is contained in the interior of the rectangle $[0,\sqrt{\pi+\delta}]\times[0,(1+t)\sqrt{\pi+\delta}]$ and $\phi^1_K(U)\cap U=\emptyset$. Cutting K off appropriately near $\bigcup_{t\in[0,1]}\phi^t_K(U)$, we get a nonnegative function $\hat{K}$ whose Hamiltonian flow still has these properties, but is now supported in $[0,\sqrt{\pi+\delta}]\times[0,2\sqrt{\pi+\delta}]$ and satisfies $\max(\hat{K})<\pi+\delta$.

One can construct a symplectic diffeomorphism ψ of $\mathbb{R}^2$ which maps $[0,\sqrt{\pi+\delta}]\times[0,2\sqrt{\pi+\delta}]$ into $B^2(\sqrt{2+2\delta})$ and for $t\in[0,1]$ maps arbitrarily large subsets of each rectangle $[0,\sqrt{\pi+\delta}]\times[0,(1+t)\sqrt{\pi+\delta}]$ onto balls centered at the origin. (Such maps are described and illustrated explicitly in Section 3.1 of [15].) We choose these arbitrarily large subsets of the rectangles $[0,\sqrt{\pi+\delta}]\times[0,(1+t)\sqrt{\pi+\delta}]$ so that they contain $\phi^t_K(U)$ for all $t\in[0,1]$. Setting $U=\psi^{-1}(B^2(1))$ and $H=\hat{K}\circ\psi$, we are done. □

Remark 4.3. It is clear from the definition of H in terms of $\hat{K}$ that for all $t\in[0,1]$ the distance between $\phi^t_H(B^2(1))$ and the boundary of $B^2(\sqrt{(1+t)(1+\delta/\pi)})$ is greater than zero and of order δ.

Consider an immersion i_δ of $\Sigma(\epsilon)$ into $\mathbb{R}^2$, as sketched in Figure 1, with the following properties:

- the double points are concentrated in an arbitrarily small region around the origin $(0,0)$.
- the vertical and horizontal sections crossing at $(0,0)$ are arbitrarily thin (and hence arbitrarily long if need be).
- the areas of the regions A and B are both equal to $\pi+2\delta$.

By the last of these properties we may assume that, for sufficiently small $\epsilon>0$, the immersion lies in a region symplectomorphic to $B^2(\sqrt{2+2\delta})$.

Let I^0_δ be the symplectic immersion of $\Sigma(\epsilon)\times B^2(1)$ into $B^2(\sqrt{2+2\delta})\times B^2(\sqrt{2+2\delta})$ which acts as i_δ on the first factor and as inclusion on the

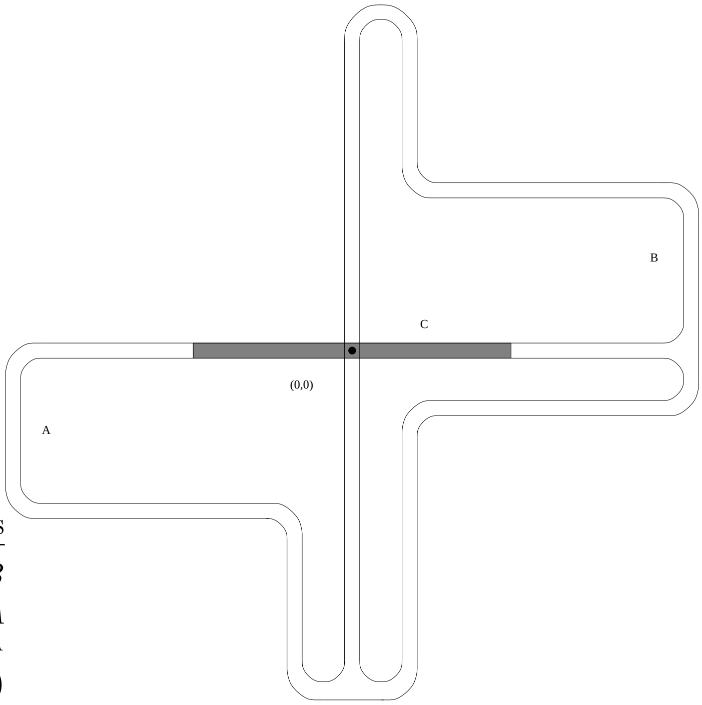

Figure 1: The symplectic immersion i_δ of the punctured torus.

second. We now alter the image of I_δ^0 to obtain the desired embedding. In what follows, we will use coordinates (x_1, y_1) on the plane containing the first copy of $B^2(\sqrt{2+2\delta})$ and coordinates (x_2, y_2) on the plane containing the second copy. The projection from $\mathbb{R}^4$ to the x_1y_1-plane will be denoted by pr_1.

The self-intersections of I_δ^0 project under pr_1 to the self intersections of i_δ. The x_1-coordinates of these points take values in an interval of the form $[-\lambda, \lambda]$. For $\Lambda > \lambda$, let C^Λ be the horizontal portion of the image of i_δ which passes through the origin and whose first coordinates satisfy $x_1 \in [-\Lambda, \Lambda]$. To remove the intersections of I_δ^0, we consider the Hamiltonian $\hat{H} = \chi(x_1)H(x_2, y_2)$ where H is the Hamiltonian from Lemma 4.2 and χ is a bump-function which equals 1 for $|x_1| \leq \lambda$ and equals 0 for $|x_1| \geq \Lambda$. The time-1 Hamiltonian flow of $\hat{H}$ is

$$\phi^1_{\hat{H}}(x_1, y_1, x_2, y_2) = (x_1, y_1 + \chi'(x_1)H(x_2, y_2), \phi_H^{\chi(x_1)}(x_2, y_2)).$$

Let I^1_δ be the symplectic immersion of $\Sigma(\epsilon) \times B^2(1)$ into $B^2(\sqrt{2+2\delta}) \times B^2(\sqrt{2+2\delta})$ obtained by applying $\phi_{\hat{H}}$ to $C^\Lambda \times B^2(1)$. Clearly, I^1_δ agrees with I^0_δ away from $i_\delta^{-1}(C^\Lambda) \times B^2(1)$, and shares none of the original double points of I^0_δ. The immersion I^1_δ can only have new double points in $\phi_{\hat{H}}(C^\Lambda_\pm \times B^2(1))$ where C^Λ_+ and C^Λ_- are the portions of C^Λ corresponding to points with x_1-values in $[\lambda, \Lambda]$ and $[-\Lambda, -\lambda]$, respectively. We now show that, for an appropriate choice of i_δ and χ, I^1_δ can be adjusted on $\phi_{\hat{H}}(C^\Lambda_\pm \times \Psi_\delta(B^2(1)))$ so that no new double points occur.

In the x_1y_1-plane, $\phi_{\hat{H}}$ only moves points in the interior of $C^\Lambda_\pm$ and does so only in the y_1–direction. Moreover, the maximum displacement in this direction is bounded from above by

$$| \int_0^1 \chi'(x_1) \max(H)\, dx_1| < \max(|\chi'|)(\pi + \delta).$$

Hence, the image of $\phi_{\hat{H}}(C^\Lambda_+ \times \Psi_\delta(B^2(1)))$ under pr_1 is contained in the set

$$C^\Lambda_+ + ([\lambda, \Lambda] \times [0, \max(|\chi'|)(\pi + \delta)])\,,$$

and the projection of $\phi_{\hat{H}}(C^\Lambda_- \times \Psi_\delta(B^2(1)))$ is contained in

$$C^\Lambda_- + ([-\Lambda, -\lambda] \times [-\max(|\chi'|)(\pi + \delta), 0])\,.$$

Choosing the width of C^Λ to be sufficiently small, and $\max(|\chi'|)$ sufficiently close to $\frac{1}{\Lambda - \lambda}$, we may assume that both $\phi_{\hat{H}}(C^\Lambda_\pm \times \Psi_\delta(B^2(1)))$ project to regions in the x_1y_1-plane whose area is less than $\pi + 2\delta$ and hence less than the area of each of the regions A and B. Acting on $\phi_{\hat{H}}(C^\Lambda_- \times \Psi_\delta(B^2(1))$ and $\phi_{\hat{H}}(C^\Lambda_+ \times \Psi_\delta(B^2(1))$ by another symplectic diffeomorphism which acts nontrivially only in the x_1y_1–directions, we may then ensure that they are mapped by pr_1 into A and B, respectively. The resulting symplectic immersion therefore has no double points and is the desired embedding of Theorem 1.6.

4.2 The proof of Theorem 1.3

Scaling things appropriately, the argument above yields a symplectic embedding of $\Sigma(\epsilon) \times B^2(r)$ into $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ for any $r < 1$ provided that ϵ is sufficiently small. In this section, we show that the previous embedding procedure can be refined to obtain a symplectic embedding of $\Sigma(\epsilon) \times B^2(r)$ into $B^4(\sqrt{3})$. This will prove Theorem 1.3 which implies that Theorem 1.1 is sharp.

Remark 4.4. The bi-disc $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ can be symplectically embedded into $B^4(2)$, by inclusion. The second Ekeland-Hofer capacity implies that this is optimal in the sense that $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ can not be symplectically embedded into a smaller ball.

As in the proof of Theorem 1.6, we start with a symplectic immersion I^0 of $\Sigma(\epsilon) \times B^2(r)$ into $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ which acts by an immersion $i\colon \Sigma(\epsilon) \hookrightarrow B^2(\sqrt{2})$ in the first factor, and by inclusion on the second factor. The immersion i is chosen so that for some $\lambda > 0$ we have:

- the vertical and horizontal crossing components of the image have length equal to $2\pi/\lambda$.
- the regions A and B have areas in the interval $(\pi r^2, \pi)$ and all but an arbitrarily small amount of this area is concentrated within a distance λ of the horizontal crossing component, C.

See Figure 2.

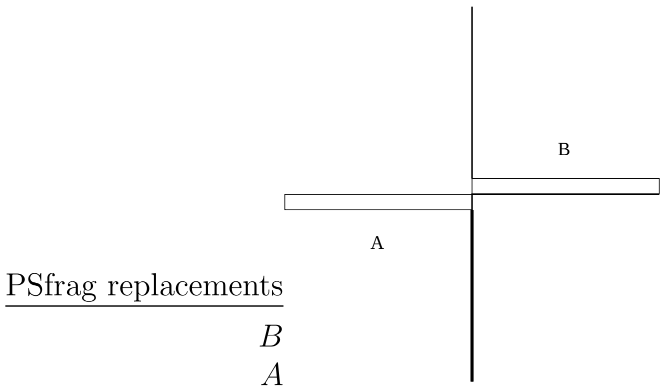

Figure 2: The symplectic immersion i, from a distance.

Set $\Lambda = \pi/\lambda$, so that $C^\Lambda = C$. Defining χ and $\hat{H}$ as in the proof of Theorem 1.6, we remove the double points of I^0 by applying the time-1 flow of $\hat{H}$ to $C \times \Psi(B^2(r))$ to obtain a new immersion I^1. Choosing λ and $\left(\max(|\chi'|) - \frac{1}{\Lambda - \lambda}\right)$ to be sufficiently small, we may assume that the projection pr_1 maps $\phi_{\hat{H}}(C \times \Psi(B^2(r)))$ to the interior of the following region of the x_1y_1-plane

$$\mathbf{C} = C + \{([-\pi/\lambda, -\lambda] \times [-\lambda, 0]) \cup ([\lambda, \pi/\lambda] \times [0, \lambda])\}\,.$$

When the width of C is small enough, the area of $\mathbf{C}$ is less than 2π. As in the proof of Theorem 1.6 we can then apply a suitable symplectic map to $\phi_{\hat{H}}(C\times\Psi(B^2(r)))$ to shift the relevant parts of its projection into A and B and hence obtain a symplectic embedding of $\Sigma(\epsilon)\times B^2(r)$ into $B^2(\sqrt{2})\times B^2(\sqrt{2})$.

We now refine this embedding procedure by choosing a new immersion of $\Sigma(\epsilon)$. To motivate this choice we first consider the fibers of the projection map pr_1 acting on $I^1(\Sigma(\epsilon)\times B^2(r))$. The points not in $\phi_{\hat{H}}(C\times\Psi(B^2(r)))$ belong to fibres which can all be identified with $B^2(r)$. For points in $\phi_{\hat{H}}(C\times\Psi(B^2(r)))$, it follows from the definition of H, that their fibers lie in a region determined by the x_1 component of their projection. Denote by $F(x_1)$ the fiber corresponding to $x_1\in[-\pi/\lambda,\pi/\lambda]$. As described in the proof of Lemma 4.2, $F(x_1)$ can be identified with a subset of $B^2(\sqrt{2})$. For $|x_1|\leq\lambda$, $F(x_1)$ is contained in a fixed subset of $B^2(\sqrt{2})$ whose closure lies in the interior of this ball. For $\lambda<|x_1|\leq\pi/\lambda$, the closure of the fiber $F(x_1)$ is contained in the interior of $B^2(\sqrt{1+\chi(x_1)})$. We can choose the bump function $\chi(x_1)$ so that for $\lambda<|x_1|\leq\pi/\lambda$ it is arbitrarily C^0-close to $1-\frac{|x_1|+\lambda}{\pi/\lambda-\lambda}$. Hence, for sufficiently small $\lambda>0$ and a suitable choice of χ we may assume that the closure of $F(x_1)$ is contained in the interior of $B^2(\sqrt{2-|x_1|\lambda/\pi})$ for all $x_1\in[-\pi/\lambda,\pi/\lambda]$.

We now apply a symplectomorphism w to the (x_1,y_1)-plane which winds $\mathbf{C}$ around itself as sketched in Figure 3. This winding is nearly tight but includes small gaps (represented by the thicker lines in Figure 3) to accommodate the winding of the rest of $i(\Sigma(\epsilon))$. Since the area of $\mathbf{C}$ is less than 2π, for sufficiently small $\epsilon>0$ we may assume that the image of $i(\Sigma(\epsilon))\cup\mathbf{C}$ under the winding map still lies in the ball $B^2(\sqrt{2})$. Replacing the embedding i in the previous construction with the composition $w\circ i$, we get a new symplectic embedding

$$I_w\colon\Sigma(\epsilon)\times B^2(r)\hookrightarrow B^2(\sqrt{2})\times B^2(\sqrt{2}).$$

We now show that the image of I_w is contained in $B^4(\sqrt{3})$.

Let $(z_1,z_2)\in\mathbb{C}^4$ be any point in the image of I_w. Then $z_1=w(x_1,y_1)$ for a unique point (x_1,y_1) in $i(\Sigma(\epsilon))$ and z_2 belongs to $F(x_1)$. Since $w(i(\Sigma(\epsilon))\cup\mathbf{C})\subset B^2(\sqrt{2})$ we have

$$|z_1|<\sqrt{2}. \tag{13}$$

By the analysis of the fibres $F(x_1)$ above, it follows from Remark 4.3 that

$$|z_2|=\sqrt{2-|x_1|\lambda/\pi}-O(|\pi-r|) \tag{14}$$

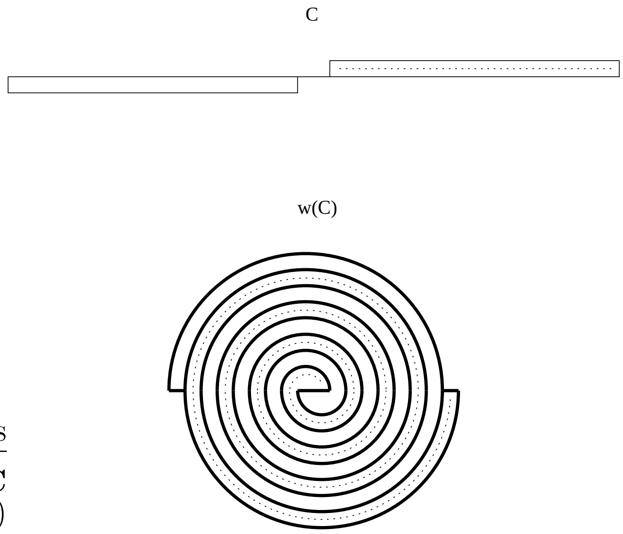

Figure 3: The winding map w acting on $\mathbf{C}$.

where the error term is positive. On the other hand it follows from the definition of the winding map w that

$$|z_1| = \sqrt{\frac{2\lambda}{\pi}|x_1|} + O(\lambda) + O(\epsilon\lambda). \tag{15}$$

The first approximation comes from equating the area of the portion of C determined by x_1 , $2|x_1|\lambda$, with $\pi|z_1|^2$. The error terms of (15) correspond, respectively, to the discrepancy caused by the width of $\mathbf{C}$, and the discrepancy caused by the gap in the winding. Together, equations (14) and (15) yield

$$(|z_2| + O(|\pi - r|))^2 = 2 - \frac{1}{2}(|z_1| - O(\lambda) - O(\epsilon\lambda))^2 \tag{16}$$

The fact that $O(|\pi - r|)$ is positive implies that for sufficiently small $\lambda > 0$ we have

$$|z_2|^2 \leq 2 - \frac{1}{2}|z_1|^2 \tag{17}$$

Together, inequalities (13) and (17) imply that

$$|z_1|^2 + |z_2|^2 \leq 3, \tag{18}$$

as desired.

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