1. Prove that the category of Abelian groups has coproducts. That is, given two abelian groups G and H show that there is a group P and homomorphisms $i : G \to P$ and $j : H \to P$ that satisfy the following property: given any Abelian group K and homomorphisms $l : G \to K$ and $r : H \to K$ there is a unique homomorphism $\phi : P \to K$ such that $\phi \circ i = l$ and $\phi \circ j = r$. This is given by the following commutative diagram:

$$\begin{tikzcd}
& K & \\
G \arrow[ur, "l"] \arrow[r, "i"'] & P \arrow[u, dashed, "\phi"'] & H \arrow[ul, "r"'] \arrow[l, "j"]
\end{tikzcd}$$

Solution: Define $P := G \times H$ and $i : G \hookrightarrow P$ as $i(g) = (g, 0)$. Similarly, define $j(h) = (0, h)$. Now assume K, $l : G \to K$, and $r : h \to k$ are given. Define

$$\phi : P \to K, \ \phi(a, b) = l(a) + r(a).$$

It's clear that this is the only map that satisfies $\phi \circ i = l$ and $\phi \circ j = r$. We need to check that ϕ is a homomorphism:

$$\begin{aligned}
\phi((a, b) + (c, d)) &= \phi(a + c, b + d) \\
&= l(a + c) + r(b + d) \\
&= l(a) + l(c) + r(b) + r(d) \\
&= l(a) + r(b) + l(c) + r(d) \\
&= \phi(a, b) + \phi(c, d).
\end{aligned}$$

2. Show that every group of order p^2q is not simple, where p and q are distinct primes.

Solution: Let n_p (resp. n_q) denote the number of $p-$Sylow subgroups of G, where $|G| = p^2q$. Suppose $p > q$, then as $n_p \mid q$ and $n_p \equiv 1 \bmod p$ it is the case $n_p = 1$ and consequently the $p - Sylow$ subgroup is normal.

Suppose $p < q$ and $n_q > 1$. As $n_q \mid p^2$ then $n_q = p$ or $n_q = p^2$, but as $q > p$ $n_q = p^2$. As $n_q \equiv 1 \bmod q$, $q \mid p^2 - 1 = (p+1)(p-1)$. q is prime so either $q \mid p - 1$ or $q \mid p + 1$ and $q > p$, so $q \mid p + 1$, that is $q = p + 1$. The only possibility is $p = 2$, $q = 3$, so $|G| = 12$.

3. Let p and q be primes such that $p > q$. If $q \nmid p-1$, then every group of order pq is isomorphic to the cyclic group $\mathbb{Z}_{pq}$. If $q \mid p-1$, then there are (up to isomorphism) exactly two distinct groups of order pq: the cyclic group $\mathbb{Z}_{pq}$ and a non-abelian group K generated by elements c and d such that

$$|c| = p; \qquad |d| = q; \qquad dc = c^s d,$$

where $s \not\equiv 1 \bmod p$ and $s^q \equiv 1 \bmod p$.

Hints: $x^q \equiv 1 \bmod p$ has exactly q distinct solutions modulo p. If r is a solution and k is the least positive integer such that $r^k \not\equiv 1 \bmod p$, then $k \mid q$.

Solution: It follows from (2) that $n_p = 1$, so let $\mathrm{Syl}_p = P = \langle a \rangle$. Let $b \in G$ have order q. Verify that $G/P = \langle bP \rangle$ a group of order q, hence every element of G can be written as $b^i a^i$ and $G = \langle a, b \rangle$.

Now n_q is either 1 or p, if it is 1 then $G \cong \mathbb{Z}_{pq}$. Suppose $n_q = p$, so that is $p \mid q-1$, then $bab^{-1} = a^r$ and $r \not\equiv 1 \bmod p$ as G is not Abelian. Since $bab^{-1} = a^r$, by induction

$$b^j a b^{-j} = a^{r^j} \text{ so for } j = q, \; a = a^{r^q} \text{ hence } r^q \equiv 1 \bmod p.$$

Employing the hint with $k = q$, it follows that $1, r, r^2, \ldots r^{q-1}$ are the distinct solutions modulo p of $x^q \equiv 1 \bmod p$. Consequently, $s \equiv r^t$ for some $1 \leq t \leq q-1$. If $b_1 = b^t \in G$, then $|b_1| = q$. Applying the preceding paragraph to b_1, we see $G = \langle a, b_1 \rangle$. One can verify that a and b_1 satisfy the relations of K and that the induced map $K \to G$ is indeed an isomorphism.

4. If p is an odd prime, then every group of order $2p$ is isomorphic either to the cyclic group $\mathbb{Z}_{2p}$ or the dihedral group D_p.

Solution: Apply question (3) with $q = 2$. If G is not cyclic, the conditions on s imply $s \equiv -1 \bmod p$. Hence

$$G = \langle c, d \mid d^2 = c^p = c^{-1}dcd^{-1} = 1 \rangle.$$

Therefore, $G \cong D_p$.