SYMPLECTIC RIGIDITY FOR ANOSOV HYPERSURFACES

D. BURNS AND R. HIND

1. Introduction

There is a canonical exact symplectic structure on the unit tangent bundle of a Riemannian manifold M given by pulling-back the symplectic two form ω and Liouville one form λ from the cotangent bundle T^*M using the Riemannian metric. The pull-back of λ gives a contact form on level-sets of the length function on TM. The geodesic flow of M is given by the Reeb vectorfield of this contact structure, and the invariants of this flow are very important invariants of the symplectic manifold with boundary, or even, in some cases, of the open symplectic manifold. In such favorable circumstances, symplectic equivalence can apply much stronger rigidity results. For example, the following result is a straightforward application of the symplectic homology theory, see [4], and a theorem of J. Otal [18] and C. Croke [5].

Theorem 1.1. *If the interiors of the unit tangent bundles of two compact Riemann surfaces of strictly negative curvature are exact symplectomorphic then the underlying Riemann surfaces are isometric.*

An exact symplectomorphism f is one for which the one form $f^*\lambda - \lambda$ is exact.

Although in this paper we will mainly be considering tangent bundles of surfaces, the above theorem 1.1 does have some generalizations to higher dimensions. For example, using a result of U. Hamenstädt in [11] we get

Theorem 1.2. *Let M and N be closed, strictly negatively curved manifolds and suppose that N is a locally symmetric space. If the interiors of the unit tangent bundles of M and N are exact symplectomorphic then M and N are isometric.*

Theorem 1.1 was first remarked by Sikorav [20], and is the paradigm of the type of result we would like to prove. For completeness,

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at the end of the introduction we detail how to obtain theorems 1.1 and 1.2 from the symplectic homology theory together with the results on length spectrum rigidity. In the meantime we will concentrate on surfaces.

Corollary 1.1. *If the interiors of the unit tangent bundles of two Riemann surfaces of strictly negative curvature are exact symplectomorphic then the closed symplectic manifolds are also symplectomorphic.*

Although it is a weaker result than the theorem, this corollary is still very interesting, especially in the light of work of Y. Eliashberg and H. Hofer, see [8], showing that there exist $\mathcal{C}^\infty$-small perturbations of the standard unit ball in $\mathbb{R}^{2n}$ whose interiors are symplectomorphic but whose boundaries have inequivalent flows.

The main purpose of this paper is to extend the above symplectic rigidity result, corollary 1.1, to a larger class of symplectic manifolds.

Fixing our underlying smooth surface M, the unit tangent bundles corresponding to different metrics can symplectically be thought of as domains in T^*M with the restricted canonical symplectic form by applying the Legendre transform. We will look at the class of symplectic manifolds obtained by deforming in T^*M the domains corresponding to negatively curved metrics and again restricting the canonical symplectic form (the exact description is given below). This class of symplectic manifolds is the same as the one obtained by fixing a unit tangent bundle and deforming the primitive of the symplectic form in the same cohomology class.

Remark. This class of symplectic manifolds is open and certainly contains domains in T^*M which are not the Legendre transform of Riemannian unit tangent bundles. It also contains domains which are not symplectomorphic to Riemannian unit tangent bundles. For example, a negatively curved Riemannian metric can be deformed to give a Finsler metric G which is non-symmetric (i.e., $G(v) \neq G(-v)$, for some v) with the property that a closed geodesic in a certain free homotopy class γ is of different length to the unique closed geodesic in the class $-\gamma$. Now, when one applies the Legendre transform to a unit (Riemannian or Finsler) tangent bundle, the geodesic flow on the tangent bundle (restricted to a fixed energy level) corresponds under the Legendre transform to the Reeb flow and in particular the lengths of closed geodesics correspond to the periods of closed orbits of the Reeb flow. By the symplectic homology theory, these periods are invariants of the exact symplectomorphism type of the domain. But for a Riemannian metric the unique closed geodesic in the opposite class to a given closed geodesic is just the original closed geodesic traversed in the opposite direction and has the same length. Thus the domain obtained from our Finsler metric cannot be exact symplectomorphic to a Riemannian domain. We do not know, on the other hand, whether there exist symmetric domains in T^*M which are not symplectomorphic to Riemannian domains.

If one deforms a Riemannian unit cotangent bundle so that the deformed domain still intersects each fiber in a convex set containing the origin, one can apply the inverse Legendre transform and associate to the domain a Finsler metric. One might hope to generalise theorem 1.1 and say that if two such domains are symplectomorphic then their associated Finsler metrics must be isometric. Unfortunately though, it is easy to construct examples showing this to be false. In particular, it is possible for a Finsler metric to have the same length spectrum as a Riemannian metric while still not being Riemannian itself. To see this, we start with a Riemannian domain W in T^*M, say corresponding to a metric g on M. Let H be a (Hamiltonian) function on T^*M supported in a sufficiently small neighbourhood of some point $x \in \partial W$. We assume that the induced Hamiltonian diffeomorphism ϕ, the time-1 flow of the Hamiltonian vector field, does not preserve ∂W, and then study the domain $W' = \phi(W)$. Applying the inverse Legendre transform to W', provided that H was sufficiently small, we will get a domain in TM which is the unit tangent bundle of a certain Finsler metric. We want to observe that this Finsler metric is not Riemannian. But for a Riemannian metric, the unit circle in each tangent space T_pM is an ellipse. In this case though, there are some tangent spaces where the unit circle of the Finsler metric coincides with that for the Riemannian metric g except in a neighbourhood of some point. Since ellipses which coincide on open sets are actually equal, the Finsler unit circle cannot be an ellipse and hence the metric is not Riemannian. Choosing the Hamiltonian H to be supported in a neighbourhood of two points, the above construction can be carried out symmetrically and gives examples of symmetric Finsler domains which are symplectomorphic to Riemannian domains but are not Riemannian.

Given these limiting observations, we will therefore seek to extend corollary 1.1. First we will be more specific about exactly which class of symplectic manifolds will be considered.

Let $\mathcal{W}$ be the class of domains W in T^*M with smooth boundary such that the canonical Liouville form λ on T^*M restricts to a contact form on ∂W whose Reeb vector field X, uniquely defined by $X \rfloor d\lambda = 0$

and $\lambda(X) = 1$, generates an Anosov flow on ∂W. We will also assume that the zero-section in T^*M lies inside W, and that the fibers in T^*M are star-shaped.

It follows from Anosov's structural stability theorem, see [1], that $\mathcal{W}$ is an open set (with a topology of smooth convergence). Furthermore, the geodesic flow of a negatively curved metric restricts to an Anosov flow on constant energy surfaces and so all the deformed domains described above lie in $\mathcal{W}$, and in fact they lie in the same connected component $\mathcal{W}^\circ$. We can now generalise the above corollary as follows.

Theorem 1.3 (**Main Theorem**). *Suppose that the interiors of two domains W_1 and W_2 in $\mathcal{W}^\circ$ are exact symplectomorphic. Then the closed symplectic manifolds are symplectomorphic and in fact the symplectomorphism can be taken to be the restriction of a smooth Hamiltonian diffeomorphism on T^*M, perhaps composed with the differential of a diffeomorphism of M.*

We emphasize the smoothness here as this relies on results in dynamical systems due to Feldman and Ornstein [9], and to R. de la Llave and R. Moriyon [16]. As far as the authors are aware, there are no other known examples of a collection of symplectic manifolds which is invariant under small perturbations and has the property given by the above theorem.

Let $\partial\mathcal{W} = \{\partial W | W \in \mathcal{W}^\circ\}$. Given a hypersurface Σ in $\partial\mathcal{W}$, each free homotopy class in $\pi_1(M)$ corresponds to the projection of a unique closed orbit on Σ. The marked length spectrum of Σ is the map associating the length of the closed orbit to the homotopy class. The only infomation needed about the interior symplectomorphism above is that it preserves the marked length spectrum of the Reeb flow on the boundary, that is, corresponding closed orbits have the same length. Therefore we also have the following result.

Theorem 1.4. *Suppose that two hypersurfaces Σ_0 and Σ_1 in $\partial\mathcal{W}$ have the same marked length spectrum. Then they are connected by a smooth 1-parameter family of hypersurfaces in $\partial\mathcal{W}$ whose Reeb flows are all smoothly time-preserving conjugate to the flow on Σ_0 by a smooth family of Hamiltonian diffeomorphisms.*

In particular, the subvarieties of constant marked length spectra are path-connected. The subsets of Riemannian hypersurfaces of constant marked spectra are of course also connected, as follows from the marked length spectrum rigidity of Croke and Otal together with the Dehn-Nielsen Theorem (Theorem 4.6.25 from [21], which gives a short history of the result). Although the theorem of Croke and Otal does not generalize to Finsler metrics, this result does.

1.1. **Proof of Theorem 1.1.** By [5] or [18], it suffices to show that the surfaces have the same marked length spectrum (up to diffeomorphism).

But suppose that the interiors are symplectomorphic via an exact symplectomorphism ϕ. Then ϕ induces an isomorphism of the first homotopy groups of the underlying surfaces. By a theorem of Nielsen, see [17], any such isomorphism can be induced by a diffeomorphism of the surface. Hence, by composing with the inverse of this diffeomorphism, we may assume that the map ϕ_* on $\pi_1(M)$ is the identity.

Now we apply the symplectic homology theory, see for example [4], to deduce that each (oriented) closed geodesic in the first surface, which gives a closed orbit of the characteristic flow on the boundary of the symplectic manifold, must correspond to a closed orbit of the characteristic flow and hence a closed geodesic in the second surface of the same length. In fact, one may define symplectic homology groups $S^a(T_rM, [\gamma])$ localized at the free homotopy class $[\gamma]$ for $a \in \mathbb{R}$ and any free homotopy class $[\gamma]$ in T_rM (or equivalently, M) as follows.

One constructs chain groups as in [4], restricting attention to Hamiltonians as in [4] together only with their period 1 closed orbits in the free homotopy class $[\gamma]$. The differential of the complexes are still defined as in [4], and the proof that this defines a complex is the same as in [4]. The key point is that the solution surfaces used to define the differentials in [4], p. 32, are cylindrical and therefore only join period-1 trajectories in the same free homotopy class. One then passes to the limit over Hamiltonians as in [4], and if all closed characteristics in the free homotopy class $[\gamma]$ are transversally non-degenrate on ∂T_rM, then one can pass to groups $S^a(T_rM, [\gamma])$, $a \in \mathbb{R}$. In this latter case, the same computation as in [4] shows that these groups are zero for a not equal to the action of a closed characteristic in the free homotopy class $[\gamma]$, and each closed characteristic with action a in this free homotopy class contributes exactly two copies of $\mathbb{Z}_2$ to the group $S^a(T_rM, [\gamma])$.

In our case, there is a unique closed characteristic in the boundary of T_rM in any free homotopy class of T_rM, and this characteristic is transversally non-degenerate. Applying the symplectic homology localized at $[\gamma]$ above, it follows that $S^b(T_rM, [\gamma]) = 0$, for $b \neq$ the action of the closed characteristic in the free homotopy class $[\gamma]$, and

$S^a(T_rM, [\gamma])$ = two copies of $\mathbb{Z}_2$ for a = the action of this closed characteristic = $(-r)\times$ the length of the corresponding closed geodesic in M. As ϕ_* preserves (localized) symplectic homology, by construction, we may identify the fundamental groups of M and M' in such a way that the length functions for the two metrics are the same on the fundamental group, as required.

1.2. **Proof of Theorem 1.2.** Again, now according to Hamenstädt, see [11], it suffices to show that the manifolds M and N have the same marked length spectrum. As M and N are strictly negatively curved, each free homotopy class of closed curves in M or N contains a unique closed geodesic. Hence the marked length spectrum can be thought of as a map from conjugacy classes in π_1 to the real numbers, assigning to each class the length of the unique closed geodesic in that class. By having the same marked length spectrum we now mean that there is an isomorphism $\Psi : \pi_1(M) \to \pi_1(N)$ which pulls back the marked length spectrum of N to that of M.

Given an exact symplectomorphism ϕ of the open unit tangent bundles, we claim that the induced map $\phi_* : \pi_1(M) \to \pi_1(N)$ gives such an isomorphism Ψ. To see this, observe again that the closed geodesic γ in a particular class $[\gamma]$ of $\pi_1(M)$ can be lifted to a closed orbit of the characteristic flow on the unit tangent bundle, since this characteristic flow is exactly the geodesic flow. Now the symplectic homology theory, restricted to considering orbits in the class $[\gamma]$, can be used to show that the class $\phi_*[\gamma]$ similarly contains a unique closed orbit of the characteristic flow of the same length. In other words, the closed geodesic of N in the class $\phi_*[\gamma]$ of $\pi_1(N)$ has the same length as the geodesic γ in M. This establishes our claim and proves the theorem.

□

1.3. **Relations with complex geometry.** The authors first were drawn to this subject by the paper of J.-C. Sikorav [19], and comments on it made to us by David Barrett concerning the translation of the symplectic rigidity results there to holomorphic rigidity results. This leads directly to the consideration of Grauert tubes [10], [13], [2], a natural complex structure on a ball bundle in the (co-)tangent bundle of a real-analytic Riemannian manifold. A translation of theorem 1.1 above in terms of complex structures is the following

Corollary 1.2. *For two compact Riemannian surfaces M, M' of stricly negative curvature, with Grauert tube complex structures defined on the respective tangent ball bundles of radius r, $T_r(M), T_r(M')$, the following are equivalent:*

(i) T_rM and T_rM' are symplectomorphic
(ii) T_rM and T_rM' are biholomorphic
(iii) M and M' are isometric.

The result of Benci and Sikorav [19] gives a similar result, but for translation invariant sets in $T(T^n)$, with fibers over T^n which have vanishing first homolgy. The only Riemannian disk bundles in this case are for flat metrics on T^n. The rigidity result theorem 1.1 for surfaces holds when the metric is allowed not to be of constant sectional curvature, but the metric has to be Riemannian. We wish to thank David Barrett for his insightful remark.

We have noted above that if T_rM and T_rM' are symplectomorphic smoothly to the boundry for two Finsler manifolds M, M', they need not be Finsler isometric. Duchamp and Kalka [6] have extended the Grauert tube construction to the case of real analytic Finsler metrics, and we observe the following:

Corollary 1.3 (Holomorphic Finsler Rigidity). *For real-analytic, symmetric Finsler manifolds M, M' with Grauert tubes T_rM, resp., T_rM', if $\Phi : T_rM \to T_rM'$ is a biholomorphism, then Φ is induced by the differential of an isometry $\phi : M \to M'$.*

Recall that a symmetric Finsler metric g is one for which $g(v) = g(-v)$, for any tangent vector v. To see the corollary 1.3, note that in this case, it follows from [6] that the antipodal map $\sigma : v \to -v$ is anti-holomorphic on the Grauert tube. Then, by [2], the mapping Φ of the corollary must take the zero section of T_rM to that of T_rM'. By [6], T_rM, T_rM' carry solutions u, u' of the homogeneous Monge-Ampére equation which are 0 on $M, resp., M'$, and r on $\partial T_rM, resp., \partial T_rM'$. Again, by [6], this implies that $\Phi\mid_M := \phi$ induces a Finsler isometry.

We remark that corollary 1.3 is false for non-symmetric Finsler metrics, at least in dimension 1. To see this consider a standard annulus $A \subset \mathbb{C}$ of finite, positive inner and outer radius. There is a circle C_0 in the interior of A and a complex conjugation σ of A which exchanges boundary components and which fixes C_0 pointwise. Choose any *other* concentric circle C in the interior of A. Since we can find a harmonic function u on $A \setminus C$ which is 1 on ∂A, and 0 on C, A is the total space of a Finsler Monge-Ampère model as in [6] for a non-symmetric Finsler metric g on C. Any biholomorphism Φ of A which exchanges boundary components of A sends C to $\sigma(C) \neq C$, violating corollary 1.3.

We do not know of examples of this last phenomenon in higher dimensions.

1.4. **The case of genus** 0. We have not said anything in this paper about metrics on the 2-sphere. It turns out that Eliashberg and Hofer's construction in [8] can be used to give an example of two arbitrarily small perturbations of the round metric on the sphere such that the corresponding symplectic domains in T^*S^2 have symplectomorphic interiors but non-conjugate Reeb flows on the boundary, hence the closed domains are not symplectomorphic.

It is not clear however whether this construction can be performed such that the perturbed metrics are Riemannian rather than just Finsler. It is also unknown whether the round sphere itself is rigid, that is, if any other Riemannian or Finsler metric has a symplectomorphic open unit tangent bundle. Symplectic homology will not provide the answer though. Ideas from [22] can be used to give examples of Riemannian metrics on S^2 whose unit tangent bundles have the same symplectic homology (and volume) as the round metric.

1.5. **Outline of sections.** The proof the main theorem is in section 3. We foliate a neighborhood of both boundaries smoothly by contact hypersurfaces whose Reeb flows will be continuously conjugate by homeomorphisms, up to parametrizations, by Anosov's theory. As above, symplectic homology will tell us that the marked length spectra of these flows will be identical near the boundaries. Then we use more recent results to deduce regularity properties of the conjugating homeomorphisms, these results being special to dimension three Anosov systems. Finally, in section 3.2, we have to extend the symplectomorphism constructed above near the boundaries to a global symplectomorphism. This requires a technical lemma about isotopies of diffeomorpisms of three manifolds which almost preserve a non-vanishing vectorfield. The vectorfield condition means that this is a basically two-dimensional problem, and the most laborious task of the paper is to use the analysis of two dimensional diffeomorphisms in [21] to analyze this situation carefully. This analysis is the content of section 2. Basically, we have to show a more or less canonical procedure for isotoping a small diffeomorphism in dimension two to the identity by an isotopy which is slow in the deformation parameter.

2. A technical lemma

The purpose of this section is to prove the following lemma which will be needed for our main result Theorem 1.3. Let N be a compact

3-manifold and X a nowhere-vanishing vector field on N. We fix a metric on N, which induces a norm on T_xN for all $x \in N$. We denote by $\mathrm{dist}(x, y)$ the corresponding distance between x and $y \in N$, while for maps $f, g : N \to N$, we set $\|f - g\|_\infty = \max_{x\in N} \mathrm{dist}(f(x), g(x))$ and $\|f_*X - X\|_\infty = \max_{x\in N} \|df(x)(X) - X\|$.

Lemma 2.1. *There exists an ϵ (depending only on N, X and the metric) such that if f is a diffeomorphism of N with $\|f - I\|_\infty < \epsilon$ and $\|f_*X - X\|_\infty < \epsilon$ then f is isotopic to the identity through maps f_t with $\|f_{t*}X - X\|_\infty < C\epsilon$, for some C independent of f and ϵ.*

In our proof the condition about the vector field is heavily used and makes this an essentially two-dimensional problem. In section 3 we will need to apply lemma 2.1 to isotopies:

Lemma 2.2. *Let $F(x, \tau)$ be an isotopy of N indexed by τ with $\|F - I\|_\infty < \epsilon$, $\|F_*X - X\|_\infty < \epsilon$, for all $\tau \in [0, 1]$, and $F(x, \tau) \equiv x$, for $\tau = 0, 1$. Then there is a one-parameter family of isotopies $F(x, \tau, t)$ indexed by t with $\|F_{t*}X - X\|_\infty < C\epsilon$ and such that $F(x, \tau, 0) = F(x, \tau)$, $F(x, \tau, 1) = I$ and $F(x, \tau, t) \equiv x$ for $\tau = 0, 1$ and all $t \in [0, 1]$.*

Proof of 2.1 and 2.2. We prove lemma 2.1 for f, making comments to show that the proof is compatible with the parameter τ in F of lemma 2.2.

We choose an open cover $B_1(x_i), 1 \le i \le n$ for N of coordinate balls of radius 1 and assume this is such that on each ball we can choose coordinates (x, y, z) with X represented by $\frac{\partial}{\partial z}$. Further we assume that in fact the balls $B_{\frac{1}{4}}(x_i)$ also cover N.

In each $B_1(x_i)$ we look at the disks $D_i^\pm = \{(x, y, \pm\frac{1}{2}) | x^2 + y^2 \le \frac{1}{2}\}$. We may assume that as i varies these disks are all disjoint in N, and even that there is a constant $\delta_0 > 0$ such that any two $D_i^\pm, D_j^\pm$ are at least $4\delta_0$ apart. Furthermore, inside any $B_1(x_i)$ we may assume that the projection along X of any collection Δ of $D_j^\pm \cap B_1(x_i)$ onto $D_i^\pm$ yields a set $U \subset D_i^\pm$ which is a smooth neighbourhood of a segment of the boundary.

Our first goal is to show that by taking ϵ sufficiently small, any given f as in the statement of lemma 2.1 can be isotoped to the identity in a neighbourhood of each $D_i^\pm \subset \{(x, y, \pm\frac{1}{2}) | x^2 + y^2 < \frac{3}{4}\} \subset B_1(x_i)$. To do this, we first note that f_*X is transverse to each $D_i^\pm$, for ϵ sufficiently small, so we can find an isotopy of f to a new diffeomorphism which doesn't change the direction of f_*X, but may change the field by a scalar factor very close to 1, so that, in local coordinates (x, y, z) on $B_1(x_i)$, the new map, still denoted by f, is given by

$$f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), z),$$

for (x,y,z) in a neighborhood of $D_i^\pm$ which contains at least $D_i^\pm \times \{\pm\frac{1}{2} - \delta_0 < z < \pm\frac{1}{2} + \delta_0\}$, where $\delta_0 > 0$ above is *independent* of f and ϵ, for ϵ sufficiently small. This can obviously be done smoothly in the parameter τ of lemma 2.2.

The maps $f|_{D_i^\pm}$ (or $F(\tau)|_{D_i^\pm}$) now give planar diffeomorphisms of a small neighborhood of $D_i^\pm$ into a 3ϵ neighborhood of $D_i^\pm$ in $\mathbb{R}^2 \times \{\pm\frac{1}{2}\}$, satisfying $|f - I| < 3\epsilon$. These maps can then be isotoped to the identity and this isotopy can be extended to a small neighbourhood of the $D_i^\pm$, at the expense perhaps of increasing $\|f_* X - X\|$ slightly: to construct this isotopy accurately requires a little care, and we exercise this care in the following lemmas.

Lemma 2.3. *Let h be a diffeomorphism of the δ_1 neighborhood of the unit square S in the plane into an η neighborhood of itself in the plane, such that $|h(x) - x| < \eta$, for all x. Then there is an isotopy h_t of h supported in a neighborhood of S, with $h_0 = h$, $h_1 = I$ on a neighborhood of S, and such that $|\frac{\partial h_t(x)}{\partial t}| \leq C_1\eta$ everywhere, where C_1 is independent of h and η. Furthermore, if we have a one parameter family $h(\tau), \tau \in [0,1]$, of such diffeomorphisms such that $h(0) = h(1) = I$, then we may construct $h_t(\tau)$ smoothly so that $h_t(0) = h_t(1) = I$, for all $t \in [0,1]$.*

Proof. We will build up our isotopy from a finite sequence of isotopies, each compactly supported in a union of squares of sidelength approximately 100η. We will see that any such isotopies can be adjusted to control their t-derivative.

First, we subdivide a neighborhood of the square S into a gridwork of squares of sidelengths 100η. By our assumptions on h, we may easily first isotope h by $h_t^{(1)}$ to $h_1^{(1)}$, so that $|h_t^{(1)}(x) - x| < 2\eta$, $h_t^{(1)}$ is constant outside some δ_2 neighborhood of S, and $h_1^{(1)}$ fixes every grid point in the lattice of vertices of our squares. By an isotopy $h^{(2)}$ supported in small, disjoint neighborhoods of each vertex, we may assume that the derivative of $h_1^{(2)}$ is the identity at each grid point and that $|h_t^{(2)}(x) - x| < 3\eta$, and then further isotope by $h_t^{(3)}$ in a neighborhood of each vertex so that $h_1^{(3)}$ is the identity in a neighborhood of each vertex, and that no point has moved more than 3η from its starting point at any stage of the above isotopies.

In the case where h depends on τ we must note that the mapping $a : [0,1] \to S^1$ given by

$$a(\tau) = \frac{\partial h}{\partial x}(p)/|\frac{\partial h}{\partial x}(p)|$$

with $a(0) = a(1) = 1 \in S^1$ has winding number zero, where $p = (0,0)$ is our grid point, and hence we can perform the above rotations so that $\frac{\partial h_t}{\partial x}|_{\tau=0} = \frac{\partial h_t}{\partial x}|_{\tau=1} = 1$ for all $t \in [0,1]$. We calculate this winding number by using the following homotopy:

$$a_\varepsilon(\tau) = \begin{cases} \frac{\partial h(0,0,\tau)}{\partial x} / |\frac{\partial h(0,0,\tau)}{\partial x}|, & \varepsilon = 0, \\ \frac{h(\varepsilon,0,\tau)-h(0,0,\tau)}{|h(\varepsilon,0,\tau)-h(0,0,\tau)|}, & \varepsilon \in (0, 100\eta]. \end{cases}$$

Notice that the map is well-defined and smooth, and that at $\varepsilon = 100\eta$, the map is identically equal to $(1,0)$, since $(100\eta, 0)$ is also a grid point, so that $h(100\eta, 0, \tau) = (100\eta, 0)$, independent of τ.

Make $h_1^{(3)}$ our new h. We will next isotope locally to make h restricted to the sides of our grid squares the identity in a uniformly large neighborhood (a disk of radius 12η) of each vertex in the grid.

To fix notation, assume the grid point is at the origin in the plane with coordinates x, y, and the grid square sides are on the coordinate axes. Consider the point $(12\eta, 0)$. (Similar considerations will apply independently to $(-12\eta, 0), (0, \pm 12\eta)$.)

By an ambient isotopy of $h([0, 12\eta], 0)$ we may assume that $|h(12\eta, 0)| > |h(x, 0)|$ for $0 < x < 12\eta$, provided that now we allow $|h(x) - x| < 7\eta$.

One way of doing this is to find an isotopy replacing $h([12\eta-\delta, 12\eta], 0)$ by a curve Γ, disjoint from $h([0, 12\eta-\delta], 0)$ and touching $|(x,y)| = 15\eta$ at a single point which we may reparameterize to be $h(12\eta, 0)$. If we choose the added curve canonically then this can also be done for one-parameter families $h(x, 0, \tau)$. A canonical choice of Γ_τ is to follow $h(x, 0, \tau)$ for $x > 12\eta$ (slightly to one side) until we reach a point when $|h(x, 0, \tau)| = 15\eta$. Once Γ_τ reaches this point we can extend it by doubling back towards $h(12\eta, 0, \tau)$. Of course, such a family of curves Γ_τ may not vary continuously with τ. Assuming the map h to be generic, discontinuities will occur at τ_0 when $h(x, 0, \tau_0)$ becomes tangent to $|(x,y)| = 15\eta$. Suppose that this tangency occurs at a single point $h(x_0, 0, \tau_0)$. We may assume that the tangency is of second order and we deal with the case when $h(x, 0, \tau)$ is disjoint from $|(x,y)| = 15\eta$ for τ slightly less that τ_0 and x close to x_0. The other case, in which two intersection points vanish, can be treated similarly. Such a tangency does not result in any discontinuity in the choice of Γ_τ unless $h(x, 0, \tau_0)$ touches $|(x,y)| = 15\eta$ first at the point $x = x_0$, so we also suppose this. The added curve then becomes suddenly shorter for $\tau \geq \tau_0$. For $\tau < \tau_0$, $h(x, 0, \tau)$ touches $|(x,y)| = 15\eta$ first at points $x = x_\tau > x_0$. We can assume that these intersections are transversal and as $\tau \to \tau_0$, the points $x_\tau \to x_1 > x_0$, where $h(x, 0, \tau_0)$ crosses $|(x,y)| = 15\eta$ transversally at

$x = x_1$. We have that the curve $h([x_0, x_1], 0, \tau_0)$ lies in $|(x, y)| \leq 15\eta$ and the Jordan disk D formed by this curve together with the corresponding portion of $|(x, y)| = 15\eta$ is disjoint from $h(x, 0, \tau_0)$ for $x < x_0$, and in particular for $x < 12\eta - \delta$. There exists an isotopy g_s, compactly supported in a neighbourhood of D, which maps Γ_{τ_0} onto some Γ_τ for $\tau < \tau_0$. Throughout the isotopy it can be arranged that $g_s(\Gamma_{\tau_0})$ touches $|(x, y)| = 15\eta$ at a single point. The isotopy fixes $h([0, 12\eta - \delta], 0, \tau_0)$ and so combining our original isotopies with g_s for τ close to τ_0 gives us a smooth choice of Γ_τ as required.

By another ambient isotopy we may rotate the angle of $\frac{\partial h}{\partial x}(12\eta, 0)$ so that the new $\frac{\partial h}{\partial x}$ is identically equal to $(1, 0)$. Now we may isotope h further in a neighborhood of $(12\eta, 0)$ so that $h(x, 0, \tau) = (x, 0)$, for x close to 12η.

As already noted, similar normalizations may be obtained independently for the other three half-axes as well.

In order to straighten out the images of our grid square sides near this vertex, consider the curve traced counterclockwise from $(12\eta, 0)$ first along the vertical segment to $(12\eta, 12\eta)$, then along the horizontal segment to $(0, 12\eta)$ then down along $h(0, y, \tau)$ to the origin, and then along $h(x, 0, \tau)$ back to $(12\eta, 0)$. Define a vectorfield along this curve which is $-\frac{\partial}{\partial y}$ along the first and second segments, equal to $-\frac{\partial h(0,y,\tau)}{\partial y}$ along the third segment and along the fourth segment it is given as the normal to $\frac{\partial h(x,0,\tau)}{\partial x}$ which is $\pi/2$ in the clockwise direction from this tangent. If the mapping $h(x, y, \tau)$ were the identity, this would just be $-\frac{\partial}{\partial y}$, and so by a degree argument, we may fill-in this vectorfield smoothly and without zeroes to the Jordan region bounded by our curve. This may be done analogously for the other three quadrants, and adjusted so that we have a smooth vectorfield which is non-vanishing on the whole square Σ centered at the origin, and of side length 24η. We may also arrange that this field is identically equal to $-\frac{\partial}{\partial y}$ on a neighborhood of $\partial\Sigma$ and of the origin. Call this field Ξ. Let H denote the non-vanishing field orthogonal to Ξ which agrees with $\frac{\partial}{\partial x}$ in a neighborhood of $\partial\Sigma$. Again by a degree argument, we may deform these orthogonal fields smoothly to $\Xi_1 \equiv -\frac{\partial}{\partial y}, H_1 \equiv \frac{\partial}{\partial x}$. Following [21], by an argument similar to that of sec. 3.10. p. 205, we may induce a compactly supported isotopy of the square Σ which straightens out the flow lines of Ξ, H to those of $-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}$. We restrict this isotopy to the images of the coordinate axes within Σ and extend it to an ambient isotopy of Σ. These straightening arguments are compatible with the parameter τ, by the obvious degree arguments.

As a result of these isotopies, we may assume that after a succession of locally supported isotopies our h satisfies $|h(\tau) - I| < 15\eta$ on S and that on the grid square sides within 12η of the vertices, $h(\tau)$ is already the identity. Further, if we examine more precisely the isotopies just performed, we have that restricted to the grid square side $L = [0, 100\eta] \times \{0\}$, the map $h(\tau)$ maps L into the rectangle $R = [0, 100\eta] \times [-12\eta, 12\eta]$, and $h(x, 0, \tau) \equiv (x, 0)$, for $x \in [0, 12\eta]$ or $x \in [88\eta, 0]$.

We now apply the following two lemmas to correct the map h within the rectangles.

Lemma 2.4. *Let $b(\tau) : [0, 1] \to \mathbb{R}^2$ be a one-parameter family of embeddings for $\tau \in [0, 1]$ with $b(\tau)(x) = (x, 0)$ near $x = 0, 1$, and $b(0), b(1) \subset [0, 1] \times [-1, 1]$. Then there exists another one-parameter family of embeddings $\tilde{b}(\tau)$ with $\tilde{b}(\tau)(x) = b(\tau)(x)$ for $\tau = 0, 1$ or x close to $0, 1$ and with $\tilde{b}(\tau) \subset [0, 1] \times [-1, 1]$.*

Proof. First we will show that the isotopy can be adjusted to lie to the right of $x = 0$. We can write $b(\tau)(x) = r_\tau(x)e^{i\theta_\tau(x)}$ in complex coordinates on $\mathbb{C} = \mathbb{R}^2$ where $r_\tau(x) > 0$ for all τ, x and $-\frac{\pi}{2} < \theta_\tau(x) < \frac{\pi}{2}$ for $\tau = 0, 1$. The smooth functions r_τ and θ_τ can be chosen to vary smoothly with τ. We let $k_\tau = \max\{1, \frac{2}{\pi} \sup_x |\theta_\tau(x)|\}$, suitably perturbed to depend smoothly on τ, and replace $b(\tau)$ by $\tilde{b}(\tau)(x) = r_\tau(x)e^{ik_\tau^{-1}\theta_\tau(x)}$. Since the maps $b(\tau)$ are immersions, it is a straightforward exercise to check that the $\tilde{b}(\tau)$ must be immersions also, and similarly embeddings.

We note that if $b(\tau)$ initially lies to the left of the line $x = 1$, an isotopy can ensure that it lies within the disk D of radius 1 centered at $(0, 0)$. If $b(\tau) \subset D$ then $\tilde{b}(\tau) \subset D$. Now a similar argument can adjust $b(\tau)$ to lie to the left of $x = 1$ while leaving the image to the right of $x = 0$. A simple scaling argument can then be applied to ensure that the range of our isotopy lies within the required rectangle. □

Lemma 2.5. *Let $h : [0, 100\eta] \to R$ be a smooth embedding which is equal to $(x, 0)$ 12η near the endpoints. Then there is a compactly supported within R isotopy of h to the mapping $i(x) \equiv (x, 0) \in R$. Such an isotopy can be constructed smoothly in the case of a one-parameter family.*

Proof. It suffices to prove the lemma for embeddings $h : [0, 1] \to R = [0, 1] \times [-1, 1]$. Certainly such an h is isotopic to the identity in $\mathbb{C} = \mathbb{R}^2$. We simply set $h_t(x) = h(t)^{-1} \cdot h(xt)$ for $t > 0$ and $h_0(x) = x$. We conclude by using lemma 2.4.

□

To complete the proof of lemma 2.3 we need one last lemma to control the t-derivatives of our isotopies.

Lemma 2.6. *Let g_t be a compactly supported isotopy from g_0 to the identity of the closed disk $B_\eta(0)$ of center 0 and radius η. Then there exists a compactly supported isotopy $\tilde{g}_t$ from g_0 to I such that $|\frac{\partial \tilde{g}_t}{\partial t}| < 3\eta$. Such an isotopy can be constructed smoothly in the case of a one-parameter family.*

Proof. Let $K = \max |\frac{\partial g_t}{\partial t}|$, and let $\psi : B_\eta(0) \to [\frac{\eta}{K}, 1]$ be a smooth radial function which is equal to $\frac{\eta}{K}$ on the support of g_t, and identically 1 in a neighborhood of the boundary of $B_\eta(0)$. For $t \in [0,1], z \in B_\eta(0)$, define

$$\phi_t(z) = (1-t)z + t\psi(z)z.$$

Note that $|\frac{\partial \phi_t}{\partial t}| < \eta$ on $B_\eta(0)$. Now define an isotopy from g_0 to I as follows:

$$\overline{g}_t = \begin{cases} \phi_t \circ g_0, & \text{for } 0 \leq t \leq 1, \\ \phi_1 \circ g_{t-1}, & \text{for } 1 \leq t \leq 2, \\ \phi_{3-t}, & \text{for } 2 \leq t \leq 3. \end{cases}$$

Smoothing the t dependence of this piecewise smooth isotopy and adjusting t to run from 0 to 1 proves the lemma. □

This lemma may be used for other configurations by conjugation:

Corollary 2.1. *Let A, B be compact, convex sets with $0 \in A \subset B$, and set $A_\eta = \eta \cdot A, B_\eta = \eta \cdot B$. Suppose $g_t : B_\eta \to B_\eta$ is a $\mathcal{C}^1$ isotopy supported in A_η. Then there exists an isotopy compactly supported in B_η, $\tilde{g}_t$ from $\tilde{g}_0 = g_0$ to $\tilde{g}_1 = g_1$, and a constant C_2 independent of η such that*

$$|\frac{\partial \tilde{g}_t}{\partial t}| < C_2\eta.$$

Proof. Let $F : \overline{B_1(0)} \to B$ be a smooth embedding of the closed unit ball into B such that $A \subset F(B_1(0))$. Let M_c denote scaling by c, and then let $F_\eta : B_\eta(0) \to B_\eta$ be given by $M_\eta \circ F \circ M_{1/\eta}$. Apply lemma 2.3 to the isotopy $h_t = F_\eta^{-1} \circ g_t \circ F_\eta$ to get $\tilde{h}_t$ on B_η, and set $\tilde{g}_t = F_\eta \circ \tilde{h}_t \circ F_\eta^{-1}$ on $F_\eta(B_\eta(0))$, and extended by g_0 to all of B_η. Then

$$|\frac{\partial \tilde{g}_t}{\partial t}| < C_2\eta,$$

where, by the chain rule,

$$C_2 = 3 \max_{B_\eta(0)} |DF_\eta| = 3 \max_{B_1(0)} |DF|$$

is independent of η. □

Returning to the case of the diffeomorphism in two dimensons, we group the sides of our grid squares, into four groups $\mathcal{G}_i, i = 1, \ldots, 4$, such that within each group, each side is surrounded by a rectangle R congruent to $[0, 100\eta] \times [-12\eta, 12\eta]$ and the interiors of all the R-rectangles in each $\mathcal{G}_i$ are pairwise disjoint. We may apply first lemma 2.5 within each R rectangle in $\mathcal{G}_1$. Then notice that within each of these rectangles we may extend the isotopy of the embedded curve in lemma 2.5 to an isotopy of the plane, which is supported in an arbitrarily small neighborhood of the curve's isotopy (in particular, within R), and which doesn't move any of the six sides which share one of the two vertices of our initial square side. Apply all of these isotopies simultaneously to $h^{(3)} = h$ within each R-rectangle in $\mathcal{G}_1$, obtaining a new diffeomorphism $h^{(4)}$. Then apply corollary 2.1 within each of the R-rectangles in $\mathcal{G}_1$ to ensure our isotopy our satisfies the t-derivative condition of lemma 2.3 in a neighborhood of all $\mathcal{G}_1$ R-rectangles. If we apply corollary 2.1 carefully, we may guarantee that $h^{(4)}$ leaves all other grid square sides fixed. Thus we may perform the same argument as for $\mathcal{G}_1$ to the group of grid square sides $\mathcal{G}_2$, using lemma 2.5 and then corollary 2.1, and so on, to arrive at $h^{(7)}$ which is the identity in a neighborhood of all grid points and grid square sides, and which is obtained from h by an isotopy $H_t^{(1)}$ which satisfies

$$|\frac{\partial H_t^{(1)}}{\partial t}| < C_4 \eta,$$

for C_4 a constant independent of f and η.

For each square Σ of side length 100η in our grid, we can enclose it symmetrically in a square $\tilde{\Sigma}$ of side length 101η. Group the squares into four sets $\mathcal{F}_i$ so that the interiors of the $\tilde{\Sigma}$-squares for squares in $\mathcal{F}_i$ are all pairwise disjoint. Following Thurston [21], sec. 3.10, p. 205, we may smoothly isotope $h^{(7)}$ restricted to an $\mathcal{F}_1$ Σ-square to the identity within each grid square by an isotopy compactly supported within that square. Within each $\tilde{\Sigma}$-square, we may apply corollary 2.1 to get an isotopy compactly supported within the $\tilde{\Sigma}$-square, and t-derivative bounded by $C_5\eta$, where C_5 is a constant independent of η and $h^{(7)}$. Again, we do this in turn for each of the groups $\mathcal{F}_i$, compose the isotopies, smooth the time parameters and put them onto the interval $t \in [0, 1]$, and we have constructed the isotopy required of lemma 2.3, completing its proof in the case of a single isotopy. The case of one-parameter families is identical. □

We can use lemma 2.3 to correct $f|_{D_i^{\pm}}$ by a planar isotopy with t-derivative bounded by C_1 independent of f, ϵ, where the constant has gotten a little bigger by the chain rule after we diffeomorph a neighborhood of $D_i^{\pm}$ to a neighborhood of the square S in lemma 2.3.

Within a "δ_0"-neighborhood of $D_i^{\pm}$, we may subsequently perform the following two isotopies: first, let ψ be a smooth, even, non-decreasing function of $z - \pm\frac{1}{2}$ which is 0 for $|z - \pm\frac{1}{2}| < \delta_0/4$ and identically 1 for $|z - \pm\frac{1}{2}| > \delta_0/2$; the isotopy is given by

$$F_t = (f_1(x, y, (1-t)z + t\psi(z)z), f_2(x, y, (1-t)z + t\psi(z)z), z),$$

which leaves $F_1(x, y, z) = (f_1(x, y, \pm\frac{1}{2}), f_2(x, y, \pm\frac{1}{2}), z)$ in a $\delta_0/4$ - neighborhood of $D_i^{\pm}$. Second, we apply lemma 2.3 to find a compactly supported planar isotopy h_t of $h_0(x, y) = (f_1(x, y, \pm\frac{1}{2}), f_2(x, y, \pm\frac{1}{2}))$ with $h_1(x, y) = (x, y)$. The second isotopy of F_1 is defined as:

$$G_t = (h_{t(1-\psi(4z))}(x, y), z).$$

It is clear from lemma 2.3 that F_t and G_t are $\mathcal{C}^0$-close to the identity, and we leave to the reader to check that

$$|\frac{\partial F_t}{\partial z} - \frac{\partial}{\partial z}| < C_5\epsilon$$

and

$$|\frac{\partial G_t}{\partial z} - \frac{\partial}{\partial z}| < C_6\epsilon$$

throughout the isotopies, where again the constants C_5, C_6 are independent of f and ϵ.

We thus have that the map f may be assumed to be equal to the identity in disjoint (if δ_0 is small enough) neighborhoods of all the $D_i^{\pm}$. The idea will next be to isotope f to the identity between the $D_i^{\pm}$.

Lemma 2.7. *Suppose a disk in some $D_i^{\pm}$ is connected by flow-lines of X, say of length no more than 1, to a disk in $D_j^{\pm}$. Let V be the union of the flow-lines. Then there is an isotopy of f, which leaves f equal to the identity near D_i and D_j and fixed away from V, such that the resulting map is the identity on V except perhaps for an ϵ-small neighbourhood of its boundary. Furthermore, if $f = I$ on a subset of V intersecting D_i or D_j on a smooth neighbourhood of a segment of its boundary then we may also leave f fixed there.*

Proof. We start with a homotopy between the vector fields f_*X and X, say X_t where $X_0 = f_*X$ and $X_1 = X$. This can be arranged so that $\|X_t - X\|_\infty < \epsilon$ for all t. We cut-off the X_t to remain as f_*X near the boundary of V. Let h_t be the diffeomorphism of a neighbourhood of V defined by leaving D_i fixed and flowing along the vector field $-X$ to D_i then back along X_t for a similar time. Then $h_0 = f$ and $h_1 = I$ away from the boundary of V, see figure 1. Unfortunately such h_t are

Figure 1. Flowlines between two disks.

not necessarily equal to the identity near D_j, except of course near its boundary and when $t = 0, 1$. However we can rescale X_t to ensure that the map preserves D_j and then compose with suitable isotopies in the (x, y)-planes to correct this. The existence of such isotopies is provided by the parameterized version of lemma 2.3. On a subset of V which is a neighbourhood of a segment of the boundary and where $f_* X = X$ we may assume that $X_t = X$ for all t and the maps h_t can be taken to be the identity all along.

□

We now apply this lemma repeatedly between different D_i and D_j. Notice that a map is equal to the identity if it is equal to the identity on the flow-lines connecting sets of the form $\tilde{D}_i^{\pm} = \{(x, y, \pm\frac{1}{2}) | x^2 + y^2 \leq \frac{1}{4}\} \subset D_i^{\pm}$ since these regions cover M. The order in which we apply the lemma must be chosen carefully however, so as not to disturb regions in which f has already been isotoped to the identity. We first isotope f to the identity on the union V_{ij} of flow-lines between all D_i and D_j such that the flow-lines are of length less than 1 and the V_{ij} do not intersect any other D_k. The disks D_k referred to here are the same as those $D_k^{\pm}$ defined above, although we are not assuming that D_i and D_j are equal to D_k^+ and D_k^- for the same k.

Next we apply the lemma to all remaining pairs of $D_i^{\pm}$. The point now is that if any D_k happens to sit between D_i^+ and D_i^- then, away from an ϵ-neighbourhood of its boundary, it must lie on a complete set of flow-lines of X from D_i^+ to D_i^- on which f has already been isotoped to the identity. Moreover, making $D_i^{\pm}$ slightly smaller if necessary, these complete flowlines will only intersect $D_i^{\pm}$ in a neighbourhood of a boundary segment and so can be left fixed by our isotopy. Thus the new isotopy will not affect f here. After all of the above isotopies then, the resulting map f is equal to the identity.

FIGURE 2. order of applying lemma 2.7

Figure 2 divides the region between two disks D_j and D_k into three numbered regions showing the order in which the isotopy provided by lemma 2.7 should be carried out if another disk D_i intersects the flowlines between them.

□

3. PROOF OF THE MAIN RESULT

3.1. **Construction of a diffeomorphism.** In this section we will construct a diffeomorphism between two closed domains whose interiors are symplectomorphic.

We start with two domains W_1 and W_2 in $\mathcal{M}^\circ$. If the interiors of W_1 and W_2 are symplectomorphic, then using the symplectic homology theory and perhaps an application of Nielsen's theorem as in the proof of theorem 1.1 we may assume that $\Sigma_1 = \partial W_1$ and $\Sigma_2 = \partial W_2$ have the same marked length spectrum. We observe that the differentials of diffeomorphisms of M preserve the set $\mathcal{M}^\circ$.

We choose a small r such that rW_1 lies in the interior of both W_1 and W_2. Then foliate the rest of W_1 by $s\Sigma_1$, $r \leq s \leq 1$, and the rest of W_2 by hypersurfaces $\Sigma_{2,s}$ where $\Sigma_{2,s} = s\Sigma_1$ for s close to r and $\Sigma_{2,s} = s\Sigma_2$ for s close to 1. Further, all $\Sigma_{2,s} \in \partial\mathcal{M}$. This is possible for r sufficiently small by the connectedness of $\mathcal{M}^\circ$ and the fact that all of our domains have star-shaped fibers.

Now, as demonstrated originally by D. Anosov in [1], there exists a continuous family of homeomorphisms $s\Sigma_1 \to \Sigma_{2,s}$ which map the Reeb flow (of the restriction of the Liouville form λ) on $s\Sigma_1$ to that on $\Sigma_{2,s}$. Actually, as is made precise in [15], the following is true. Fix a smooth family ψ_s of diffeomorphisms $\Sigma_{2,s} \to rW_1 = \Sigma$ for $r \leq s \leq 1$ with ψ_r the identity. Then the ψ_s push-forward the Reeb flows on $\Sigma_{2,s}$ to a family of Anosov flows on Σ generated, say, by X_s. There are a family of homeomorphisms h_s mapping $\frac{s}{r}X_r$ onto a_sX_s, where a_s is a function on Σ. Now, h_s is the flow of a continuous time-dependent vector field, Y_s say, on Σ, where we may assume that Y_s can be differentiated only in the direction of the Reeb flow at time s. We arrange things so that $Y_s \equiv 0$ for s close to r and to 1.

The homeomorphism $h_s = h$ for s close to 1 is a conjugacy between Anosov flows which by assumption have the same marked length spectrum. This homeomorphism is in fact Hölder continuous (see for instance Chapter 9 of [12]) and hence we can apply a theorem of Livsic, see [14], which constructs a function g on Σ such that the homeomorphism ϕ defined by shifting a point x a distance $g(x)$ along the flow-line through x makes $\phi \circ h$ a time-preserving conjugacy, that is, it preserves the Anosov vector field itself as opposed to just the flow-lines.

Such a conjugacy must in fact be of class C^1 by a result of J. Feldman and D. Ornstein, see [9], and so preserve the contact form λ, see for instance [12], lemma 18.3.7. We now use the theorem of R. de la Llave and R. Moriyon in [16] which says that our time-preserving conjugacy must be C^∞. Now let h be this diffeomorphism. The next step is to extend $f(x) = \psi_1^{-1}h(rx)$, which is a diffeomorphism between Σ_1 and Σ_2, to a diffeomorphism f between the domains W_1 and W_2. It extends trivially as $f(x) = sf(\frac{1}{s}x)$ on levels $s\Sigma_1$ for s close to 1.

The homeomorphism ϕ above is clearly the flow of a vector field, which must be differentiable along the Reeb flow, so we may still assume that h is the time-1 map of a continuous vector field Y_s, still identically zero for s close to 1.

We now approximate Y_s by a smooth vector field. This can be done in such a way that the resulting one-parameter family of diffeomorphisms, say $\tilde{h}_s$, C^0-approximate the original homeomorphisms and map the Reeb vectorfield on $r\Sigma_1$ to a vectorfield C^0 close to the a_sX_s. We remark here that the inverse maps $\tilde{h}_s^{-1}$ will also map a_sX_s to a vectorfield C^0 close to $\frac{s}{r}X_r$ since the inverse is given simply by flowing along Y_s in the opposite direction. In our current situation we have $a_s \equiv 1$ for s close to 1. Now for s less than 1, define $f|_{sW_1}(x) = f_s(x) = \psi_s^{-1}\tilde{h}_s(\frac{r}{s}x)$. Suppose that f_s is sufficiently close to $sf(\frac{1}{s}x)$ on a level $1-2\delta$ that we can apply Lemma 2.1. Then the f_s can be redefined for $1-2\delta \le s \le 1-\delta$ to be the derived isotopy between $f_{1-2\delta}$ and f. These f_s extend f smoothly over all levels $s\Sigma_1$ and f further extends as the identity inside $r\Sigma_1$.

3.2. **Isotopy to a smooth symplectomorphism.** In this section we find an isotopy of W_1 which, composed with the f of the previous section, gives a smooth symplectomorphism between W_1 and W_2. We recall that f mapped the hypersurfaces $s\Sigma_1$ onto $\Sigma_{2,s}$, approximately preserving the Reeb vectorfields. It was the identity near the zero-section and near the boundary preserved the Liouville form, and so in particular is already a symplectomorphism.

Lemma 3.1. *Suppose ω_0 and ω_1 are two symplectic (nondegenerate, antisymmetric) bilinear forms on $\mathbb{R}^4$ giving the same orientation and $\Sigma^3 \subset \mathbb{R}^4$ a linear subspace. Let X_0 and X_1 be nonzero vectors in $\ker \omega_0|_\Sigma$ and $\ker \omega_1|_\Sigma$. Suppose that*

(i) there exist $v, w \in \Sigma$ such that $\omega_i(v, w) > 0$ for $i = 0, 1$;

(ii) there exists a u transverse to Σ such that $\omega_i(X_j, u) > 0$ for $i, j = 0, 1$.

Then, for all $0 \leq t \leq 1$, $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic.

Proof. If not, since $\omega_t(v, w) > 0$, then $\ker \omega_t$ is 2-dimensional and intersects Σ in a 1-dimensional kernel. Let Y be a nonzero vector in $(\ker \omega_t)|_\Sigma = \ker(\omega_t|_\Sigma)$. Then we can write $Y = aX_0 + bX_1$ with $a, b > 0$. In fact, if when restricted to the plane spanned by v and w we have $\omega_0 = k\omega_1$, then we can take $a = (1-t)k$ and $b = t$. But then $\omega_t(Y, u) > 0$, a contradiction.

□

Corollary 3.1. *For all $0 \leq t \leq 1$, $(1-t)\omega + tf^*\omega$ is a symplectic form on W_1, where $\omega = d\lambda$ is the canonical symplectic form on the cotangent bundle and f is suitably chosen as above.*

Proof. We study $\omega_0 = \omega$ and $\omega_1 = f^*\omega$ on T_xW_1 with $\Sigma = T_xs\Sigma_1$. Then for $i = 0, 1$, $\ker \omega_i|_\Sigma$ are generated by the Reeb vectorfield X_0 on $s\Sigma_1$ and the image X_1 of the Reeb vectorfield on $\Sigma_{2,s}$ under f^{-1}. Choosing v, w to satisfy the condition (i) with respect to ω_0, provided that f is chosen so that X_1 is close to X_0 then v, w will also satisfy (i) with respect to ω_1. We recall that f is orientation preserving on the level sets. Also, our construction ensures that both f and f^{-1} approximately preserve the Reeb vectorfields. We choose $u = (f_*^{-1})(\frac{\partial}{\partial s})$, the image under f^{-1} of the radial vectorfield in W_2. This satisfies $\omega_1(X_1, u) > 0$. Also, $\omega_0(X_0, u) > K > 0$, where the number K is independent of any specific diffeomorphism mapping the $s\Sigma_1$ onto $\Sigma_{2,s}$. Therefore $\omega_0(X_1, u) > 0$ for f suitably chosen to approximately preserve the Reeb vectorfields and $\omega_1(X_0, u) > 0$ similarly.

□

Now, since $f^*\lambda$ and λ agree near ∂W_1, we can apply Moser's method to find an isotopy of W_1, fixed near ∂W_1, which generates a symplectomorphism between ω and $f^*\omega$. Specifically, the isotopy can be taken to be the time-1 flow of the time-dependent vector field Z_t uniquely defined by $Z_t \rfloor ((1-t)\omega + tf^*\omega) = \lambda - f^*\lambda$. Note that $Z_t \equiv 0$ both near ∂W_1 and near the zero-section.

The composition of this isotopy with our original diffeomorphism, denoted again by f, is now the required symplectomorphism between W_1 and W_2.

We now represent f explicitly as a Hamiltonian diffeomorphism. Observe that associated to any 1-form μ on T^*M is, in the terminology of [7], a 'contracting' vector field X_μ defined by $X_\mu \rfloor \omega = -\mu$. In the case of $\mu = \lambda$ or $\mu = f^*\lambda$, this vector field vanishes only along the zero-section M and the associated flow contracts a disk towards each point on M. For X_λ, these disks are just the cotangent fibers. Now, f maps $X_{f^*\lambda}$ into X_λ. The only map doing this which is fixed near M is defined as follows. Flow along $X_{f^*\lambda}$ until we are in the region where $f = id$, then flow out along $-X_\lambda$ for the same time.

Let ϕ_t and $\phi_t^{'}$ denote the time-t flows of X_λ and $X_{f^*\lambda}$ respectively. Assume that $\phi_T^{'}(W_1)$ lies in the region where $f = id$. Note that as $f^*\lambda = \lambda$ near ∂W_1 we can extend $f^*\lambda$ and $X_{f^*\lambda}$ smoothly to T^*M.

Define an isotopy h_t, $0 \leq t \leq T$ by $h_t = \phi_t^{-1} \circ \phi_t^{'}$. Then $h_0 = id$ and $h_T = f$.

Now, $\mathcal{L}_{X_\lambda}\omega = d(X_\lambda \rfloor \omega) = -\omega$ and similarly $\mathcal{L}_{X_{f^*\lambda}}\omega = -\omega$ so we have $\phi_t^*\omega = \phi_t^{'*}\omega = e^{-t}\omega$ and the h_t are all symplectomorphisms.

Let $V_t = \frac{dh_t}{dt}$, then $0 = L_{V_t}\omega = d(V_t \rfloor \omega)$. Hence the form $V_t \rfloor \omega$ is closed and the isotopy is Hamiltonian if it is exact. But V_t vanishes near the zero-section and so we can use a parameterized version of the Relative Poincaré Lemma to construct a smooth family of H_t on T^*M such that $V_t \rfloor \omega = dH_t$ as required.

□

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M.I.T., Cambridge, MA, *and* Univ. of Michigan, Ann Arbor, MI
E-mail address: dburns@umich.edu

University of Notre Dame, Notre Dame, IN
E-mail address: Hind.1@nd.edu