HOMEWORK 8

Due Friday July 24

(1) Let $\text{rad}(R) = \sqrt{0} = \{a \in R | a^n = 0 \text{ for some } n \in \mathbb{Z}^+\}$. Show that

$$\sqrt{0} = \bigcap_{\substack{P \leq R \\ P \text{ prime}}} P.$$

(2) Show that $\mathbb{Z}[\sqrt{2}]$ is an Euclidean domain (hint: $\phi(a+b\sqrt{2}) = |a^2 - b^2|$) and that $(\mathbb{Z}[\sqrt{2}])^*$ is a direct product of a cyclic group of order 2 (generated by -1) and an infinite cyclic group (generated by $1+\sqrt{2}$).

(3) Show $x^4 + 1$ is irreducible in $\mathbb{Z}[x]$.

(4) Show $x^4 - 5$ is irreducible in $\mathbb{Z}_3[x]$.

(5) Show $x^3 - 5x - 1$ is irreducible in $\mathbb{Z}[x]$.

(6) Show $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Q}[x]$.

(7) Factor the following in $\mathbb{Z}[i]$:
   (a) 30;
   (b) $1 - 3i$;
   (c) 10;
   (d) $6 + 9i$.

(8) Describe $\mathbb{Z}[i]/(p)$ for
   (a) $p = 2$;
   (b) $p \equiv 1 \bmod 4$.

(9) Is $\frac{1}{2}(1 + \sqrt{3})$ an algebraic integer?