HOMEWORK 7

Due Tuesday July 14

(1) Prove that a finite integral domain is a field.

(2) Prove that every vector space has a basis.

(3) Show that the category of rings has products.

(4) Prove that the ideal $(2, x)$ is a maximal ideal in $\mathbb{Z}[x]$, but (x) is not. Are these ideals prime?

(5) Let R be the ring of functions from $[0, 1]$ to $\mathbb{R}$. For each $a \in [0, 1]$ let M_a be the kernel of evaluation at a. Prove that M_a is a maximal ideal. What if R is the ring of continuous functions?

(6) Let R be a non-zero commutative ring with unit and let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group (with group operation written multiplicatively). Define the *group ring*, RG, of G with coefficients in R to be the set of all formal sums

$$a_1g_1 + a_2g_2 + \cdots + a_ng_n \text{ with } a_i \in R \text{ and } 1 \leq i \leq n.$$

Addition is defined componentwise and multiplication by

$$(a_1g_1 + \cdots + a_ng_n)(b_1g_1 + \cdots + b_ng_n) := \sum_{g_k \in G} \left(\sum_{g_ig_j = g_k} a_ib_j \right) g_k.$$

Note that RG is commutative only if G is. Define the *augmentation map* $\epsilon : RG \to R$ defined by

$$\sum_{i=1}^{n} a_ig_1 \mapsto \sum_{i=1}^{n} a_i.$$

The kernel of ϵ is the augmentation ideal I. Show that if R is a field, then I is a maximal ideal.

Due Friday July 17

(1) Let R be a ring and M an R-module. Prove that a subset N of M is a submodule if and only if

(a) $N \neq \emptyset$, and

(b) $x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.

(2) Let M and N be R-modules. A map $\varphi : M \to N$ is an R-module homomorphism if and only if

$$\varphi(rx + y) = r\varphi(x) + \varphi(y) \text{ for all } x, y \in M \text{ and all } r \in R.$$

(3) Let R be a ring and let $n \in \mathbb{Z}$ such that $n > 0$. Define a non-trivial module structure on $R^n = \{(a_1, a_2, \ldots a_n) | a_i \in R\}$.

(4) Prove the Short Five Lemma, that is, let R be a ring and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

(a) If α and γ are injective, then β is injective;

(b) If α and γ are surjective, then β is surjective;

(c) If α and γ are isomorphisms, then β is an isomorphism.

(5) Prove that if P is a prime ideal of a commutative ring R and P contains no zero divisors then R is an integral domain.

(6) Let R be a UFD and P a prime ideal of $R[x]$ with $P \cap R = 0$. Show that P is principal.