HOMEWORK 6

Due Wednesday July 6

(1) Let A be an Abelian group and let $\mathrm{End}(A)$ be the set of homomorphisms $f : A \to A$. Define addition in $\mathrm{End}(A)$ by $(f+g)(a) = f(a) + g(a)$. Let multiplication in $\mathrm{End}(A)$ be given by composition. Verify $\mathrm{End}(A)$ is a (possibly noncommutative) ring.

(2) Show $\mathrm{End}(\mathbb{Z} \times \mathbb{Z})$ is a noncommutative ring.

(3) An element of a ring is nilpotent if $a^n = 0$ for some n. Prove that in a commutative ring $a + b$ is nilpotent if a and b are. Show this may fail if R is not commutative.

(4) Prove that the sum of a nilpotent element and a unit is again a unit.

(5) Identify the group of units in the following rings:
 (a) $\mathbb{C}$;
 (b) $\mathrm{M}_n(\mathbb{Z})$, $\mathrm{M}_n(\mathbb{R})$;
 (c) $\mathbb{R}[x]$.

Due Friday July 10

(1) Let F be a field. The only ideals of F are the zero ideal and the unit ideal.

(2) If a ring R has exactly two ideals, then R is a field.

(3) Find all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_{30}$. What if we do not require that the image of the unit is the unit? In each case describe the kernel and the image.

(4) Let I, J, K be ideals of a ring R.
 (a) Prove that IJ is an ideal contained in $I \cap J$.
 (b) Prove that if R is commutative and if $I + J = R$, then $IJ = I \cap J$.
 (c) Prove that if $J \subset I$ then $I \cap (J + K) = J + (I \cap K)$.

(5) Is $\mathbb{Z}/(10)$ isomorphic to $\mathbb{Z}/(2) \times \mathbb{Z}/(5)$? What about $\mathbb{Z}/(8)$ and $\mathbb{Z}/(2) \times \mathbb{Z}/(4)$?

(6) Describe the following rings.
 (a) $\mathbb{Z}[x]/(x^2 - 3, 2x + 4)$;
 (b) $\mathbb{Z}[i]/(2 + i)$.