Nodal Sets of Solutions of Elliptic Differential Equations

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To Yansu, Raymond and Thomas
from Q. H.

Contents

Preface

The subject of nodal sets, or level sets in general, is an important research topic for solutions of differential equations. In some cases, properties of nodal sets of solutions are themselves the primary concern. In other cases, nodal sets provide an important tool in the study of properties of solutions.

In this book, we discuss the nodal sets of solutions of homogeneous linear elliptic equations of the second order. Our primary concern is the measure theoretic properties of these sets and their relation to the growth of solutions. There are two important conjectures. The first conjecture was raised by S.-T. Yau and asks whether the nodal sets of the eigenfunctions of the Laplacian operator in a compact manifold have an area on the order of the square root of the corresponding eigenvalues. Eigenfunctions on flat torus illustrate that the order of the square root is the correct one. The second conjecture was proposed by F.-H. Lin and concerns the size of the nodal sets and the critical nodal sets of solutions of general homogeneous elliptic differential equations.

An important aspect of the discussion is the local growth of solutions and eigenfunctions. We will show that solutions and eigenfunctions are well approximated by polynomials, where the degree of these polynomials can be controlled. Then a natural question arises as to whether the nodal sets of solutions and eigenfunctions are approximated by nodal sets of these approximating polynomials.

This book has seven chapters. In Chapter 1, we briefly review prerequisite results which will be used throughout this book. These include basic a priori estimates for solutions of linear elliptic equations of the second order, basic knowledge on Hausdorff measures and several well-known results on zeroes of analytic functions. We describe these results without proof and provide references.

In Chapter 2, we discuss harmonic functions exclusively. This chapter sets the tone for the rest of the book. In this chapter, we first introduce the frequency function for harmonic functions and prove the important monotonicity formula. An important consequence of this monotonicity formula is the control of vanishing orders of harmonic functions by the frequency. In the rest of this chapter, we estimate the size of nodal sets and critical nodal sets of harmonic functions by the frequency. All results are optimal.

In Chapter 3, we generalize the notion of the frequency to solutions of general linear elliptic equations of the second order. We prove a modified monotonicity formula for solutions of linear elliptic equations under the assumptions that the leading coefficients are Lipschitz and other coefficients are bounded. An important consequence is a quantitative version of the unique continuation, which plays an essential role in the rest of this book.

In Chapter 4, we discuss the structure of nodal sets of solutions of general linear elliptic equations of the second order with the same assumptions under which unique continuation holds. We will prove that the nodal set is countably rectifiable of codimension 1 and that the critical nodal set is countably rectifiable of codimension 2. The proof is based on a pointwise a priori estimate for these solutions.

In Chapter 5, we study the size of nodal sets of solutions of general linear elliptic equations of the second order. We prove optimal estimates if coefficients are analytic and less than optimal estimates if coefficients are non-analytic. An important aspect in the discussion is to analyze the local growth of solutions in terms of its frequency.

In Chapter 6, we study the size of nodal sets of eigenfunctions of the Laplace-Beltrami operator on compact Riemannian manifolds. We will prove optimal upper bounds and lower bounds if the Riemannian manifold is analytic.

In Chapter 7, we study the size of critical nodal sets of solutions of general linear elliptic equations of the second order. We prove a uniform estimate on the measure of critical nodal sets in terms of the frequency. This result is far from satisfactory. An optimal estimate exists only for planar harmonic functions.

Many results in Chapters 5, 6 and 7 are far from optimal, leaving plenty of room for improvements. We hope this book will inspire others to work on this fascinating area of mathematics.

Acknowledgments.

Qing Han, Fang-Hua Lin

CHAPTER 1

Prerequisite Knowledge

The main concern in this book is the measure theoretic properties of the nodal sets of solutions of the elliptic differential equations. There is no surprise that we will employ frequently a priori estimates in the theory of elliptic differential equations and various integral formulas in the geometric measure theory. We will also use very often some basic results concerning zeroes of polynomials. In this chapter, we will give a brief review of these results. Specifically, we will review some important results of interior estimates of elliptic differential equations, integral formulas related to Hausdorff measures and zeros of analytic functions. We will quote these results without proof. Those who are not familiar with these results should not feel intimidated. They can start with Chapter 2 and refer back to this chapter when necessary.

1.1. Elliptic Differential Equations

In this section, we will give a quick review of some basic estimates of solutions of elliptic differential equations. The simplest elliptic differential operator is the Laplacian operator Δ. The following result yields the interior estimates for harmonic functions.

Lemma 1.1.1. *Let $u \in C\big(\bar{B}_R\big)$ be a harmonic function in $B_R \subset \mathbb{R}^n$. Then for any multi-index α with $|\alpha| = m$*

$$|D^\alpha u(0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_R} |u|.$$

Lemma 1.1.1 will be used frequently in Chapter 2.

Now we consider general elliptic differential equations of the form

$$a_{ij}u_{ij} + b_i u_i + cu = f \quad \text{in } B_1 \subset \mathbb{R}^n. \tag{1.1.1}$$

We always assume (a_{ij}) satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \text{for any } x \in B_1, \xi \in \mathbb{R}^n,$$

for a positive constant λ, which is often called the ellipticity constant.

Concerning solutions of (1.1.1), we have the following Schauder estimates and $W^{2,p}$-estimates.

Theorem 1.1.2. *Let a_{ij}, b_i and c be C^α-functions in B_1 and u be a $C^{2,\alpha}$-solution of (1.1.1) for some $\alpha \in (0,1)$. Then*

$$|u|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C\big(|u|_{L^\infty(B_1)} + |f|_{C^\alpha(B_1)}\big),$$

where C is a positive constant depending only on n, λ, α and the C^α-norms of a_{ij}, b_i and c.

THEOREM 1.1.3. *Let* a_{ij} *be continuous functions,* b_i *and* c *be bounded functions in* B_1 *and* u *be a* $W^{2,p}$*-solution of (1.1.1) for some* $p > 1$*. Then*

$$\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \leq C\big(\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}\big),$$

where C *is a positive constant depending only on* n*,* λ*,* p*, the module of continuity of* a_{ij} *and the* L^∞*-norms of* a_{ij}*,* b_i *and* c*.*

Here we only stated interior estimates. Global estimates also hold if we assume, in addition, that $u|_{\partial B_1}$ can be extended to a $C^{2,\alpha}$-function in B_1 or a $W^{2,p}$-function in B_1 respectively. Theorem 1.1.2 and Theorem 1.1.3 will be used throughout the book. Their proofs can be found in [**38**].

Now we consider elliptic differential equations of the divergence form

$$\big(a_{ij}u_j\big)_i = 0. \tag{1.1.2}$$

A function $u \in H^1(B_1)$ is a weak solution of (1.1.2) if

$$\int_{B_1} a_{ij}u_i\varphi_j = 0 \quad \text{for any } \varphi \in H_0^1(B_1).$$

The following result is referred to as the Moser's local boundedness estimate. It is one of the most fundamental estimates in the theory of elliptic differential equations.

THEOREM 1.1.4. *Let* a_{ij} *be bounded in* B_1 *and* $u \in H^1$ *be a weak solution of (1.1.2) in* B_1*. Then*

$$\sup_{B_{\frac{1}{2}}} u^2 \leq C \fint_{B_1} u^2, \tag{1.1.3}$$

where C *is a positive constant depending only on* n *and* λ*.*

Here and thereafter, $\fint$ denotes the average.

Another estimate we will use is the following Caccioppoli's inequality.

LEMMA 1.1.5. *Let* a_{ij} *be bounded in* B_1 *and* $u \in H^1$ *be a weak solution of (1.1.2) in* B_1*. Then*

$$\fint_{B_{\frac{1}{2}}} |\nabla u|^2 \leq C \fint_{B_1} |u - \bar{u}|^2, \tag{1.1.4}$$

where $\bar{u}$ *is the average of* u *in* B_1 *and* C *is a positive constant depending only on* n *and* λ*.*

The proof of Theorem 1.1.4 can be found in [**38**]. Lemma 1.1.5 follows from a simple integration with the help of cutoff functions. They will be used in Chapter 3.

1.2. Hausdorff Measures

In this section, we review some basic knowledge concerning Hausdorff measures, a class of *lower dimensional* measures on $\mathbb{R}^n$, which allow us to measure certain *very small* subsets of $\mathbb{R}^n$. A good reference is [**33**].

We first recall the definition of Hausdorff measures.

Definition 1.2.1. Let $A \subseteq \mathbb{R}^n$ be a nonempty subset and s be a nonnegative integer.

(i) For any $\delta \in (0, \infty]$, define

$$\mathcal{H}^s_\delta(A) = \inf\big\{ \sum_{j=1}^{\infty} \omega(s)\big(\frac{\mathrm{diam} C_j}{2}\big)^s; \ A \subset \bigcup_{j=1}^{\infty} C_j, \mathrm{diam} C_j \leq \delta \big\},$$

where

$$\omega(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}, \quad \text{for } s \in [0, \infty),$$

and

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \text{for } s \in (0, \infty).$$

Note that $\Gamma(s)$ is the usual gamma function for $s \in (0, \infty)$.

(ii) Define

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

We call $\mathcal{H}^s$ the *s-dimensional Hausdorff measure* on $\mathbb{R}^n$.

For the empty set, we simply define $\mathcal{H}^s_\delta(\emptyset) = 0$ for any $\delta > 0$ and $\mathcal{H}^s(\emptyset) = 0$.

Note that $\omega(s)$ is the volume of the unit ball in $\mathbb{R}^s$ if s is a positive integer.

Remark 1.2.2. For $\delta > \delta'$, $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s_{\delta'}(A)$. Hence $\mathcal{H}^s_\delta(\cdot)$ is a monotone decreasing function of $\delta \in (0, \infty]$ and the limit in (ii) makes sense. In particular, we have for any subset $A \subset \mathbb{R}^n$, $\delta > 0$ and $s \geq 0$

$$\mathcal{H}^s(A) \geq \mathcal{H}^s_\delta(A) \geq \mathcal{H}^s_\infty(A).$$

We note that it is necessary to require $\delta \to 0$ in order to force the coverings to follow the local geometry of the set A. This is well illustrated by a spiral in $\mathbb{R}^2$.

Theorem 1.2.3. *For any $s \in [0, \infty)$, $\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^n$. In other words, for any subset $A \subset \mathbb{R}^n$, there is a Borel subset $B \subset \mathbb{R}^n$ with $A \subset B$ such that $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.*

The next result asserts that, for any subset $A \subset \mathbb{R}^n$, there is only one possible $s \geq 0$ such that $\mathcal{H}^s(A)$ is meaningful.

Lemma 1.2.4. *Let $A \subset \mathbb{R}^n$ and $0 \leq s < t < \infty$.*
(i) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.
(ii) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$.

Definition 1.2.5. The *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$ is defined by

$$\dim_{\mathcal{H}}(A) = \inf\{s \in [0, \infty); \ \mathcal{H}^s(A) = 0\}.$$

It is easy to see that $\dim_{\mathcal{H}}(A) \leq n$. By setting $s = \dim_{\mathcal{H}}(A)$, we have $\mathcal{H}^t(A) = 0$ for any $t > s$ and $\mathcal{H}^t(A) = \infty$ for any $t < s$. We note that $\mathcal{H}^s(A)$ may be any number between 0 and ∞ inclusive. Furthermore, $\dim_{\mathcal{H}}(A)$ need not be an integer. Even if $\dim_{\mathcal{H}}(A) = k$ is an integer and $0 < \mathcal{H}^k(A) < \infty, A$ need not be a k-dimensional surface in any sense.

Next, we discuss some properties of Lipschitz functions. The following theorem of Rademacher concerns the differentiability of Lipschitz functions.

THEOREM 1.2.6. *Let* $f : \mathbb{R}^m \to \mathbb{R}^n$ *be a locally Lipschitz map. Then* f *is differentiable* $\mathcal{L}^m$*-almost everywhere in* $\mathbb{R}^m$*, i.e., for* $\mathcal{L}^m$*-almost every* $x \in \mathbb{R}^m$*,* $Df(x) = (\partial_1 f(x), \cdots, \partial_m f(x))$ *exists and*

$$\lim_{y \to x} \frac{1}{|y-x|}\big(f(y) - f(x) - Df(x)\cdot(y-x)\big) = 0.$$

Now let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map and $x \in \mathbb{R}^m$ be a point where f is differentiable at x. Hence $Df(x) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map. We can view $Df(x)$ as an $n \times m$ matrix. With $L = Df(x)$, we define the *Jacobian* of f at x by

$$Jf(x) = \begin{cases} \sqrt{\det(L^*L)}, & m \le n, \\ \sqrt{\det(LL^*)}, & n \le m. \end{cases}$$

Sometimes, in order to emphasize the maximal possible rank, we denote the Jacobian by $J_m f(x)$ for $m \le n$ and $J_n f(x)$ if $m \ge n$. Obviously, $Jf(x) = 0$ if $\operatorname{rank}(Df(x)) < \min\{m, n\}$. Now we intend to calculate

$$\int_A Jf(x) d\mathcal{L}^m(x).$$

THEOREM 1.2.7. *Let* $f : \mathbb{R}^m \to \mathbb{R}^n$ *be a Lipschitz map and* $m \le n$*. Then for any* $u \in L^1(\mathbb{R}^m)$

$$\int_{\mathbb{R}^m} u(x) J_m f(x) dx = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}(y)} u(x) d\mathcal{H}^m(y). \tag{1.2.1}$$

In particular, if f is one-to-one on an $\mathcal{L}^m$-measurable set $A \subseteq \mathbb{R}^m$, then we obtain

$$\int_A J_m f(x) dx = \mathcal{H}^m\big(f(A)\big),$$

by taking $u = \chi_A$.

THEOREM 1.2.8. *Let* $f : \mathbb{R}^m \to \mathbb{R}^n$ *be a Lipschitz map and* $m \ge n$*. Then for any* $u \in L^1(\mathbb{R}^m)$

$$\int_{\mathbb{R}^m} u(x) J_n f(x) dx = \int_{\mathbb{R}^n} \int_{f^{-1}(y)} u d\mathcal{H}^{m-n} dy. \tag{1.2.2}$$

We note that (1.2.1) and (1.2.2) are referred to as the area formula and the coarea formula respectively.

Now we introduce countably rectifiable sets, which provide an appropriate notion of *generalized submanifolds in Euclidean spaces.* For the rest of this section, we always assume $1 \le m \le n-1$.

DEFINITION 1.2.9. Let $E \subset \mathbb{R}^n$ be an $\mathcal{H}^m$-measurable subset. E is countably m-rectifiable if

$$E \subset E_0 \cup \Big(\bigcup_{j=1}^{\infty} f_j(\mathbb{R}^m)\Big),$$

where $\mathcal{H}^m(E_0) = 0$ and $f_j : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function for $j = 1, 2, \cdots$.

Countably rectifiable sets are important in the geometric measure theory and have many nice geometric properties. For example, countably rectifiable sets have approximate tangent spaces almost everywhere and the area and coarea formulas hold for Lipschitz maps between countably rectifiable sets. For our purpose, we will only review the integral geometric formula ([**33**], 3.2.22), an important method in calculating the measure of countably rectifiable sets.

Let $\Lambda(n,m)$ be the collection of $(\lambda_1,\cdots,\lambda_m)$ with $1\le\lambda_1<\cdots<\lambda_m\le n$. For any $\lambda\in\Lambda(n,m)$, we denote by $P_\lambda:\mathbb{R}^n\to\mathbb{R}^m$ the orthogonal projection given by

$$P_\lambda(x_1,\cdots,x_n)=(x_{\lambda_1},\cdots,x_{\lambda_m}).$$

THEOREM 1.2.10. *Suppose E is a countably m-rectifiable subset of $\mathbb{R}^n$, $m\le n$ and*

$$a_\lambda=\int\mathcal{H}^0\big(E\cap P_\lambda^{-1}(y)\big)d\mathcal{L}^m y\quad\textit{for any }\lambda\in\Lambda(n,m).$$

Then

$$\Big(\sum_{\lambda\in\Lambda(n,m)}a_\lambda^2\Big)^{\frac{1}{2}}\le\mathcal{H}^m(E)\le\sum_{\lambda\in\Lambda(n,m)}a_\lambda.$$

The integral geometry formula plays a fundamental role in estimating measures of nodal sets throughout this book. It has a clear geometric interpretation. For example, in order to estimate the $\mathcal{H}^1$-measure of a countably 1-rectifiable subset in $\mathbb{R}^2$, we need only examine the intercepts of this subset with all straight lines parallel to axis.

1.3. Zeros of Analytic Functions

In this section, we collect some results concerning zeros of analytic functions.

We begin with the Bezout formula ([**7**], Corollary 1, P200), which can be viewed as the two-dimensional version of the fundamental theorem of algebra.

LEMMA 1.3.1. *Suppose P and Q are homogeneous polynomials of degree k and l in $\mathbb{C}^2$ respectively. If P and Q have no common factors, then*

$$\#\{(z_1,z_2);P(z_1,z_2)=Q(z_1,z_2)=0\}=kl,$$

including the multiplicity.

The next result is the 2-dimensional version of the Rouché Theorem. For a general form and a proof, refer to Lupacciolu [**71**].

LEMMA 1.3.2. *Let $\Omega\subset\mathbb{C}^2$ be a bounded C^1-domain and $f,g:\Omega\to\mathbb{C}^2$ be holomorphic in Ω and C^1 up to the boundary $\partial\Omega$. If*

$$|f(z_1,z_2)-g(z_1,z_2)|<|g(z_1,z_2)|\quad\textit{for any }(z_1,z_2)\in\partial\Omega,$$

then $f^{-1}(0)$ and $g^{-1}(0)$ are isolated in Ω and the number of points in $f^{-1}(0)$ is the same as that in $g^{-1}(0)$, counting the multiplicity.

A good choice for g in Lemma 1.3.2 is given by $g=(P,Q)$, where P and Q are as in Lemma 1.3.1.

Next, we collect some results on zeros of functions in real Euclidean spaces. It is well known that real zeros do not behave as well as complex zeros. For example, a perturbation of a function with isolated zeros may produce functions with complicated zeros. In some special cases, we may preserve the finiteness of zeros.

Lemma 1.3.3. *Let* P_1 *and* P_2 *be homogeneous polynomials of degree* $d \geq 1$ *in* $\mathbb{R}^2$ *such that* P *and* Q *can be written as products of distinct linear factors. Then there exists a positive constant* δ*, depending only on* P_1 *and* P_2*, such that for any* C^{2d^2}*-functions* u_1 *and* u_2 *on* $B_1 \subset \mathbb{R}^2$ *with*

$$|u_i - P_i|_{C^{2d^2}(B_1)} < \delta \quad \textit{for any } i = 1, 2,$$

there holds

$$\#(\{x \in B_{\frac{1}{2}}; u_1(x) = u_2(x) = 0\}) \leq cd^2,$$

where c *is a universal constant.*

Lemma 1.3.3 can be viewed as Lemma 1.3.2 in the real space with $g = (P_1, P_2)$. The proof of Lemma 1.3.3 is based on the Weierstrass-Malgrange Preparation Theorem for finitely differentiable functions. It is quite complicated although the result is easily visualized to be true. See [**47**] for details.

Last, we state a result concerning zeros of polynomials. See Theorem 2.1 in [**51**].

Lemma 1.3.4. *Let* P *be a (real) polynomial of degree* d *in* $\mathbb{R}^n$*. If* $dim P^{-1}(0) \leq k$*, then*

$$\mathcal{H}^k\big(P^{-1}\{0\} \cap B_1\big) \leq cd^{n-k},$$

where c *is a positive constant depending only on* n*.*

CHAPTER 2

Nodal Sets of Harmonic Functions

In this chapter, we discuss the relation between the growth of harmonic functions and the growth of their nodal sets. The growth of harmonic functions is measured by their frequency. For any harmonic function u in the unit ball $B_1 \subset \mathbb{R}^n$, the *frequency* in B_1 is defined by

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}.$$

If u is a homogenous harmonic polynomial, its frequency is exactly its degree. In general, the frequency controls the growth of the harmonic functions.

As we know, planar harmonic functions are simply the real parts of holomorphic functions in the complex plane. This simple identification is not enjoyed by harmonic functions in higher dimensional spaces. However, harmonic functions in $\mathbb{R}^n$, as analytic functions with interior estimates on derivatives, can be extended to holomorphic functions in $\mathbb{C}^n$. It turns out that such an extension is extremely important in the discussion of nodal sets. This is because the nodal sets of (general analytic) functions in $\mathbb{R}^n$ are not stable in the sense that a simple perturbation may change the structure of nodal sets. In particular, the dimension of nodal sets may change by perturbations. However, this does not happen for holomorphic functions in $\mathbb{C}^n$. In order to discuss nodal sets of harmonic functions, we first discuss complex nodal sets of their holomorphic extensions. It is not surprising that complex analysis plays an important role in our study. For example, we use repeatedly Rouché Theorem in $\mathbb{C}$ and in $\mathbb{C}^2$, which asserts in particular that, if an equidimensional holomorphic map has isolated zeroes, then its holomorphic perturbation enjoys the same property and the number of zeroes is preserved. Another property we use is the behavior of polynomials away from their zeroes. For a suitably normalized polynomial, a positive lower bound can be established for the modulus of the polynomial outside some balls around its zeroes.

The foundation of our discussion is a monotonicity formula for frequencies of harmonic functions. Corollaries of such a monotonicity include the doubling condition of L^2-integrals and the finite vanishing order. In fact, an integral quantity of harmonic functions in the unit ball controls the vanishing order of harmonic functions inside the ball.

2.1. Nodal Domains of Spherical Harmonics

We begin our discussion of nodal sets of harmonic functions with those of homogeneous harmonic polynomials. Because of the homogeneity, homogeneous harmonic polynomials in $\mathbb{R}^n$ can be identified with the corresponding spherical harmonics, their restrictions on the unit sphere $\mathbb{S}^{n-1}$. For any integer $m \geq 1$, we denote by $\mathcal{H}_m(\mathbb{R}^n)$ the collection of all homogeneous harmonic polynomials of degree m in $\mathbb{R}^n$. Obviously, $\mathcal{H}_m(\mathbb{R}^n)$ is a linear space.

We start with $n = 2$ and denote points in $\mathbb{R}^2$ by (x, y). The general homogeneous polynomial P_m of degree m in $\mathbb{R}^2$ contains $m + 1$ coefficients. Then ΔP_m is a homogeneous polynomial of degree $m - 2$ and therefore contains $m - 1$ terms. These terms have to vanish if P_m is a harmonic function. This gives $m-1$ relations which must hold among the $m + 1$ constants if P_m is a harmonic function; so that these constants can be expressed linearly in terms of $(m+1)-(m-1)$ or 2 of them. Therefore, $\mathcal{H}_m(\mathbb{R}^2)$ is a linear space of dimension 2. We note that $\mathrm{Re}(x + iy)^m$ and $\mathrm{Im}(x + iy)^m$ are linearly independent homogeneous harmonic polynomials and hence form a basis in $\mathcal{H}_m(\mathbb{R}^2)$. In fact, in polar coordinates (r, θ), any homogeneous harmonic polynomial P_m of degree m is of the form

$$P_m = ar^m \cos(m\theta + \theta_0),$$

for some constants a and $\theta_0 \in [0, 2\pi)$. Obviously, $P_m^{-1}(0)$ consists of m straight lines passing the origin and forming equal angles by any two consecutive lines. It is natural to ask whether nodal sets of homogeneous harmonic polynomials in $\mathbb{R}^n$ enjoy similar simple characterizations for $n \geq 3$. In the rest of this section, we only discuss the case $n = 3$.

The general homogeneous polynomial P_m of degree m in $\mathbb{R}^3$ contains $\frac{1}{2}(m + 1)(m + 2)$ coefficients. Then ΔP_m is a homogeneous polynomial of degree $m - 2$ and therefore contains $\frac{1}{2}m(m - 1)$ terms. These terms have to vanish if P_m is a harmonic function. This gives $\frac{1}{2}m(m - 1)$ relations which must hold among the $\frac{1}{2}(m+1)(m+2)$ constants if P_m is a harmonic function; so that these constants can be expressed linearly in terms of $\frac{1}{2}\{(m+1)(m+2) - m(m-1)\}$ or $2m+1$ of them. Therefore, $\mathcal{H}_m(\mathbb{R}^3)$ is a linear space of dimension $2m + 1$. A basis of this linear space is given in terms of Legendre functions. We now discuss how to construct such a basis. The following brief discussion follows MacRobert [**72**].

Let (x, y, z) and (r, θ, ϕ) be the rectangular coordinate and the corresponding polar coordinate in $\mathbb{R}^3$, i.e.,

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta.$$

Let $P_m = r^m Q_m(\theta, \varphi)$ be a homogeneous harmonic polynomial of degree m in $\mathbb{R}^3$. Then

$$m(m + 1)Q_m + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial Q_m}{\partial\theta}\big) + \frac{1}{\sin^2\theta}\frac{\partial^2 Q_m}{\partial\varphi^2} = 0.$$

By writing $\mu = \cos\theta$, we have

$$m(m + 1)Q_m + \frac{\partial}{\partial\mu}\big((1 - \mu^2)\frac{\partial Q_m}{\partial\mu}\big) + \frac{1}{1 - \mu^2}\frac{\partial^2 Q_m}{\partial\varphi^2} = 0.$$

By seting $Q_m(\theta, \varphi) = f(\theta)g(\varphi)$, we obtain

$$m(m + 1) + \frac{1}{f}\frac{\partial}{\partial\mu}\big((1 - \mu^2)\frac{\partial f}{\partial\mu}\big) + \frac{1}{1 - \mu^2}\frac{1}{g}\frac{\partial^2 g}{\partial\varphi^2} = 0.$$

The first two terms in this equation are independent of g, and therefore so is the last. Hence, the value of $\frac{1}{g}\frac{\partial^2 g}{\partial\varphi^2}$ must be a constant. Since g is 2π-periodic in φ, we may take this constant to be $-k^2$ for an integer k. Thus

$$\frac{\partial^2 g}{\partial\varphi^2} = -k^2 g,$$

and hence

$$g(\varphi) = A\cos(k\varphi) + B\sin(k\varphi),$$

where A and B are constants. Then f satisfies

$$\frac{\partial}{\partial\mu}\big((1-\mu^2)\frac{\partial f}{\partial\mu}\big) + \big(m(m+1) - \frac{k^2}{1-\mu^2}\big)f = 0, \tag{2.1.1}$$

which is known as *Legendre's associated equation.* If $f_{m,k}(\mu)$ is a solution of (2.1.1), then

$$r^m\big(A\cos(k\varphi) + B\sin(k\varphi)\big)f_{m,k}(\cos\theta) \tag{2.1.2}$$

is a harmonic function wherever it is defined in $\mathbb{R}^3$. We are interested in only those $f_{m,k}$ such that (2.1.2) gives a homogeneous polynomial of degree m in $\mathbb{R}^3$. A lengthy calculation then yields for $k = 0, 1, \cdots, m$

$$f_{m,k}(\mu) = (1-\mu^2)^{\frac{k}{2}}\frac{d^{m+k}}{d\mu^{m+k}}(1-\mu^2)^m. \tag{2.1.3}$$

For $k = 0$, we have

$$f_m(\mu) = f_{m,0}(\mu) = \frac{d^m}{d\mu^m}(1-\mu^2)^m. \tag{2.1.4}$$

This is the *Legendre function.* We also note that $f_{m,m}$ is a constant. For each fixed positive integer m, the collection in (2.1.2) with $f_{m,k}$ given by (2.1.3) for $0 \le k \le m$ consists of $2m+1$ linearly independent homogeneous harmonic polynomials of degree m and hence forms a basis of $\mathcal{H}_m(\mathbb{R}^3)$. (We note that there is only one function for $k = 0$ in (2.1.2).)

Now, we examine nodal sets of these homogenous harmonic polynomials in (2.1.2). Because of the homogeneity, it is convenient to consider the restriction of the nodal sets to the unit sphere $\mathbb{S}^2$, which is called *the nodal curves.* These nodal curves divide $\mathbb{S}^2$ into a certain number of domains, called *nodal domains*, where defining functions are not zero.

If $k = 0$, the corresponding harmonic polynomial is a constant multiple of the Legendre function $f_m(\cos\theta)$ as in (2.1.4). A simple calculus argument shows that $f_m(\mu)$ has m distinct zeros between -1 and 1, arranged symmetrically about $\mu = 0$. Hence $f_m(\cos\theta)$ has m distinct zeros between 0 and π, arranged symmetrically about $\theta = \frac{\pi}{2}$. Accordingly on $\mathbb{S}^2$, the function $f_m(\cos\theta)$ vanishes on m latitude circles. These circles are symmetrically situated with respect to the equator $\theta = \frac{\pi}{2}$, and, if m is an odd number, the equator itself is one of these circles. Similarly, level sets of this function consist of latitude circles. Because of this division of the sphere into zones by sets of latitude circles, the function $f_m(\cos\theta)$ and its constant multiples are called *zonal harmonics.*

If $0 < k < m$, the spherical harmonic is of the form

$$\big(A\cos(k\varphi) + B\sin(k\varphi)\big)\sin^k\theta\frac{d^{m+k}}{d\mu^{m+k}}(\mu^2-1)^m|_{\mu=\cos\theta}.$$

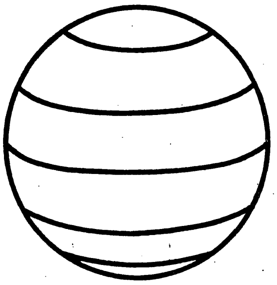

FIGURE 2.1.1. Nodal curves of zonal harmonics.

The first factor vanishes when $A\cos(k\varphi) + B\sin(k\varphi) = 0$, i.e., when $\tan(k\varphi) = -A/B$, and on $\mathbb{S}^2$ this corresponds to k great circles through the pole $\theta = 0$, the angle between the planes of any two consecutive great circles being π/k. The second factor vanishes at the points $\theta = 0$ and $\theta = \pi$, and the third on $m-k$ latitude circles, arranged like the corresponding circles in the case of zonal harmonics. Since the two sets of circles intersect orthogonally, these harmonics are called *tesseral harmonics*.

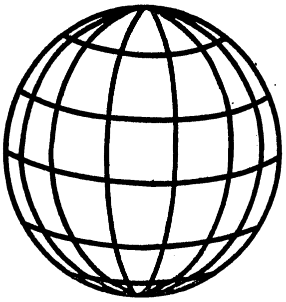

FIGURE 2.1.2. Nodal curves of tesseral harmonics.

Finally, if $k = m$, the spherical harmonic is of the form

$$\big(A\cos(m\varphi) + B\sin(m\varphi)\big)\sin^m\theta,$$

which vanishes when $\theta = 0$ or π, or when $\tan(m\varphi) = -A/B$. This corresponds on $\mathbb{S}^2$ to the points $\theta = 0$ and $\theta = \pi$, and to m great circles through these points, the angle between the planes of any two consecutive being π/m. As $\mathbb{S}^2$ is thus divided up into $2m$ sectors, these functions called *sectorial harmonics*. We note that sectorial harmonics are simply linear combinations of $\mathrm{Re}(x+iy)^m$ and $\mathrm{Im}(x+iy)^m$.

Now we examine nodal sets and nodal domains of these special harmonics closely. It is clear that the length of the nodal curves grow as the degree grows. This in fact is a general result. The measure estimate of nodal sets constitutes an important part of this book. Figures 2.1-2.1 (**Fix the cross referencing.**) also seem to suggest that the number of nodal domains grows according to the degree. This in fact is false. We now present examples of spherical harmonics such that the

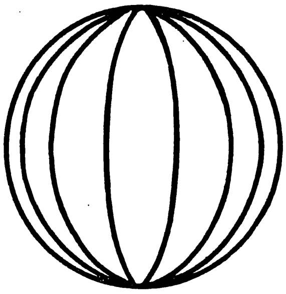

FIGURE 2.1.3. Nodal curves of sectorial harmonics.

number of their nodal domains remains constant as degrees grow. The rest of this section follows Lewy [**68**].

Let u be a spherical harmonic of degree m in $\mathbb{R}^3$. There is an obvious lower bound for the number of nodal domains $N_m(u)$. Evidently, $N_m(u) \geq 2$ since u must change sign on $\mathbb{S}^2$, for $0 = u(0) = \fint_{\mathbb{S}^2} u$. If m is even, then $N_m(u) \geq 3$. For if $N_m(u) = 2$, the sets $\{u > 0\}$ and $\{u < 0\}$ of $\mathbb{S}^2$ would be connected and each must contain a point p and also its diametrically opposite point p'. Suppose, say, $u(p) > 0$, then there is an arc σ joining p to p' on which $u > 0$ and the diametrically opposite arc σ' must also belong to $u > 0$. If now $u(q) < 0$, then also $u(q') < 0$ (q' opposite to q), but it is impossible to connect q to q' without intersecting $\sigma \cup \sigma'$.

In the following, we prove that these lower bounds for N_m are actually assumed. In other words, we construct spherical harmonics of degree m whose nodal curves divide $\mathbb{S}^2$ in 2 resp. 3 domains for m odd resp. even. We achieve this by starting from a particular spherical harmonic P, whose nodal curves are connected and easy to describe, adding ε times a suitable other such of the same degree where ε is small, and investigating the change of the nodal curves due to this addition. The sphere $\mathbb{S}^2$ is the union of finitely many neighborhoods each contained in a hemisphere H. After a rotation, H becomes $z > 0$. The homogeneity relation

$$P(x, y, z) = z^m P(\frac{x}{z}, \frac{y}{z}, 1)$$

reduces the study of the nodal curves of P in H to that of the zeros of polynomials in two variables.

LEMMA 2.1.1. *Suppose $m \geq 2$ is an integer and $f(x)$ is an analytic function in a neighborhood of $0 \in \mathbb{R}$ with $f(0) = 1$, $|f(x) - 1| \leq M|x|$ and $|f'(x)| \leq M$. Then there exists a $\rho > 0$ such that, for any $\varepsilon > 0$ small, there is exactly one $x_0 > 0$ satisfying $x_0^m = \varepsilon f(x_0)$, and $x^m < \varepsilon f(x)$ for $x < x_0$ and $x^m > \varepsilon f(x)$ for $x_0 < x < \rho$.*

PROOF. Note that $F(x) = x^m - \varepsilon f(x)$ satisfies $F(0) < 0$ and $F\big((2\varepsilon)^{1/m}\big) > 2\varepsilon - \varepsilon - \varepsilon M(2\varepsilon)^{1/m} > 0$ if ε small. Hence there exists an $x_0 \in \big(0, (2\varepsilon)^{1/m}\big)$ with $F(x_0) = 0$. Set $x_0 = a_0 \varepsilon^{1/m}$ for some constant a_0 depending on ε. Obviously,

$a_0 < \sqrt[m]{2}$. Then we have

$$0 = \varepsilon^{-1}F(x_0) = a_0^m - f(a_0\varepsilon^{1/m}) \leq a_0^m - 1 + M(2\varepsilon)^{1/m},$$

or

$$a_0^m \geq 1 - M(2\varepsilon)^{1/m}.$$

Hence, for any given $\delta > 0$, we have $a_0 > 1 - \delta$ if ε is small enough. Then for any $x \geq x_0$

$$\begin{aligned} F'(x) =& mx^{m-1} - \varepsilon f'(x) \geq mx_0^{m-1} - \varepsilon M \\ \geq& ma_0^{m-1}\varepsilon^{1-1/m} - \varepsilon M \geq m(1-\delta)^{m-1}\varepsilon^{1-1/m} - \varepsilon M > 0, \end{aligned}$$

for small ε. This shows that there is no root of $F(x) = 0$ above x_0. Obviously, such an x_0 is unique. ☐

Lemma 2.1.2. *Suppose* $\psi(x,y) = Im(x+iy)^m$ *for some* $m \geq 2$ *and* f *is analytic near the origin* $0 \in \mathbb{R}^2$ *with* $f(0) = 1$. *Then there is a* $\rho > 0$ *such that for small* $\varepsilon > 0$

$$\psi(x,y) - \varepsilon f(x,y) = 0 \tag{2.1.5}$$

has no solution in $0 < y \leq x\tan\frac{\pi}{2m}$ *for any* $x \in [0, x_1)$ *and has exactly one solution* $y(x)$ *in* $0 < y \leq x\tan\frac{\pi}{2m}$ *for any* $x \in [x_1, \rho]$, *where* x_1 *is the solution of*

$$\psi(x_1, x_1\tan\frac{\pi}{2m}) = \varepsilon f(x_1, x_1\tan\frac{\pi}{2m}).$$

Proof. Set

$$F(x,y) = \psi(x,y) - \varepsilon f(x,y) = \mathrm{Im}(x+iy)^m - \varepsilon f(x,y).$$

It is easy to see that

$$\psi(x, x\tan\frac{\pi}{2m}) = \mathrm{Im}(x + ix\tan\frac{\pi}{2m})^m = \left(\frac{x}{\cos\frac{\pi}{2m}}\right)^m.$$

With

$$x' = \frac{x}{\cos\frac{\pi}{2m}},$$

we have

$$F(x, x\tan\frac{\pi}{2m}) = x'^m - \varepsilon\tilde{f}(x') = x'^m - \varepsilon f(x'\cos\frac{\pi}{2m}, x'\sin\frac{\pi}{2m}).$$

We apply Lemma 2.1.1 to find a ρ such that there exists a unique x'_0 solving $x_0'^m = \varepsilon\tilde{f}(x'_0)$ in $(0,\rho)$ for sufficiently small ε. With $x_1 = x'_0\cos\frac{\pi}{2m}$, we have

$$\psi(x_1, x_1\tan\frac{\pi}{2m}) = \varepsilon f(x_1, x_1\tan\frac{\pi}{2m}),$$

and

$$\psi(x, x\tan\frac{\pi}{2m}) - \varepsilon f(x, x\tan\frac{\pi}{2m}) < 0 \quad \text{for any } x \in [0, x_1).$$

Consider any $x \in [0,\rho]$ and $0 < y \leq \tan\frac{x}{2m}$, and set $r = \sqrt{x^2+y^2}$. Then we have $\psi(x,y) = \mathrm{Im}(x+iy)^m \leq r^m$ and

$$F(x,y) = \psi(x,y) - \varepsilon f \leq r^m - \frac{\varepsilon}{2}.$$

First, consider $x \in (0, x_1/2)$. Since $x_0' < (2\varepsilon)^{1/m}$, we have

$$r < \frac{1}{2}(2\varepsilon)^{1/m}, \quad r^m < \frac{\varepsilon}{2^{m-1}}.$$

This implies $F(x,y) < 0$ for any $x \in (0, x_1/2)$, if ε is small. On the other hand, if $x \geq x_1/2$, $r \geq (1-\delta)\varepsilon^{1/m}/2$ with $\delta > 0$ small, we have

$$\begin{aligned} \partial_y F &= \psi_y - \varepsilon f_y = m r^{m-1} \cos(m-1)\theta - \varepsilon f_y \\ &\geq m\varepsilon^{1-1/m}\big(\frac{1-\delta}{2}\big)^{m-1} \sin\frac{\pi}{2m} - \varepsilon M' > 0, \end{aligned} \tag{2.1.6}$$

for small ε. Thus $F(x, x\tan\theta) \leq F(x, x\tan\frac{\pi}{2m})$ if $x \geq x_1/2$, $\theta \in [0, \frac{\pi}{2m}]$; while $F(x, x\tan\frac{\pi}{2m}) < 0$ if $x_1/2 \leq x < x_1$. Hence, there is no solution of $F(x,y) = 0$ unless $x \geq x_1$. Note that (2.1.6) also shows that for $x > x_1$ there can be no more than one solution of $F(x,y) = 0$; it cannot lie on $y = x\tan\frac{\pi}{2m}$ by Lemma 2.1.1, since $F(x,y) > 0$ there. Since $F(x,0) = -\varepsilon f(x,0) < 0$ for $x \in [0,\rho]$, there is a unique solution $y(x)$ of $F(x,y) = 0$ in $x \in [x_1, \rho]$, $0 \leq y \leq x\tan\frac{\pi}{2m}$ if ε is small. Note that $y(x)$ is continuous and tends to 0 as $\varepsilon \to 0+$. ☐

REMARK 2.1.3. In Lemma 2.1.2, ψ may be replaced by $q(x,y)\psi$ where $q(x,y)$ is analytic with $q(0,0) > 0$ and the condition $f(0,0) = 1$ by $f(0,0) > 0$. If $f(0,0) < 0$ we have analogous results on replacing $\varepsilon > 0$ by $\varepsilon < 0$. Moreover, if ψ is of degree m and $\psi = 0$ has a simple singular point at $0 \in \mathbb{R}^2$ where exactly two regular branches of $\psi = 0$ meet at an angle $\theta \in (0,\pi)$, then we can introduce an analytic transformation which carries branches into $x' = 0$ and $y' = 0$ and $\psi = 0$ into $x'y'\psi'(x',y')$, where ψ' is analytic and $\psi'(0,0) \neq 0$. Then $\psi - \varepsilon f = 0$ becomes $x'y' - \varepsilon f'\psi'^{-1} = 0$ and Lemma 2.1.2 applies with $m = 2$.

THEOREM 2.1.4. *Let $m > 0$ be an odd integer. There is a spherical harmonic of degree m whose nodal curves divide $\mathbb{S}^2$ into two domains.*

It will be clear from the proof that such a spherical harmonic is an appropriate perturbation of a sectorial harmonic.

PROOF. Consider

$$F(x,y,z) = \mathrm{Im}(x+iy)^m - \varepsilon f(x,y,z),$$

where f is a spherical harmonic of degree m with $f(0,0,1) > 1$. We show that F has the required property for small $\varepsilon > 0$.

Cover $\mathbb{S}^2$ by finitely many closed neighborhoods Ω each contained in a hemisphere and of one of the three properties

(i) $\mathrm{Im}(x+iy)^m > 0$ on Ω, or $\mathrm{Im}(x+iy)^m < 0$ on Ω;

(ii) Ω intersects with $\mathrm{Im}(x+iy)^m = 0$ exactly in one non-singular arc;

(iii) Ω contains $(0,0,1)$ or $(0,0,-1)$.

In addition, the diameter of Ω is required not to exceed limitations which depend on f and m, but not on ε. We note that, in Ω of (iii), the gradient of $\mathrm{Im}(x+iy)^m$ is not zero on $\mathrm{Im}(x+iy)^m = 0$.

In each of the m sectors

$$0 < \theta < \frac{\pi}{m},\ \frac{2\pi}{m} < \theta < \frac{3\pi}{m},\ \cdots,\ \frac{2m-2}{m} < \theta < \frac{2m-1}{m}$$

of the Ω of (iii) containing $(0,0,1)$, the equation $F(x,y,z) = z^m F\big(\frac{x}{z}, \frac{y}{z}, 1\big) = 0$ defines for small ε a U-shaped curve tending to the two sides of the sector as $\varepsilon \to 0$, according to Lemma 2.1.2. Similarly, the Ω containing $(0,0,-1)$ of (iii) intersects with $F(x,y,z) = 0$ in the m sectors

$$\frac{\pi}{m} < \theta < \frac{2\pi}{m}, \ \cdots, \ \frac{2m-1}{m} < \theta < 2\pi,$$

since $f(0,0,-1) = -f(0,0,1) < 1$ as m is odd. The omitted sectors and the points $(0,0,1)$ resp. $(0,0,-1)$ belong to $F(x,y,z) \neq 0$. Outside the Ω's of (iii), we either have $F \neq 0$, or $F = 0$ on arcs tending to $\operatorname{Im}(x+iy)^m = 0$ as $\varepsilon \to 0$. Thus, we see that $F = 0$ is a non-singular curve of $\mathbb{S}^2$ beginning, say, in the sector $0 < \theta < \frac{\pi}{m}$ near $(0,0,1)$, arriving in $\frac{\pi}{m} < \theta < \frac{2\pi}{m}$ near $(0,0,-1)$, continuing and again arriving in $\frac{2\pi}{m} < \theta < \frac{3\pi}{m}$ near $(0,0,1)$, etc., and finally arriving back in $0 < \theta < \frac{\pi}{m}$ near $(0,0,1)$, and thus defines a closed Jordan curve of $\mathbb{S}^2$ which divides $\mathbb{S}^2$ into exactly 2 domains. □

The case of even $m > 0$ is more involved.

Lemma 2.1.5. *There is a spherical harmonic of even degree* $m > 0$

$$\psi(x,y,z) = xyq(r^2, z^2)$$

with $q(0,1) > 0$.

Proof. We first note

$$0 = \Delta\psi = xy\big(\Delta q + 8\frac{\partial q}{\partial r^2}\big) = xy\big(4r^2\frac{\partial^2 q}{\partial (r^2)^2} + 12\frac{\partial q}{\partial r^2} + \frac{\partial^2 q}{\partial z^2}\big).$$

The expression in parenthesis vanishes for polynomials $q(r^2, z^2)$ which are solutions of

$$\Delta_7 q = \big(\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{6} \frac{\partial^2}{\partial x_i^2}\big) q = 0,$$

with $r^2 = \sum_{i=1}^6 x_i^2$. We set

$$q = (z^2 + r^2)^{m+\frac{1}{2}} \frac{\partial^{m-2}}{\partial z^{m-2}} (z^2 + r^2)^{-\frac{5}{2}}.$$

This finishes the proof. □

It follows that $\psi(x,y,z) = 0$ on $\mathbb{S}^2$ is the union of the great circles $x = 0$ and $y = 0$, and latitude circles $z = z_j$ with $1 > z_1 > z_2 > \cdots > z_{m-2} > -1$ where $z_j = -z_{m-1-j}$ and $z_j > 0$ for $1 \leq j \leq \frac{m-2}{2}$. Each of the points $(0,0,\pm 1)$, $(\pm\sqrt{1-z_j^2}, 0, z_j)$, $(0, \pm\sqrt{1-z_j^2}, z_j)$ is a singularity of ψ which, after a suitable rotation of $\mathbb{S}^2$, takes the form $xyp(x,y,z)$ with $p(0,0,z) \neq 0$ and p analytic in the new variables.

Lemma 2.1.6. *There is a spherical harmonic $g(x, y, z)$ of degree m satisfying $g(0, 0, 1) > 0$ and for $j = 1, \cdots, \dfrac{m-2}{2}$*

$$g(\sqrt{1-z_j^2}, 0, z_j) > 0, \quad g(-\sqrt{1-z_j^2}, 0, z_j) < 0,$$
$$(-1)^{j+1} g(0, \sqrt{1-z_j^2}, z_j) > 0, \quad (-1)^j g(0, -\sqrt{1-z_j^2}, z_j) < 0.$$

Proof. There exists a polynomial

$$p_1 = \sum_{j=1,3,5,\cdots}^{m-1} a_j x^j z^{m-j},$$

which equals 1 on $\mathbb{S}^2$ at $z = z_j$, $j = 1, \cdots, \dfrac{m-2}{2}$, $y = 0$, $x > 0$. We have to solve an interpolation problem for a polynomial of degree $(m-2)/2$ in $x^2 z^{-2}$. Since p_1 is odd in x, we have $p_1(-x, z) = -p_1(x, z)$ so that p_1 assumes on $\mathbb{S}^2$ the value -1 at $g(-\sqrt{1-z_j^2}, 0, z_j)$. There is a spherical harmonic u_1 of degree m for which

$$u_1(x, 0, z) = p_1(x, z), \quad \frac{\partial}{\partial y} u_1(x, 0, z) = 0.$$

It is given by

$$u_1(x, y, z) = p_1(x, z) - \frac{y^2}{2!} \Delta p_1(x, z) + \frac{y^4}{4!} \Delta^2 p_1(x, z) + \cdots .$$

Since the coefficients of the powers of y are odd in x, we conclude $u_1(0, y, z) = 0$. Similarly, we construct a spherical harmonic u_2 of degree m which is odd in y and assumes the values $(-1)^{j+1}$ at $(0, \sqrt{1-z_j^2}, z_j)$, $j = 1, \cdots, \dfrac{m-2}{2}$, hence $= (-1)^j$ at $(0, -\sqrt{1-z_j^2}, z_j)$, $j = 1, \cdots, \dfrac{m-2}{2}$. We have $u_2(x, 0, z) = 0$. Thus we set $g = u_1 + u_2$.

The g constructed above vanishes at $(0, 0, 1)$ but there are points $(x', 0, z')$ near $(0, 0, 1)$ where $g > 0$. Let ω be a rotation of $\mathbb{S}^2$ about the y-axis by a small angle for which $(x', 0, z') = \omega(0, 0, 1)$. Then $g\big(\omega(x, y, z)\big)$ is a spherical harmonic with the required properties by the continuity. □

Theorem 2.1.7. *Let $m > 0$ be an even integer. There is a spherical harmonic of degree m whose nodal curves divide $\mathbb{S}^2$ into three domains.*

Proof. The required function is

$$u = \psi + \varepsilon g$$

with ψ from Lemma 2.1.5, g from Lemma 2.1.6 and $\varepsilon > 0$ a small constant.

Figure 2.1 (**Fix the cross referencing**) shows schematically the nodal curves of u on the hemisphere $z \geq 0$ for $m = 10$; their picture on $z \leq 0$ is obtainable by reflection in the center $x = y = 0$. Note that in this reflection domains with $u > 0$ ($u < 0$) are preserved. The nodal curves of the figure are two Jordan arcs, say σ_1 and σ_2, their respective reflections σ_1' and σ_2'. On $\mathbb{S}^2$, $\sigma_1 \cup \sigma_2'$ and $\sigma_2 \cap \sigma_1'$ are the two closed Jordan curves which divide $\mathbb{S}^2$ in three domains.

For the figure we can likewise read the nodal curves for even $m > 10$ and $m < 10$. For this we note that we obtain the case $m = 8$ by disregarding the

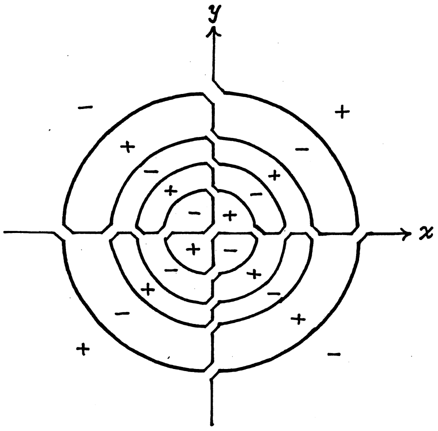

FIGURE 2.1.4. An illustration for $m = 10$.

outermost circle and its exterior, $m = 6$ by disregarding also the next to outermost circle and its exterior, etc. On the other hand, we observe that $m = 12$ requires the same type of addition of curves to our figure (of $m = 10$) which are required to pass from $m = 6$ to $m = 8$, that $m = 14$ requires the same type of addition of curves to the figure of $m = 12$ as leads from $m = 8$ to $m = 10$, and so on. □

2.2. A Monotonicity Formula for Harmonic Functions

By examining the nodal curves of zonal, tesseral and sectorial harmonics in Section 2.1, we note that their lengths grow according to degrees. This is true even for spherical harmonics in Theorem 2.1.4 and Theorem 2.1.7 and turns out to be a general fact. To prove this, we should first generalize the notion of degrees of harmonic polynomials to those of arbitrary harmonic functions.

In this section, we discuss a monotonicity formula for harmonic functions, first discovered by Almgren [**5**]. Important corollaries include doubling conditions and the control of vanishing order by the frequency. The entire section follows closely Garofalo and Lin [**35**], [**36**] and Lin [**69**].

Throughout this section, we always assume that u is a harmonic function in $B_1 \subset \mathbb{R}^n$, i.e.,

$$\triangle u = 0 \quad \text{in } B_1. \tag{2.2.1}$$

Define for any $r \in (0, 1)$

$$D(r) = \int_{B_r} |\nabla u|^2,$$

$$H(r) = \int_{\partial B_r} u^2,$$

and

$$N(r) = \frac{rD(r)}{H(r)}. \tag{2.2.2}$$

DEFINITION 2.2.1. $N(r)$ is called the frequency of u in B_r.

We first note that $D(r)$ can be written as a surface integral. By (2.2.1), we have $\triangle u^2 = 2|\nabla u|^2$. With Green's Theorem, we rewrite $D(r)$ as

$$D(r) = \frac{1}{2}\int_{B_r} \triangle u^2 = \int_{\partial B_r} uu_n, \tag{2.2.3}$$

where u_n is the normal derivative of u.

As an example, we calculate $N(r)$ for homogeneous harmonic polynomials.

EXAMPLE 2.2.2. If u is a homogeneous harmonic polynomial of degree k, then $N(r)$ is a constant and $N(r) = k$. To see this, we write

$$u(x) = r^k\varphi(\theta),$$

where φ is the restriction of u to $\mathbb{S}^{n-1}$. Then we have

$$u_n = kr^{k-1}\varphi(\theta),$$

and hence for any $r > 0$

$$N(r) = \frac{rD(r)}{H(r)} = \frac{r\int_{\partial B_r} uu_n}{\int_{\partial B_r} u^2} = k.$$

Now we prove the following basic result referred to as the monotonicity formula, which was first observed by F. J. Almgren, Jr. [**5**].

THEOREM 2.2.3. *Let u be a harmonic function in $B_1 \subset \mathbb{R}^n$. Then $N(r)$ is a nondecreasing function of $r \in (0,1)$.*

PROOF. First, we have

$$\begin{aligned} N'(r) &= \frac{D(r)}{H(r)} + \frac{rD'(r)}{H(r)} - \frac{rD(r)}{H^2(r)}H'(r) \\ &= N(r)\left\{\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\right\}. \end{aligned} \tag{2.2.4}$$

We need to calculate $D'(r)$ and $H'(r)$. A simple differentiation yields

$$D'(r) = \int_{\partial B_r} |\nabla u|^2 = \frac{1}{r}\int_{\partial B_r} < x|\nabla u|^2, \frac{x}{r} > .$$

By applying Green's Theorem for each $i = 1, \cdots, n$, we have

$$\begin{aligned} &\int_{\partial B_r} x_i|\nabla u|^2 \cdot \frac{x_i}{r} = \int_{B_r} \partial_i(x_i|\nabla u|^2) \\ &= \int_{B_r} |\nabla u|^2 + 2\sum_j \int_{B_r} x_i u_j u_{ij} \quad \text{(integration by parts of the second term)} \\ &= \int_{B_r} |\nabla u|^2 - 2\sum_j \int_{B_r} \partial_j(x_i u_j)u_i + 2\sum_j \int_{\partial B_r} x_i u_j u_i \nu_j \quad (\nu = \frac{x}{r} \text{ on } \partial B_r) \\ &= \int_{B_r} |\nabla u|^2 - 2\int_{B_r} u_i^2 - \sum_j \int_{B_r} x_i u_{jj} u_i + 2r\sum_j \int_{\partial B_r} \nu_i u_i u_j \nu_j \quad (\triangle u = 0) \\ &= \int_{B_r} |\nabla u|^2 - 2\int_{B_r} u_i^2 + 2r\int_{\partial B_r} (\nu_i u_i)u_n. \end{aligned}$$

Summing over i, we get

$$D'(r) = \frac{1}{r}\left\{(n-2)\int_{B_r} |\nabla u|^2 + 2r\int_{\partial B_r} u_n^2\right\},$$

or

$$D'(r) = \frac{n-2}{r}D(r) + 2\int_{\partial B_r} u_n^2. \tag{2.2.5}$$

With (2.2.3), we get

$$\frac{D'(r)}{D(r)} = \frac{n-2}{r} + \frac{2\int_{\partial B_r} u_n^2}{\int_{\partial B_r} uu_n}. \tag{2.2.6}$$

Next, we write $H(r)$ as

$$H(r) = \int_{|x|=r} u^2(x)dS_x = r^{n-1}\int_{|y|=1} u^2(ry)dS_y.$$

This implies

$$\begin{aligned} H'(r) &= (n-1)r^{n-2}\int_{|y|=1} u^2(ry)dS_y + 2r^{n-1}\int_{|y|=1} u(ry)\frac{\partial u}{\partial n}(ry)dS_y \\ &= \frac{n-1}{r}H(r) + 2\int_{\partial B_r} uu_n. \end{aligned}$$

Hence, we have

$$\frac{H'(r)}{H(r)} = \frac{n-1}{r} + \frac{2\int_{\partial B_r} uu_n}{\int_{\partial B_r} u^2}. \tag{2.2.7}$$

By substituting (2.2.6) and (2.2.7) in (2.2.4), we get

$$N'(r) = 2N(r)\left\{\frac{\int_{\partial B_r} u_n^2}{\int_{\partial B_r} uu_n} - \frac{\int_{\partial B_r} uu_n}{\int_{\partial B_r} u^2}\right\},$$

which is nonnegative by the Cauchy inequality. ☐

In the following, we provide another proof of (2.2.5) by a radial deformation, which has its roots in the study of minimal surfaces.

ANOTHER PROOF OF (2.2.5). For any fixed $r, \triangle r \in (0,1)$ with $r + \triangle r < 1$ and a fixed $t \in (0, 1 + \frac{\triangle r}{r+\triangle r})$, define a function $w_t : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$w_t(\rho) = \begin{cases} t & \text{for } \rho \le r, \\ 1 & \text{for } \rho \ge r + \triangle r, \\ t\dfrac{r+\triangle r-\rho}{\triangle r} + \dfrac{\rho - r}{\triangle r} & \text{for } r \le \rho \le r + \triangle r, \end{cases}$$

and define $l_t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$l_t(x) = w_t(|x|)x.$$

It is easy to see that l_t is bi-Lipschitz and maps B_1 to B_1. Then in B_1 we set

$$u^t = u \circ l_t^{-1}.$$

Note u^t is independent of t in $B_1 \setminus B_{r+\triangle r}$. Hence $\frac{d}{dt}\big|_{t=1} u^t$ has compact support in B_1. Consider

$$I(u^t) = \int_{B_1} |\nabla u^t|^2.$$

Since u is harmonic, we have

$$\frac{d}{dt}\Big|_{t=1} I(u^t) = 0. \tag{2.2.8}$$

To continue, we write

$$I(u^t) = \int_{B_{rt}} |\nabla u^t|^2 + \int_{B_{r+\triangle r}\setminus B_{rt}} |\nabla u^t|^2 + \int_{B_1 \setminus B_{r+\triangle r}} |\nabla u^t|^2 = I_1 + I_2 + I_3,$$

and

$$u^t(y) = u(x),$$

with $y = l_t(x)$. For I_3, we have $y = x$ with $r + \triangle r \leq |y| \leq 1$. This implies

$$I_3 = \int_{B_1 \setminus B_{r+\triangle r}} |\nabla u^t|^2 = \int_{B_1 \setminus B_{r+\triangle r}} |\nabla u|^2,$$

and then

$$\frac{d}{dt}\Big|_{t=1} I_3 = 0. \tag{2.2.9}$$

For I_1, we have $y = l_t(x) = tx$ for $|y| < tr$ and $|x| < r$, and hence

$$u^t(y) = u\left(\frac{y}{t}\right).$$

The volume element is given by $dy = t^n dx$. Hence we obtain

$$I_1 = \int_{B_{rt}} |\nabla u^t(y)|^2 dy = t^{n-2} \int_{B_r} |\nabla u(x)|^2 dx = t^{n-2} D(r),$$

and then

$$\frac{d}{dt}\Big|_{t=1} I_1 = (n-2)D(r). \tag{2.2.10}$$

Finally, for I_2, we have

$$y = \left(t\frac{r + \triangle r - |x|}{\triangle r} + \frac{|x| - r}{\triangle r} \right) x,$$

for $tr \leq |y| \leq r + \triangle r$ and $r \leq |x| \leq r + \triangle r$. We note that the transformation between x and y preserves the radial direction. By setting $s = |y|$ and $\rho = |x|$ and using polar coordinates, we have

$$|\nabla u^t(y)|^2 = |\partial_s u^t|^2 + \frac{1}{s^2}|\nabla_\theta u^t|^2 = |\partial_\rho u|^2 \left(\frac{\partial \rho}{\partial s}\right)^2 + \frac{1}{s^2}|\nabla_\theta u|^2.$$

Note that the volume element is given by $dy = s^{n-1} ds d\theta = s^{n-1}\frac{\partial s}{\partial \rho} d\rho d\theta$. Then we have

$$I_2 = \int_{B_{r+\triangle r}\setminus B_{rt}} |\nabla u^t|^2 = \int_r^{r+\triangle r} \int_{\mathbb{S}^{n-1}} \left(s^{n-1}\frac{\partial \rho}{\partial s}|\partial_\rho u|^2 + s^{n-3}\frac{\partial s}{\partial \rho}|\nabla_\theta u|^2 \right) d\rho d\theta.$$

Note

$$s = \rho\left(t\frac{r+\triangle r-\rho}{\triangle r} + \frac{\rho - r}{\triangle r}\right),$$

and

$$\frac{\partial s}{\partial \rho} = t\frac{r+\triangle r-2\rho}{\triangle r} + \frac{2\rho - r}{\triangle r}.$$

A straightforward calculation yields

$$\begin{aligned}\frac{d}{dt}\Big|_{t=1} I_2 =& \frac{1}{\triangle r}\int_r^{r+\triangle r}\int_{\mathbb{S}^{n-1}} \big\{\rho^{n-1}\big((n-1)(r+\triangle r-\rho) - (r+\triangle r-2\rho)\big)|\partial_\rho u|^2 \\ &+ \rho^{n-3}\big((n-3)(r+\triangle r-\rho) + (r+\triangle r-2\rho)\big)|\nabla_\theta u|^2\big\} d\theta d\rho.\end{aligned}$$

By letting $\triangle r \to 0+$, we obtain

$$\begin{aligned}\frac{d}{dt}\Big|_{t=1} I_2 =& \int_{\mathbb{S}^{n-1}} \big(r^n|\partial_r u|^2 - r^{n-2}|\nabla_\theta u|^2\big)d\theta = r\int_{\partial B_r}\left(|\partial_r u|^2 - \frac{1}{r^2}|\nabla_\theta u|^2\right) \\ =& r\int_{\partial B_r}\big(2|\partial_r u|^2 - |\nabla u|^2\big),\end{aligned}$$

or

$$\frac{d}{dt}\Big|_{t=1} I_2 = 2r\int_{\partial B_r} u_n^2 - rD'(r). \tag{2.2.11}$$

With (2.2.8)-(2.2.11), we get

$$(n-2)D(r) + 2r\int_{\partial B_r} u_n^2 - rD'(r) = 0.$$

This is (2.2.5). ☐

Now we discuss some corollaries of Theorem 2.2.3.

Corollary 2.2.4. *Let u be a harmonic function in $B_1 \subset \mathbb{R}^n$. Then the limit of $N(r)$ as $r \to 0+$ exists and is equal to the order of vanishing of u at 0.*

Proof. The existence of the limit of $N(r)$ as $r \to 0+$ follows easily from the monotonicity of $N(r)$, by Theorem 2.2.3. Now we calculate this limit. We first note that u does not vanish up to infinite order because of the analyticity. With the Taylor expansion, we write

$$u = P + R,$$

where P is a nonzero homogeneous polynomial of degree k and R is the remainder. Both P and R are harmonic. Then we have by Example 2.2.2

$$\lim_{r\to 0} N(r) = \frac{\int_{B_1}|\nabla P|^2}{\int_{\partial B_1} P^2} = k.$$

All terms involving R in the above limit have a higher order of r and hence vanish as $r \to 0$. ☐

Corollary 2.2.5. *Let u be a harmonic function in $B_1 \subset \mathbb{R}^n$. Then for any $r < 1$*

$$\frac{d}{dr}\log\frac{H(r)}{r^{n-1}} = 2\frac{N(r)}{r}, \tag{2.2.12}$$

and for any $0 < r_1 < r_2 < 1$

$$\frac{H(r_2)}{r_2^{n-1}} = \frac{H(r_1)}{r_1^{n-1}} \exp\{2\int_{r_1}^{r_2} \frac{N(r)}{r} dr\}, \tag{2.2.13}$$

and

$$\frac{H(r_2)}{r_2^{n-1}} \leq \left(\frac{r_2}{r_1}\right)^{2N(r_2)} \frac{H(r_1)}{r_1^{n-1}}. \tag{2.2.14}$$

Proof. We write (2.2.7) as

$$\frac{H'(r)}{H(r)} - \frac{n-1}{r} = 2\frac{D(r)}{H(r)} = 2\frac{N(r)}{r},$$

or

$$\frac{d}{dr}\log H(r) - \frac{d}{dr}\log r^{n-1} = 2\frac{N(r)}{r}.$$

This implies (2.2.12). A simple integration of (2.2.12) yields (2.2.13). To get (2.2.14), we simply note by Theorem 2.2.3

$$\exp\{2\int_{r_1}^{r_2} \frac{N(r)}{r} dr\} \leq \exp\{2N(r_2)\int_{r_1}^{r_2} \frac{1}{r} dr\} = \left(\frac{r_2}{r_1}\right)^{2N(r_2)}.$$

This finishes the proof. □

The identity (2.2.12) plays an important role in subsequent discussions. An integration of (2.2.12) relates two surface integrals of u^2 through the function $N(r)$, as shown in (2.2.13). In particular, the L^2-integral over a large sphere can be controlled by that over a small sphere.

Corollary 2.2.6. *Let* u *be a harmonic function in* $B_1 \subset \mathbb{R}^n$. *Then for any* $R \in (0, 1/2)$ *and* $\eta \in (1, 2]$

$$\fint_{\partial B_{\eta R}} u^2 \leq \eta^{2N(1)} \fint_{\partial B_R} u^2, \tag{2.2.15}$$

$$\fint_{B_{\eta R}} u^2 \leq \eta^{-1}\eta^{2N(1)} \fint_{B_R} u^2. \tag{2.2.16}$$

Proof. By Theorem 2.2.3 and taking $r_1 = R$ and $r_2 = \eta R$ in (2.2.14), we obtain

$$\frac{H(\eta R)}{(\eta R)^{n-1}} \leq \eta^{2N(1)} \frac{H(R)}{R^{n-1}},$$

which is (2.2.15). To get (2.2.16), we simply integrate (2.2.15) from 0 to R. □

Note that (2.2.15) and (2.2.16) are often referred to as *the doubling condition* if $\eta = 2$.

Corollary 2.2.7. *Let* u *be a harmonic function in* $B_1 \subset \mathbb{R}^n$. *Then*

$$\frac{1}{n + 2N(1)} \int_{\partial B_1} u^2 \leq \int_{B_1} u^2 \leq \frac{1}{n} \int_{\partial B_1} u^2.$$

Proof. First, for any $r \in (0,1)$, we have

$$H(1) = \frac{H(r)}{r^{n-1}} \ \exp\{2\int_r^1 \frac{N(r)}{r}dr\} \geq \frac{H(r)}{r^{n-1}},$$

or

$$H(r) \leq r^{n-1}H(1).$$

This implies

$$\int_{B_1} u^2 = \int_0^1 H(r)dr \leq H(1)\int_0^1 r^{n-1}dr = \frac{1}{n}H(1).$$

On the other hand, for any $r \in (0,1)$, we have

$$\begin{aligned} H(1) &= \frac{H(r)}{r^{n-1}} \ \exp\{2\int_r^1 \frac{N(s)}{s}ds\} \leq \frac{H(r)}{r^{n-1}} \ \exp\{2N(1)\int_r^1 \frac{1}{s}ds\} \\ &= \frac{H(r)}{r^{n-1}} \exp\{-2N(1)\log r\} = \frac{1}{r^{n-1+2N(1)}}H(r), \end{aligned}$$

or

$$H(r) \geq r^{2N(1)+n-1}H(1).$$

Hence,

$$\int_{B_1} u^2 = \int_0^1 H(r)dr \geq H(1)\int_0^1 r^{2N(1)+n-1}dr = \frac{1}{n+2N(1)}H(1).$$

This finishes the proof. □

We should remark that the estimate in Corollary 2.2.7 is optimal. For example, consider in $\mathbb{R}^2$ the homogeneous harmonic polynomial u of degree k given by

$$u(x) = r^k \cos k\theta,$$

where (r,θ) is the polar coordinate in $\mathbb{R}^2$. Then it is easy to see that

$$\begin{aligned} \int_{\partial B_1} u^2 &= \int_0^{2\pi} \cos^2 k\theta d\theta, \\ \int_{B_1} u^2 &= \int_0^1 r^{2k+1}dr \ \int_0^{2\pi} \cos^2 k\theta d\theta = \frac{1}{2k+2}\int_0^{2\pi} \cos^2 k\theta d\theta. \end{aligned}$$

Now, we compare the Dirichlet energy with the L^2-norm of u. First, we have

$$\frac{\int_{B_1} |\nabla u|^2}{\int_{B_1} u^2} = N(1) \ \frac{\int_{\partial B_1} u^2}{\int_{B_1} u^2}.$$

Then by Corollary 2.2.7, we have

$$nN(1) \leq \frac{\int_{B_1} |\nabla u|^2}{\int_{B_1} u^2} \leq N(1)\big(n+2N(1)\big).$$

Next, we provide another proof of Theorem 2.2.3 and Corollary 2.2.5. We assume u is given by

$$u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} a_k r^k \varphi_k,$$

where u_k is a homogeneous harmonic polynomial of degree k in $\mathbb{R}^n$ and φ_k is its restriction to $\mathbb{S}^{n-1}$. We may assume $\{\varphi_k\}$ is orthonormal in $L^2(\mathbb{S}^{n-1})$, i.e,

$$\int_{\mathbb{S}^{n-1}} \varphi_k \varphi_l = \delta_{kl}.$$

Since each u_k is harmonic, we have by Green's Theorem

$$0 = \int_{B_r} u_k \triangle u_k = \int_{\partial B_r} u_k \partial_n u_k - \int_{B_r} |\nabla u_k|^2.$$

Note $u_k \partial_n u_k = k r^{2k-1} \varphi_k^2$. This implies

$$\int_{B_r} |\nabla u_k|^2 = k \int_{\partial B_r} r^{2k-1} \varphi_k^2 = k r^{2k-1} r^{n-1}.$$

Therefore, we obtain

$$D(r) = r^{n-1} \sum_{k=0}^{\infty} k a_k^2 r^{2k-1},$$
$$H(r) = r^{n-1} \sum_{k=0}^{\infty} a_k^2 r^{2k},$$

and

$$N(r) = \frac{rD(r)}{H(r)} = \frac{\sum_{k=0}^{\infty} k a_k^2 r^{2k}}{\sum_{k=0}^{\infty} a_k^2 r^{2k}}.$$

Then, we have by the Cauchy inequality

$$N'(r) = 2 \frac{\left(\sum_{k=0}^{\infty} k^2 a_k^2 r^{2k}\right)\left(\sum_{k=0}^{\infty} a_k^2 r^{2k}\right) - \left(\sum_{k=0}^{\infty} k a_k^2 r^{2k}\right)^2}{r\left(\sum_{k=0}^{\infty} a_k^2 r^{2k}\right)^2} \geq 0.$$

Hence, we conclude $N(r)$ is increasing.

Next, we note

$$\frac{H(r)}{r^{n-1}} = \sum_{k=0}^{\infty} a_k^2 r^{2k}.$$

Then we have

$$\frac{d}{dr}\left(\log \frac{H(r)}{r^{n-1}}\right) = \frac{\sum_{k=0}^{\infty} 2k a_k^2 r^{2k-1}}{\sum_{k=0}^{\infty} a_k^2 r^{2k}} = \frac{2D(r)}{H(r)},$$

or

$$\frac{d}{dr}\left(\log \frac{H(r)}{r^{n-1}}\right) = 2\frac{N(r)}{r}.$$

This is (2.2.12).

For any $p \in B_1$ and any $r \in (0, 1-|p|)$, we define

$$N(p;r) = \frac{r \int_{B_r(p)} |\nabla u|^2}{\int_{\partial B_r(p)} u^2}. \tag{2.2.17}$$

The quantity $N(p;r)$ is called the *frequency* of u in $B_r(p)$.

Theorem 2.2.8. *Let u be a harmonic function in $B_1 \subset \mathbb{R}^n$. Then for any $R \in (0,1)$, there exists a constant $N_0 = N_0(R) \ll 1$ such that the following holds. If $N(0,1) \le N_0$, then u does not vanish in B_R. If $N(0,1) \ge N_0$, then*

$$N\big(p; \frac{1}{2}(1-R)\big) \le CN\big(0;1\big) \quad \text{for any } p \in B_R,$$

where C is a positive constant depending only on n and R. In particular, the vanishing order of u at any point in B_R never exceeds $CN(0;1)$.

Proof. The monotonicity of $N(0;r)$ implies that the vanishing order of u at 0 never exceeds $N(0;1)$. In the following, we prove for $R = 1/4$.

Note that $B_{3/4}(p) \subset B_1$ and $B_{1/4} \subset B_{1/2}(p)$ for any $p \in B_{1/4}$. By Corollary 2.2.6, we have for any $p \in B_{1/4}$

$$\fint_{B_{\frac{3}{4}}(p)} u^2 \le c(n) 4^{2N(0,1)} \fint_{B_{\frac{1}{2}}(p)} u^2.$$

Now, we claim

$$\fint_{\partial B_{\frac{5}{8}}(p)} u^2 \le c(n) 4^{2N(0,1)} \fint_{\partial B_{\frac{1}{2}}(p)} u^2. \tag{2.2.18}$$

In fact, we have by (2.2.12)

$$\frac{d}{dr} \log \fint_{\partial B_r(p)} u^2 = \frac{2N(p,r)}{r}. \tag{2.2.19}$$

Hence, the function

$$r \mapsto \fint_{\partial B_r(p)} u^2$$

is increasing with respect to r. Then, we have

$$\int_{B_{\frac{3}{4}}(p)} u^2 \ge \int_{B_{\frac{3}{4}}(p) \setminus B_{\frac{5}{8}}(p)} u^2 = \int_{\frac{5}{8}}^{\frac{3}{4}} r^{n-1} \fint_{\partial B_r(p)} u^2 dS\, dr \ge c(n) \fint_{\partial B_{\frac{5}{8}}(p)} u^2,$$

and

$$\int_{B_{\frac{1}{2}}(p)} u^2 = \int_0^{\frac{1}{2}} r^{n-1} \fint_{\partial B_r(p)} u^2 dS\, dr \le c(n) \fint_{\partial B_{\frac{1}{2}}(p)} u^2.$$

This yields (2.2.18). Integrating (2.2.19), we obtain

$$\log \fint_{\partial B_{\frac{5}{8}}(p)} u^2 - \log \fint_{\partial B_{\frac{1}{2}}(p)} u^2 = \int_{\frac{1}{2}}^{\frac{5}{8}} \frac{2N(p;r)}{r} dr \ge 2N\big(p; \frac{1}{2}\big) \left(\log \frac{5}{8} - \log \frac{1}{2} \right).$$

This implies with (2.2.18)

$$c(n) N\big(p; \frac{1}{2}\big) \le \log \left(c(n) 4^{2N(0;1)} \right),$$

or

$$N\big(p; \frac{1}{2}\big) \le c(n) N(0;1) + c(n).$$

By the monotonicity of $N(p;r)$, we obtain for any $p \in B_{1/4}$ and any $r \le 1/2$

$$N(p;r) \le c(n) N(0;1) + c(n).$$

Therefore, the vanishing order of u at p never exceeds $c(n)N(0;1) + c(n)$.

To finish the proof, we claim $u(p) \neq 0$ for any $p \in B_{1/4}$ if $N(0;1) \leq \varepsilon(n) \ll 1$. To prove this, we assume

$$\int_{\partial B_1} u^2 = 1,$$

which implies

$$\int_{B_1} |\nabla u|^2 \leq \varepsilon(n).$$

Interior estimates yield

$$\sup_{B_{\frac{1}{2}}} |\nabla u| \leq c_1(n)\sqrt{\varepsilon(n)}.$$

By (2.2.15), we have

$$1 = \int_{\partial B_1} u^2 \leq 2^{n-1} 2^{2N(0;1)} \int_{\partial B_{\frac{1}{2}}} u^2 \leq c(n) \int_{\partial B_{\frac{1}{2}}} u^2.$$

Hence there exists a $p_0 \in \partial B_{1/2}$ such that $|u(p_0)| \geq c_2(n)$. Then, we have for any $p \in B_{1/2}$

$$|u(p)| \geq c_2(n) - c_1(n)\sqrt{\varepsilon(n)} > 0,$$

if $\varepsilon(n)$ is small. $\square$

Most of the results in this section can be generalized to analytic functions in $\mathbb{C}$. Suppose $f = f(z)$ is an analytic function in the unit ball in $\mathbb{C}$ given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then we have

$$f'(z) = \sum_{k=0}^{\infty} k a_k z^k.$$

Similarly, we may define $H(r), D(r)$ and $N(r)$ by

$$\begin{aligned} H(r) &= \int_{|z|=r} |f|^2 d\sigma = 2\pi \sum_{k=0}^{\infty} |a_k|^2 r^{2k+1}, \\ D(r) &= \int_{|z|\leq r} |f'(z)| dz = \sum_{k=0}^{\infty} k^2 |a_k|^2 \int_{|z|\leq r} |z|^{2k-2} dz = \pi \sum_{k=0}^{\infty} k |a_k|^2 r^{2k}, \end{aligned}$$

and

$$N(r) = \frac{rD(r)}{H(r)} = \frac{\sum_{k=0}^{\infty} k|a_k|^2 r^{2k}}{2\sum_{k=0}^{\infty} |a_k|^2 r^{2k}}.$$

Then, we conclude as before that $N'(r) \geq 0$.

2.3. Measure Estimates of Nodal Sets of Harmonic Functions

In this section, we estimate the measure of nodal sets of harmonic functions in terms of the frequency. The main result, Theorem 2.3.1, was proved by Lin [**69**].

We first examine an example. Consider in $\mathbb{R}^2$ the homogeneous harmonic polynomial u_d given by

$$u_d(x,y) = \mathrm{Re}(z^d) = r^d \cos(d\theta).$$

The nodal set of u_d consists of d straight lines intersecting at the origin. Hence, we have

$$\mathcal{H}^1(u_d^{-1}(0) \cap B_1) = 2d.$$

Note that the measure of the nodal set depends on the degree linearly.

In general, we have the following result.

Theorem 2.3.1. *Suppose u is a harmonic function in $B_1 \subset \mathbb{R}^n$. Then*

$$\mathcal{H}^{n-1}\{x \in B_{\frac{1}{2}}; u(x) = 0\} \leq cN, \tag{2.3.1}$$

where c is a positive constant depending only on n, and N is the frequency of u in B_1 defined by

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}.$$

We first prove a result concerning zeros of complex analytic functions following Donnelly and Fefferman [**27**].

Lemma 2.3.2. *Suppose $f : B_1 \subset \mathbb{C} \to \mathbb{C}$ is analytic with*

$$|f(0)| = 1 \quad \text{and} \quad \sup_{B_1} |f| \leq 2^N,$$

for some positive constant N. Then for any $r \in (0,1)$

$$\#\{z \in B_r; f(z) = 0\} \leq cN,$$

where c is a positive constant depending only on r.

The proof is based on Blaschke factors.

Proof. Without loss of generality, we assume that f has at most countably many zeroes in B_1 which are denoted by $\{z_k\}$ and f is continuous up to ∂B_1 with $f(z) \neq 0$ for $|z| = 1$. Then we write

$$f(z) = e^{g(z)} \prod \frac{z - z_k}{1 - \bar{z}_k z}.$$

With

$$\max_{|z|=1} |f(z)| \leq |f(0)| 2^N,$$

we obtain

$$\max_{|z|=1} e^{\mathrm{Re} g} \leq e^{\mathrm{Re} g(0)} 2^N \prod |z_k|.$$

Note that $\mathrm{Re} g$ is a harmonic function in B_1. Hence by the maximum principle, we obtain

$$1 \leq 2^N \prod |z_k|.$$

Let M be the number of zeroes of f in B_r. We then have $1 \leq 2^N \cdot r^M$ and hence $M \leq N \log 2/(-\log r)$. □

To prove Theorem 2.3.1, we also need the complexification of harmonic functions. Suppose u is a harmonic function in $B_1 \subset \mathbb{R}^n$. Then there is an $R \in (0,1)$, depending only on n, such that $u(x)$ extends to a holomorphic function $\tilde{u}(z)$ on

$$\Omega = \{z = x + iy \in \mathbb{C}^n; x \in B_{\frac{3}{4}}, y \in B_R\}.$$

Moreover,

$$\sup_{\Omega} |\tilde{u}| \leq c\|u\|_{L^2(\partial B_1)}, \tag{2.3.2}$$

where c is a positive constant depending only on n. To see this, we simply consider the Taylor expansion of $u = u(x)$ at any point $p \in B_{3/4}$ and replace $x - p \in \mathbb{R}^n$ by $z - p \in \mathbb{C}^n$. With the estimate of the derivatives of harmonic functions, the new complex series converges for $|z - p| < R$, for some positive constant R depending only on n. In the following, R will be fixed such that the above extension property and (2.3.2) hold. Hence, the constant c is also fixed, independent of u.

Now we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. Without loss of generality, we assume

$$\fint_{\partial B_1} u^2 = 1.$$

By Corollary 2.2.5, Corollary 2.2.6 and Theorem 2.2.8, we have

$$\fint_{B_{\frac{1}{16}}(p)} u^2 \geq 4^{-cN} \quad \text{for any } p \in B_{\frac{1}{4}},$$

where c is a positive constant depending only on n. In particular, there is a point $x_p \in B_{1/16}(p)$ such that $|u(x_p)| \geq 2^{-cN}$.

Now we choose $p_1, \cdots, p_n \in \partial B_{1/4}$, with p_j on the x_j-axis, $j = 1, \cdots, n$. Let $x_{p_j} \in B_{1/16}(p_j)$ be such that

$$|u(x_{p_j})| \geq 2^{-cN} \quad \text{for any } j = 1, \cdots, n.$$

For each j and $w \in \mathbb{S}^{n-1}$, we consider

$$f_j(w;t) = u(x_{p_j} + tw) \quad \text{for } t \in (-\frac{5}{8}, \frac{5}{8}).$$

It is obvious $f_j(w;t)$ is an analytic function of $t \in (-5/8, 5/8)$. Moreover, $f_j(w;t)$ extends to an analytic function $f_j(w;z)$ for $z = t + iy$, $|t| < 5/8$ and $|y| < y_0$. Then we have

$$|f_j(w;0)| \geq 2^{-cN},$$

and

$$|f_j(w;t+iy)| \leq C,$$

for some positive constant C depending only on n. Applying Lemma 2.3.2, we obtain

$$\#\{t; u(x_{p_j} + tw) = 0, \ |t| < \frac{1}{2}\} \leq C(n)N,$$

and in particular

$$N_j(w) \equiv \#\{t; u(x_{p_j} + tw) = 0, \ x_{p_j} + tw \in B_{\frac{1}{16}}\} \leq C(n)N.$$

By Theorem 1.2.10, the integral geometric formula, we have

$$\mathcal{H}^{n-1}\{x \in B_{\frac{1}{16}}; u(x) = 0\} \le c(n) \sum_{j=1}^{n} \int_{\mathbb{S}^{n-1}} N_j(w)dw \le C(n)N.$$

Now Theorem 2.3.1 follows simply by a suitable finite covering of $B_{1/2}$ by balls of radius 1/16. ☐

In the rest of this section, we prove the following result similar to Lemma 2.3.2.

LEMMA 2.3.3. *Let $f(z)$ be a nonzero analytic function in $B_1 = \{z \in \mathbb{C}; |z| < 1\}$. Then*

$$\#\{f^{-1}\{0\} \cap B_r\} \le 2N,$$

where $r \in (0, 1)$ is universal and N is defined by

$$N = \frac{\int_{B_1} |f'|^2}{\int_{\partial B_1} |f|^2}.$$

Lemma 2.3.3 was first proved by Lin using Taylor expansion [**69**]. Here, we prove it by Rouché Theorem. The proof here follows Han [**46**]. We need a result of H. Cartan, which provides an estimate from below for the modulus of a polynomial in $\mathbb{C}$ away from its zeroes. See Theorem 10, P19 in Levin [**67**].

LEMMA 2.3.4. *For any given number $\varepsilon > 0$ and complex numbers $a_1, \cdots, a_d$, there is a collection of at most d circles in $\mathbb{C}$, with the sum of the radii equal to 2ε, such that for each z lying outside these circles there holds*

$$|z - a_1| \cdot |z - a_2| \cdots |z - a_d| > \left(\frac{\varepsilon}{e}\right)^d.$$

PROOF. We divide the proof into several steps.

Step 1. We choose the quantity ε/d as the unit of measurement and show that there are closed circles in the complex plane, having radius equal to the number of points $\{a_k\}$ contained within the circle. Indeed, form the smallest convex polygon containing all the points $\{a_k\}$, and choose any vertex a_j of this polygon. Clearly there are circles of arbitrary radius that contain this point but do not contain any other of the points $\{a_k\}$. In particular, the radius can be chosen to equal the multiplicity of the point a_j.

Step 2. From among all circles with radius equal to the number of points $\{a_k\}$ lying inside the circle choose one with the largest radius, $m_1\varepsilon/d$, and call it C_1. Note that no circle in the plane with radius greater than or equal to $m_1\varepsilon/d$ can contain more points of the set $\{a_k\}$ than the number of units of measurement in its radius. Indeed, suppose a circle of radius $m\varepsilon/d$, with $m \ge m_1$, contains $m' > m$ points of the set $\{a_k\}$. The concentric circle of radius $m'\varepsilon/d$ either contains m' points or $m'' > m'$ points. In the second case, we consider the concentric circle of radius $m''\varepsilon/d$ and so forth. Since the set $\{a_k\}$ is finite, we eventually come to a circle of radius $m\varepsilon/d$, larger than $m_1\varepsilon/d$, that contains m points. This is impossible, since C_1 is the largest circle having this property. The points of the set $\{a_k\}$ that lie inside C_1 will be said to be of rank m_1.

Step 3. We remove the points of rank m_1, and for the remaining $d - m_1$ points we construct the largest circle C_2 that contains the same number of points as there are units in its radius. Let its radius be $m_2\varepsilon/d$. We show that $m_2 \le m_1$. Indeed, if this were not the case then the circle C_2 would have a radius larger than $m_1\varepsilon/d$, and would contain at least as many of the original d points as there are units in its radius. But this contradicts the result of Step 2. The points in C_2 will be said to be of rank m_2. Now remove these points too, and for the remaining $d - m_1 - m_2$ points we find the largest circle C_3 containing the same number of points as there are units in its radius. Suppose that its radius is $m_3\varepsilon/d$. Clearly $m_3 \leq m_2$. The points in C_3 will be said to be of rank m_3, and so forth.

We thus obtain a sequence of circles $C_1, \cdots, C_l$ with radii that contain $m_1, \cdots, m_l$ units of measurement, respectively, where $m_1 \geq \cdots \geq m_l$ and

$$\frac{\varepsilon}{d}(m_1 + \cdots + m_l) = \varepsilon.$$

Step 4. We now form circles $\Gamma_1, \cdots, \Gamma_l$ concentric with the circles $C_1, \cdots, C_l$ but with radii twice as large. Let z be an arbitrary point lying outside all of the new circles. Describe about z the circle C_z of radius $m\varepsilon/d$, where m is some natural number. This circle does not intersect any of the circles C_j that have a radius larger than or equal to m. Thus this circle can only contain points whose rank is less than m. From the definition of rank, it follows that after removing all points of rank greater than or equal to m, no circle of radius greater than or equal to $m\varepsilon/d$ can contain as many of the remaining points as there are units of measurement in its radius. It follows from Step 2 that the circle C_z can contain at most $m - 1$ points.

Step 5. Enumerating the points $\{a_k\}$ in the order of increasing distance from z, we have

$$|z - a_k| > \frac{k\varepsilon}{d},$$

and

$$|z - a_1| \cdots |z - a_k| > d!\big(\frac{\varepsilon}{d}\big)^d > \big(\frac{\varepsilon}{e}\big)^d.$$

This ends the proof. □

Next, we state a simple corollary.

COROLLARY 2.3.5. *Suppose $p(z) = \sum_{k=0}^d c_k z^k$ is a polynomial in $\mathbb{C}$ with*

$$\sum_{k=0}^d |c_k|^2 \geq 1.$$

Then for any $\varepsilon \in (0, 1)$, there is a collection of at most d circles in $\mathbb{C}$, with the sum of the radii $\leq 2\varepsilon$, such that for any z with $|z| \leq 1$ lying outside these circles

$$|p(z)| > \left(\frac{\varepsilon}{10}\right)^d.$$

PROOF. We note

$$\frac{1}{2\pi}\int_{\partial D_1} |p|^2 = \sum_{k=0}^d |c_k|^2 \geq 1.$$

Since p has d zeroes in $\mathbb{C}$, we may assume

$$p(z) = c(z - a_1) \cdots (z - a_d).$$

Then for some $z_0 \in \partial D_1$, we have

$$1 \leq |p(z_0)| = |c| \cdot |z_0 - a_1| \cdots |z_0 - a_d| \leq |c|(1 + |a_1|) \cdots (1 + |a_d|),$$

which implies

$$|p(z)| \geq \frac{|z-a_1|}{1+|a_1|} \cdots \frac{|z-a_d|}{1+|a_d|}.$$

Note that we only consider z with $|z| \leq 1$. We assume, for some integer $d' \leq d$, $|a_k| \leq 2$ for $k \leq d'$ and $|a_k| > 2$ for $d' < k \leq d$. Then for $d' < k \leq d$, we have

$$\frac{|z-a_k|}{1+|a_k|} \geq \frac{|a_k|-1}{|a_k|+1} > \frac{1}{3} \quad \text{for any } z \text{ with } |z| \leq 1.$$

Obviously, $1+|a_k| \leq 3$ for $k \leq d'$. Hence we obtain

$$|p(z)| \geq \left(\frac{1}{3}\right)^{d-d'} |z-a_1| \cdots |z-a_{d'}|.$$

A simple application of Lemma 2.3.4 yields the desired result. □

Now we are ready to prove Lemma 2.3.3.

Proof of Lemma 2.3.3. Set $g(z) = f(z/M)$ for some $M > 0$ to be determined. Then we have

$$N = \frac{M \int_{B_M} |g'|^2}{\int_{\partial B_M} |g|^2}.$$

We claim

$$\#\{z \in B_{\frac{1}{2}}; g(z) = 0\} \leq 2N.$$

We set

$$g(z) = \sum_{m=0}^{\infty} a_m z^m.$$

We assume, without loss of generality, that

$$\frac{1}{2\pi} \int_{\partial B_1} |g|^2 = \sum_{m=0}^{\infty} |a_m|^2 = 1. \tag{2.3.3}$$

In the following, we set $N_* = [N]$, the integral part of N. Obviously, we have

$$N_* \leq N \leq N_* + 1.$$

By (2.2.14) for analytic functions, we get

$$\frac{1}{2\pi M} \int_{\partial B_M} |g|^2 \leq \frac{M^{2N}}{2\pi} \int_{\partial B_1} |g|^2 = M^{2N},$$

which implies

$$\sum_{m=0}^{\infty} |a_m|^2 M^{2m} \leq M^{2N}.$$

By $N \leq N_* + 1$, we have obviously

$$\sum_{m=0}^{\infty} |a_m|^2 M^{2m} \leq M^{2(N_*+1)}.$$

Therefore, we obtain

$$|a_m| \leq M^{N_*-m+1} \quad \text{for any } m \geq 0. \tag{2.3.4}$$

Note that

$$\frac{1}{2\pi}\int_{\partial B_1}|\sum_{m\geq 2N_*+1}a_mz^m|^2=\sum_{m\geq 2N_*+1}|a_m|^2\leq\frac{c}{M^{2N_*}},$$

for some universal constant $c>0$. We then get by interior estimates

$$\sup_{B_{\frac{3}{4}}}|\sum_{m\geq 2N_*+1}a_mz^m|\leq\frac{c}{M^{N_*}}. \tag{2.3.5}$$

We choose M large, independent of N_*, such that

$$\sum_{m=2N_*+1}^{\infty}|a_m|^2\leq\frac{1}{2}. \tag{2.3.6}$$

Set

$$P_*(z)=\sum_{m=0}^{2N_*}a_mz^m,\quad R_*(z)=\sum_{m=2N_*+1}^{\infty}a_mz^m. \tag{2.3.7}$$

Then $g=P_*+R_*$. Obviously, we have by (2.3.3) and (2.3.6)

$$\sum_{m=0}^{2N_*}|a_m|^2\geq\frac{1}{2}.$$

Then by Corollary 2.3.5 with $d=2N_*$ and possibly a different normalization constant, we conclude

$$\inf_{\partial B_r}|P_*|>\varepsilon^{2N_*},$$

for some $r\in(1/2,3/4)$ and a universal constant $\varepsilon\in(0,1)$. By choosing M large enough, independent of N_*, we conclude by (2.3.5)

$$\sup_{B_{3/4}}|R_*|<\varepsilon^{2N_*}.$$

This implies

$$|g(z)-P_*(z)|<|P_*(z)|\quad\text{for any }|z|=r.$$

Then the Rouché Theorem implies

$$\#\{z\in B_r;g(z)=0\}\leq 2N_*,$$

and in particular

$$\#\{z\in B_{\frac{1}{2}};g(z)=0\}\leq 2N_*.$$

This finishes the proof, since $N_*\leq N$. □

Lemma 2.3.2 can also be proved similarly.

2.4. Singular Sets of Planar Harmonic Functions

Let u be a harmonic function in $B_1\subset\mathbb{R}^2$. For any $p\in B_1$ with $u(p)=0$ and $Du(p)\neq 0$, it is easy to see that $u^{-1}(0)$ is an analytic curve in a neighborhood of p. Now we set

$$\mathcal{S}(u)=\{p\in B_1;u(p)=0,Du(p)=0\}.$$

LEMMA 2.4.1. *Let u be a nontrivial harmonic function in $B_1\subset\mathbb{R}^2$. Then $\mathcal{S}(u)$ is isolated.*

PROOF. *Method 1.* We identify $\mathbb{C} = \mathbb{R}^2$ and write $z = x_1 + ix_2$. Setting $f(z) = u_{x_1} - iu_{x_2}$, we note that f is (complex) analytic and $f^{-1}(0) = |Du|^{-1}(0)$, which is isolated.

Method 2. Now we provide a proof which does not rely on the complex structure. For any $d \geq 2$, we set

$$\mathcal{S}_d(u) = \{p \in B_1;\ \partial^\nu u(p) = 0 \text{ for any } |\nu| < d,\\ \partial^{\nu_0} u(p) \neq 0 \text{ for some } |\nu_0| = d\}.$$

Then we have

$$\mathcal{S}(u) = \bigcup_{d\geq 2} \mathcal{S}_d(u).$$

This is a finite union by the finite vanishing order due to the analyticity. We prove that $\mathcal{S}_d(u)$ is isolated for each $d \geq 2$.

For any $p \in \mathcal{S}_d(u)$, there exists a nontrivial homogeneous harmonic polynomial P of degree d such that

$$u(x+p) = P(x) + O(|x|^{d+1}).$$

By a simple rotation, we assume in polar coordinate (r, θ)

$$P(x) = ar^d \cos d\theta,$$

for some nonzero constant a. A simple calculation yields

$$|Du(x+p)| = (d-1)|a|r^{d-1} + O(r^d).$$

Hence $\mathcal{S}(u)$ has no other points besides p in a neighborhood of p. □

Lemma 2.4.1 illustrates that $\mathcal{S}(u)$ is indeed the singular part of $\{u = 0\}$. We usually call $\mathcal{S}(u)$ *the singular set* of u. Our primary interest is to estimate the number of points in singular sets. The entire section follows closely Han [**46**].

We first prove the following result.

THEOREM 2.4.2. *Let u be a nontrivial harmonic function in $B_1 \subset \mathbb{R}^2$. Then*

$$\#\left(\mathcal{S}(u) \cap B_{\frac{1}{2}}\right) \leq cN,$$

where $c > 0$ is a universal constant and N is the frequency of u in B_1 defined by

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}$$

.

To prove Theorem 2.4.2, we identify $\mathbb{C} = \mathbb{R}^2$ and prove the following result.

THEOREM 2.4.3. *There exists a universal constant $M > 1$ such that, for any nontrivial harmonic function u in $B_M \subset \mathbb{R}^2$, with $u(0) = 0$, satisfying*

$$\frac{M \int_{B_M} |\nabla u|^2}{\int_{\partial B_M} u^2} \leq N,$$

there holds

$$\#\{x \in B_{\frac{1}{2}}; u_{x_1}(x) = u_{x_2}(x) = 0\} \leq 2N.$$

PROOF. We denote by (r, θ) the polar coordinates in $\mathbb{R}^2$ and write u in the form

$$u(r,\theta) = \sum_{m=1}^{\infty} a_m r^m \cos(m\theta + \theta_m),$$

where $\theta_m \in [0, 2\pi)$. We assume, without loss of generality, that

$$\frac{1}{\pi}\int_{\partial B_1} u^2 = \sum_{m=1}^{\infty} a_m^2 = 1. \tag{2.4.1}$$

In the following, we set

$$N_* = \inf\{n \in \mathbb{Z}_+;\ n \geq N\}.$$

In other words, $N_* = N$ if N is an integer and $N_* = [N] + 1$ otherwise. Here $[N]$ is the integral part of N. Obviously, we have

$$N_* - 1 \leq N \leq N_*.$$

By (2.2.14), we get

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} = \frac{1}{\pi M}\int_{\partial B_M} u^2 \leq \frac{M^{2N(0;M)}}{\pi}\int_{\partial B_1} u^2 = M^{2N(0;M)}.$$

By $N(0; M) \leq N \leq N_*$, we have

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \leq M^{2N_*},$$

and hence

$$|a_m| \leq M^{N_* - m} \quad \text{for any } m \geq 1. \tag{2.4.2}$$

We first choose M large, independent of N_*, such that

$$\sum_{m=2N_*}^{\infty} |a_m|^2 \leq \frac{1}{2}. \tag{2.4.3}$$

Now we identify $\mathbb{C} = \mathbb{R}^2$ and write $z = x_1 + ix_2$. Setting $f(z) = u_{x_1} - iu_{x_2}$, we note that f is (complex) analytic and $f^{-1}(0) = |Du|^{-1}(0)$. A straightforward calculation yields

$$f(z) = \sum_{m=1}^{\infty} m a_m e^{i\theta_m} z^{m-1}.$$

Set

$$P(z) = \sum_{m=1}^{2N_*-1} m a_m e^{i\theta_m} z^{m-1}, \quad R(z) = \sum_{m=2N_*}^{\infty} m a_m e^{i\theta_m} z^{m-1}. \tag{2.4.4}$$

Then we get $f = P + R$. Note that P is a polynomial in $\mathbb{C}$ of degree $2N_* - 2$ and its coefficients satisfy

$$\sum_{m=1}^{2N_*-1} |m a_m|^2 \geq \frac{1}{2}.$$

By Lemma 2.3.5, there exists an $r \in (1/2, 1)$ and a universal $\varepsilon > 0$ such that

$$|P(z)| > \varepsilon^{2N_*-2} \quad \text{for any } |z| = r.$$

Moreover, by choosing a universal M large enough, independent of N_*, we have by (2.4.2)

$$|R(z)| \le \sum_{m\ge 2N_*} |ma_m| \le \sum_{m\ge 2N_*} \frac{m}{M^{m-N_*}} \le \frac{c}{M^{N_*}} < \varepsilon^{2N_*-2},$$

for any $|z|<1$. This implies

$$|f(z)-P(z)| < |P(z)| \quad \text{for any } |z|=r.$$

By the Rouché Theorem, we have

$$\#\{f^{-1}(0)\cap B_r\} \le 2N_*-2,$$

or

$$\#\{f^{-1}(0)\cap B_{\frac{1}{2}}\} \le 2N_*-2.$$

This finishes the proof, since $N_*-1\le N$. $\square$

Theorem 2.4.2 follows from Theorem 2.4.3 and Theorem 2.2.8 easily. The proof of Theorem 2.4.2 makes an essential use of the identification $\mathbb{R}^2=\mathbb{C}$. In the following, we study the singular set from another point of view. Instead of identifying $\mathbb{R}^2$ as $\mathbb{C}$, we put $\mathbb{R}^2$ into $\mathbb{C}^2$ and then consider the complexification of harmonic functions.

Suppose u is a harmonic function in $B_1\subset\mathbb{R}^2$. As discussed in Section 2.3, u extends to a holomorphic function $\tilde{u}(z)$ in $D_R\subset\mathbb{C}^2$ for some universal $R\in(0,1)$. Moreover, for some universal constant $c>0$

$$\sup_{D_R}|\tilde{u}| \le c\|u\|_{L^2(\partial B_1)}. \tag{2.4.5}$$

In the following, we always denote by $\tilde{u}$ the complexification of u. We also denote by $B_r(x)$ and $D_r(z)$ open balls of radius r centered at x and z in $\mathbb{R}^2$ and $\mathbb{C}^2$, respectively. When the center is the origin, we simply write B_r and D_r. The (complex) singular set of $\tilde{u}$ is defined by

$$\mathcal{S}(\tilde{u}) = \{z\in D_R; \tilde{u}(z)=\tilde{u}_{z_1}(z)=\tilde{u}_{z_2}(z)=0\}.$$

THEOREM 2.4.4. *Let u be a (real) nontrivial harmonic function in $B_1\subset\mathbb{R}^2$. Then for some universal constants $R_0\in(0,1)$ and $c>0$,*

$$\#\left(\mathcal{S}(\tilde{u})\cap D_{R_0}\right) \le cN^2,$$

where N is the frequency of u in B_1 defined by

$$N=\frac{\int_{B_1}|\nabla u|^2}{\int_{\partial B_1}u^2}.$$

A significant aspect of Theorem 2.4.4 is that a property of the complexified $\tilde{u}$ is determined by its restriction on the real space $u=\tilde{u}|_{\mathbb{R}^2}$. Here we make an important remark about the complexification $\tilde{u}$. Since u is a harmonic function, the holomorphic function $\tilde{u}$ satisfies $\partial_{z_1z_1}\tilde{u}+\partial_{z_2z_2}\tilde{u}=0$. Theorem 2.4.4 asserts that the singular set of $\tilde{u}$ is isolated and that the number of singular points can be estimated in terms of the frequency of the (real) function u. This result does not hold for general holomorphic functions v satisfying

$$\partial_{z_1z_1}v+\partial_{z_2z_2}v=0. \tag{2.4.6}$$

The following example was constructed by M. Hoffman-Ostenhof, T. Hoffman-Ostenhof and Nadirashvili [**53**].

Example 2.4.5. Let $v(z) = (z_1 - iz_2)^2$. Obviously v satisfies (2.4.6). The singular set of v is given by $\{(z_1, z_2); z_1 = iz_2\}$, which is not isolated.

Hence in order to have an isolated singular set for a holomorphic function $v = v(z_1, z_2)$ satisfying (2.4.6), all the coefficients in the Taylor expansion of v have to be real.

To prove Theorem 2.4.4, we first provide a simple but crucial calculation for harmonic polynomials in $\mathbb{R}^2$ and their complexifications in $\mathbb{C}^2$. In polar coordinates (r, θ) in $\mathbb{R}^2$, the homogeneous polynomial $P_d(x) = r^d \cos d\theta$ is harmonic. If we consider a linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with } a^2 + b^2 = 1,\ a, b \in \mathbb{R}, \tag{2.4.7}$$

then $P_d(T\cdot)$ is also harmonic. In fact, any homogeneous harmonic polynomial of degree d can be written in this way. Note that T in (2.4.7) is simply a rotation in $\mathbb{R}^2$.

Now we consider the gradient of homogeneous harmonic polynomials. We identify $\mathbb{R}^2 = \mathbb{C}$ and use the complex coordinate $z = x_1 + ix_2$. Consider the homogeneous polynomial

$$\bar{z}^d = (x_1 - ix_2)^d = r^d \cos d\theta - ir^d \sin d\theta.$$

We use its real part and complex part to construct a homogeneous polynomial map $Q_d : \mathbb{R}^2 \to \mathbb{R}^2$ as follows

$$Q_d(x) = Q_d(x_1, x_2) = \begin{pmatrix} r^d \cos d\theta \\ -r^d \sin d\theta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z^d + \bar{z}^d \\ i(z^d - \bar{z}^d) \end{pmatrix},$$

or

$$Q_d(x) = \frac{1}{2} \begin{pmatrix} (x_1 + ix_2)^d + (x_1 - ix_2)^d \\ i((x_1 + ix_2)^d - (x_1 - ix_2)^d) \end{pmatrix}. \tag{2.4.8}$$

Each component is a homogeneous harmonic polynomial. In fact, Q_d is the gradient of some homogeneous harmonic polynomial of degree $d + 1$.

As before, we consider a linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ given in (2.4.7). If Q_d is a homogeneous polynomial map given in (2.4.8), then $T^t Q_d(T\cdot)$ is also a homogeneous polynomial map given by the gradient of some homogeneous harmonic polynomial of degree $d + 1$. In fact, the converse is also true. A homogeneous polynomial map of degree d can be expressed as $T^t Q_d(T\cdot)$ for some linear transform T in (2.4.7) if it is the gradient of some homogeneous harmonic polynomial of degree $d + 1$.

Now we extend the map $Q_d : \mathbb{R}^2 \to \mathbb{R}^2$ to $Q_d : \mathbb{C}^2 \to \mathbb{C}^2$ simply by replacing $x = (x_1, x_2)$ by $z = (z_1, z_2)$,

$$Q_d(z) = Q_d(z_1, z_2) = \frac{1}{2} \begin{pmatrix} (z_1 + iz_2)^d + (z_1 - iz_2)^d \\ i((z_1 + iz_2)^d - (z_1 - iz_2)^d) \end{pmatrix}. \tag{2.4.9}$$

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transform given in (2.4.7). Set

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 - bz_2 \\ bz_1 + az_2 \end{pmatrix}.$$

Then we have

$$z_1' + iz_2' = (a+ib)(z_1+iz_2), \quad z_1' - iz_2' = (a-ib)(z_1-iz_2).$$

Setting $\alpha = a+ib$, we obtain

$$Q_d(Tz) = \frac{1}{2}\begin{pmatrix} \alpha^d(z_1+iz_2)^d + \bar{\alpha}^d(z_1-iz_2)^d \\ i\big(\alpha^d(z_1+iz_2)^d - \bar{\alpha}^d(z_1-iz_2)^d\big)\end{pmatrix},$$

and

$$T^t Q_d(Tz) = \frac{1}{2}\begin{pmatrix} \alpha^{d+1}(z_1+iz_2)^d + \bar{\alpha}^{d+1}(z_1-iz_2)^d \\ i\big(\alpha^{d+1}(z_1+iz_2)^d - \bar{\alpha}^{d+1}(z_1-iz_2)^d\big)\end{pmatrix}. \tag{2.4.10}$$

We then conclude easily

$$|T^t Q_d(Tz)|^2 = \frac{1}{2}\big(|z_1+iz_2|^{2d} + |z_1-iz_2|^{2d}\big).$$

A direct calculation shows

$$|z_1 \pm iz_2|^2 = |z_1|^2 + |z_2|^2 \pm 2(y_1x_2 - x_1y_2).$$

Then we obtain

$$\begin{aligned}|T^t Q_d(Tz)|^2 =& \frac{1}{2}\Big(\big(|z_1|^2+|z_2|^2+2(y_1x_2-x_1y_2)\big)^d \\ &+\big(|z_1|^2+|z_2|^2-2(y_1x_2-x_1y_2)\big)^d\Big).\end{aligned}$$

Note that only the even powers of $y_1x_2 - x_1y_2$ appear in the right side. Hence we get

$$|T^t Q_d(Tz)| \geq |z|^d. \tag{2.4.11}$$

Next, we generalize (2.4.11) to nonhomogeneous harmonic polynomial maps.

LEMMA 2.4.6. *Suppose* $Q : \mathbb{C}^2 \to \mathbb{C}^2$ *is given by*

$$Q(z) = \sum_{k=0}^{d} c_k T_k^t Q_k(T_k z), \tag{2.4.12}$$

where, for $k = 0, 1, \cdots, d$, $Q_k : \mathbb{C}^2 \to \mathbb{C}^2$ *is the homogeneous harmonic polynomial map given by (2.4.9),* $T_k : \mathbb{C}^2 \to \mathbb{C}^2$ *is a linear transform given by*

$$T_k = \begin{pmatrix} a_k & -b_k \\ b_k & a_k \end{pmatrix}, \quad \text{with } a_k^2 + b_k^2 = 1,\ a_k, b_k \in \mathbb{R},$$

and c_k *is a complex number such that* $\sum_{k=0}^d |c_k|^2 \geq 1$. *Then there exists an* $r \in (1/2, 1)$ *such that*

$$|Q(z)| > \varepsilon^d \quad \text{for any } z \in \partial D_r,$$

for some universal constant $\varepsilon \in (0,1)$.

PROOF. Set $\alpha_k = a_k + ib_k$, for $k = 0, 1, \cdots, d$. We claim

$$|Q(z)|^2 = \frac{1}{2}\left(|\sum_{k=0}^d c_k\alpha_k^{k+1}(z_1+iz_2)^k|^2 + |\sum_{k=0}^d c_k\bar{\alpha}_k^{k+1}(z_1-iz_2)^k|^2\right). \tag{2.4.13}$$

To prove this, we set for simplicity

$$w_1 = z_1 + iz_2, \quad w_2 = z_1 - iz_2.$$

Then we obtain by (2.4.10)

$$\begin{aligned} Q(z) &= \sum_{k=0}^{d} c_k T_k^t Q_k(T_k z) = \frac{1}{2}\sum_{k=0}^{d} c_k \begin{pmatrix} \alpha_k^{k+1} w_1^k + \bar{\alpha}_k^{k+1} w_2^k \\ i(\alpha_k^{k+1} w_1^k - \bar{\alpha}_k^{k+1} w_2^k) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sum_{k=0}^{d} c_k\alpha_k^{k+1} w_1^k + \sum_{k=0}^{d} c_k\bar{\alpha}_k^{k+1} w_2^k \\ i(\sum_{k=0}^{d} c_k\alpha_k^{k+1} w_1^k - \sum_{k=0}^{d} c_k\bar{\alpha}_k^{k+1} w_2^k) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I + II \\ I - II \end{pmatrix}. \end{aligned}$$

This implies

$$\begin{aligned} |Q(z)|^2 &= \frac{1}{4}\big(|I+II|^2 + |I-II|^2\big) = \frac{1}{2}\big(|I|^2 + |II|^2\big) \\ &= \frac{1}{2}\left(|\sum_{k=0}^{d} c_k \alpha_k^{k+1} w_1^k|^2 + |\sum_{k=0}^{d} c_k \bar{\alpha}_k^{k+1} w_2^k|^2 \right). \end{aligned}$$

This finishes the proof of (2.4.13).

We now apply Lemma 2.3.5 to polynomials

$$\sum_{k=0}^{d} c_k \alpha_k^{k+1} w^k \quad \text{and} \quad \sum_{k=0}^{d} c_k \bar{\alpha}_k^{k+1} w^k.$$

For any $\varepsilon \in (0,1)$, there is a collection of discs $\{D_{r_k}(p_k)\}$ and $\{D_{s_l}(q_l)\}$ in $\mathbb{C}$, with $\sum r_k \le 2\varepsilon$ and $\sum s_l \le 2\varepsilon$, such that

$$\begin{aligned} |Q(z)| > \left(\frac{\varepsilon}{10}\right)^d \quad &\text{for any } z=(z_1,z_2)\in D_1 \text{ with} \\ z_1 + iz_2 \notin \cup D_{r_k}(p_k) \quad &\text{or} \quad z_1 - iz_2 \notin \cup D_{s_l}(q_l). \end{aligned}$$

Now we consider the set

$$\mathcal{B}_{r,s}(p,q) = \{(z_1,z_2)\in\mathbb{C}^2; z_1 + iz_2 \in D_r(p), z_1 - iz_2 \in D_s(q)\},$$

and the linear transform in $\mathbb{C}^2$ from (z_1,z_2) to (w_1,w_2)

$$w_1 = \frac{1}{\sqrt{2}}(z_1 + iz_2), \quad w_2 = \frac{1}{\sqrt{2}}(z_1 - iz_2).$$

In the new coordinate system, $\mathcal{B}_{r,s}(p,q)$ is a polydisc

$$D_{\frac{r}{\sqrt{2}}}\left(\frac{p}{\sqrt{2}}\right) \times D_{\frac{s}{\sqrt{2}}}\left(\frac{q}{\sqrt{2}}\right) \subset \mathbb{C}\times\mathbb{C} = \mathbb{C}^2.$$

Hence by setting $r = r_k$ and $s = s_l$, there is a collection of polydiscs $\{D_{r_k/\sqrt{2}}(p_k/\sqrt{2}) \times D_{s_l/\sqrt{2}}(q_l/\sqrt{2})\}$ such that if $w=(w_1,w_2)$ is not in these polydiscs then $|Q(z)| > (\varepsilon/10)^d$ for the corresponding $z=(z_1,z_2)$. By choosing $\varepsilon > 0$ small enough, we find an $r \in (1/2,1)$ such that

$$\partial D_r \cap \left(\cup\{D_{\frac{r_k}{\sqrt{2}}}\left(\frac{p_k}{\sqrt{2}}\right) \times D_{\frac{s_l}{\sqrt{2}}}\left(\frac{q_l}{\sqrt{2}}\right)\}\right) = \emptyset.$$

Therefore we obtain

$$|Q(z)| > \left(\frac{\varepsilon}{10}\right)^d \quad \text{for any } z \in \partial D_r.$$

This completes the proof. □

The next result relies on Lemma 1.3.2 of Section 1.3, the Rouché Theorem in $\mathbb{C}^2$.

Corollary 2.4.7. *Suppose that $f : D_1 \subset \mathbb{C}^2 \to \mathbb{C}^2$ is holomorphic in D_1 and continuous up to the boundary ∂D_1 and that Q is given in Lemma 2.4.6. If, for the universal $\varepsilon > 0$ in Lemma 2.4.6,*

$$|f(z_1, z_2) - Q(z_1, z_2)| < \varepsilon^d \quad \textit{for any } (z_1, z_2) \in D_1 \setminus D_{\frac{1}{2}},$$

then

$$\#\{f^{-1}(0) \cap D_{\frac{1}{2}}\} \le d^2.$$

Proof. By Lemma 2.4.6, there exists an $r \in (1/2, 1)$ such that

$$|f(z_1, z_2) - Q(z_1, z_2)| < |Q(z_1, z_2)| \quad \text{for any } (z_1, z_2) \in \partial D_r.$$

Lemma 1.3.1 of Section 1.3, the Bezout formula, implies

$$\#\{Q^{-1}(0)\} \le d^2.$$

Here the multiplicity is counted. Hence by Lemma 1.3.2 of Section 1.3, we obtain

$$\#\{f^{-1}(0) \cap D_r\} \le d^2.$$

This finishes the proof. □

Remark 2.4.8. Suppose P is a harmonic polynomial of degree $d + 1$, with $P(0) = 0$. We write

$$P = \sum_{m=1}^{d+1} a_m \Phi_m,$$

where Φ_m is a homogeneous harmonic polynomial of degree m with $\int_{\mathbb{S}^1} \Phi_m^2 = 1$, for any $m = 1, \cdots, d+1$. Obviously, $\{\Phi_m|_{\mathbb{S}^1}\}$ is orthogonal in $L^2(\mathbb{S}^1)$. Now we assume $\int_{\mathbb{S}^1} P^2 \ge 1$, which implies $\sum_{m=1}^{d+1} a_m^2 \ge 1$. Then it is easy to see that DP, considered as a map from $\mathbb{C}^2$ to $\mathbb{C}^2$, can be written as in (2.4.12), with $\sum_{k=0}^{d} |c_k|^2 \ge 1/2$.

For Theorem 2.4.4, we prove the following result instead. The constant N in Theorem 2.4.9 is not the frequency.

Theorem 2.4.9. *There are universal constants $M > 1$ and $r \in (0, 1)$ such that for a harmonic function u in $B_M \subset \mathbb{R}^2$, with $u(0) = 0$, satisfying*

$$\frac{M \int_{B_M} |\nabla u|^2}{\int_{\partial B_M} u^2} \le N,$$

there holds

$$\#\{z \in D_r; \tilde{u}_{z_1}(z) = \tilde{u}_{z_2}(z) = 0\} \le 4N^2.$$

The proof of Theorem 2.4.9 is similar to that of Theorem 2.4.3.

Proof. For simplicity, we use the same notation to denote harmonic functions and their complexifications. We denote by (r, θ) the polar coordinates in $\mathbb{R}^2$ and write u in the following form

$$u(r, \theta) = \sum_{m=1}^{\infty} a_m \Phi_m(r, \theta) \quad \text{and} \quad \Phi_m(r, \theta) = r^m \varphi_m(\theta),$$

where $\varphi_m(\theta)$ satisfies

$$\int_{\mathbb{S}^1} \varphi_m^2(\theta) d\theta = 1 \quad \text{and} \quad \varphi_m''(\theta) + m^2 \varphi_m(\theta) = 0.$$

Without loss of generality, we assume

$$\int_{\partial B_1} u^2 = \sum_{m=1}^{\infty} a_m^2 = 1. \tag{2.4.14}$$

In the following, we set

$$N_* = \inf\{n \in \mathbb{Z}_+;\ n \geq N\}.$$

In other words, $N_* = N$ if N is an integer and $N_* = [N] + 1$ otherwise. Here $[N]$ is the integral part of N. Obviously, we have

$$N_* - 1 \leq N \leq N_*.$$

By (2.2.14), we get

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} = \frac{1}{M} \int_{\partial B_M} u^2 \leq M^{2N(0;M)} \int_{\partial B_1} u^2 = M^{2N(0;M)}.$$

By $N(0;M) \leq N \leq N_*$, we have

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \leq M^{2N_*},$$

and hence

$$|a_m| \leq M^{N_*-m} \quad \text{for any } m \geq 1. \tag{2.4.15}$$

Since $\{\varphi_m\}$ is orthonormal in $L^2(\mathbb{S}^1)$, we have for some universal constant $c > 0$

$$\int_{\partial B_1} |\sum_{m \geq 2N_*} a_m \Phi_m|^2 = \sum_{m \geq 2N_*} |a_m|^2 \leq \frac{c}{M^{2N_*}}.$$

We first choose M large, independent of N_*, such that

$$\sum_{m=2N_*}^{\infty} |a_m|^2 \leq \frac{1}{2}. \tag{2.4.16}$$

By (2.4.5), we get for some universal $R \in (0,1)$

$$\sup_{D_R} |\sum_{m \geq 2N_*} a_m \Phi_m| \leq \frac{c}{M^{N_*}}.$$

Interior estimates for holomorphic functions imply

$$\sup_{D_{\frac{R}{2}}} |D(\sum_{m \geq 2N_*} a_m \Phi_m)| \leq \frac{c}{RM^{N_*}}. \tag{2.4.17}$$

Set

$$P_* = \sum_{m=1}^{2N_*-1} a_m \Phi_m, \quad R_* = \sum_{m=2N_*}^{\infty} a_m \Phi_m. \tag{2.4.18}$$

Then $u = P_* + R_*$. Obviously, we have by (2.4.14) and (2.4.16)

$$\sum_{m=1}^{2N_*-1} |a_m|^2 \geq \frac{1}{2}.$$

Then DP_* satisfies the assumptions in Lemma 2.4.6, with $d = 2N_* - 2$ and possibly a different normalization constant. See Remark 2.4.8. By choosing M large enough, independent of N_*, we conclude by (2.4.17)

$$\sup_{D_{\frac{R}{2}}} |DR_*| < \varepsilon^{2N_*-2},$$

where ε is the universal constant as in Corollary 2.4.7, or Lemma 2.4.6. This implies

$$|Du(z) - DP_*(z)| < \varepsilon^{2N_*-2} \quad \text{for any } z \in D_{\frac{R}{2}}.$$

By applying Corollary 2.4.7 to Du and DP_* in $D_{R/2}$, we conclude

$$\#\{|Du|^{-1}(0) \cap D_{R/4}\} \leq (2N_* - 2)^2.$$

This finishes the proof, since $N_* - 1 \leq N$. ☐

Now we are ready to prove Theorem 2.4.4.

Proof of Theorem 2.4.4. First, we consider the case when N is small. Let $N_0 = N_0(1/4)$ be the constant in Theorem 2.2.8. If $N \leq N_0$, then u is never zero in $B_{1/4}$ by Theorem 2.2.8. Then Harnack inequality and interior estimates for harmonic functions and holomorphic functions imply that $\tilde{u}$ has no zeroes in D_{R_1}, for some universal $R_1 < 1$. Therefore we have $\mathcal{S}(\tilde{u}) \cap D_{R_1} = \emptyset$.

Next, we consider $N \geq N_0$. By Theorem 2.2.8, there holds for any $p \in B_{1/4}$

$$\frac{\int_{B_{\frac{1}{4}}(p)} |\nabla u|^2}{4 \int_{\partial B_{\frac{1}{4}}(p)} u^2} \leq CN,$$

for some positive constant C independent of u. For any $p \in B_{1/4}$, with $u(p) = 0$, by the scaled version of Theorem 2.4.9, we have

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{R_2}(p)\} \leq cN^2,$$

for some positive constants $R_2 < 1$ and c, independent of u and p. To finish the proof, we consider two cases. If u is never zero in $B_{R_2/2}$, then $\tilde{u}$ is never zero in $D_{2R_1R_2}$, as in the first part of the proof. This implies that $\mathcal{S}(\tilde{u}) \cap D_{2R_1R_2} = \emptyset$. If $u(p) = 0$ for some $p \in B_{R_2/2}$, then we have

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{R_2}(p)\} \leq cN^2,$$

which implies

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{\frac{R_2}{2}}\} \leq cN^2.$$

We finish the proof by taking $R_0 = \min\{R_1, 2R_1R_2, R_2/2\}$. ☐

To finish this section, we provide an example showing that the number of complex singular points is indeed in the quadratic order of the frequency. Hence the estimate in Theorem 2.4.4 is optimal.

Example 2.4.10. For any integer $d \geq 2$ and any small $\varepsilon > 0$, consider the harmonic polynomial u in the polar coordinate

$$u(x) = \varepsilon r \cos\theta - \frac{1}{d+1} r^{d+1} \cos(d+1)\theta.$$

Then it is easy to see

$$Du(x) = \begin{pmatrix} \varepsilon - r^d \cos d\theta \\ r^d \sin d\theta \end{pmatrix}.$$

By (2.4.9), we have

$$D\tilde{u}(z) = \begin{pmatrix} \varepsilon - \frac{1}{2}\big((z_1 + iz_2)^d + (z_1 - iz_2)^d\big) \\ -\frac{i}{2}\big((z_1 + iz_2)^d - (z_1 - iz_2)^d\big) \end{pmatrix}.$$

A simple calculation shows that $D\tilde{u}(z) = 0$ has d^2 solutions close to the origin. Obviously, the frequency of u is in the order of d.

CHAPTER 3

Monotonicity Formulas and Doubling Conditions

In this chapter, we discuss local properties of solutions of elliptic differential equations. Topics include a monotonicity formula and its various corollaries.

As we have seen, the monotonicity formula proved in Section 2.2 plays an essential role in the discussion of nodal sets of harmonic functions in the rest of Chapter 2. In this chapter, we generalize this monotonicity formula to general linear elliptic differential equations of the second order.

We first note that important consequences of the monotonicity formula are the doubling condition and hence the unique continuation. As is well known, the unique continuation only holds for solutions of homogeneous linear elliptic differential equations of the second order with Lipschitz leading coefficients. In fact, there exists an example of a nonzero solution which vanishes in an open set for some homogeneous linear elliptic differential equations of the second order with only Hölder continuous leading coefficients. Hence the monotonicity formula is not expected to hold for equations with only Hölder continuous leading coefficients.

In this chapter, we discuss the monotonicity formula and its consequences for solutions of homogeneous linear elliptic differential equations of the second order with Lipschitz leading coefficients. Garofalo and Lin [**35**] proved that the generalized frequency is indeed monotone for solutions of homogeneous linear elliptic differential equations of the second order with Lipschitz leading coefficients and no lower order terms. A consequence of such a monotonicity formula is the doubling condition. The situation is more complicated when the lower order terms are present. Garofalo and Lin [**36**] proved that the doubling conditions hold when coefficients of the lower order terms are bounded. Hence the unique continuation holds for solutions of homogeneous linear elliptic differential equations of the second order with Lipschitz leading coefficients and bounded lower order coefficients. We should remark here that the unique continuation, an important topic in the theory of elliptic differential equations, is not the main focus of this chapter. Our primary concern is on the growth of solutions in relation to the frequency. We refer readers to references at the end of this book for general results on the unique continuation.

3.1. A Monotonicity Formula

In this section, we discuss a monotonicity formula for solutions of elliptic differential equations of divergence form. The entire section follows closely Garofalo and Lin [**35**]. A similar formula was derived for harmonic functions in Section 2.2.

Let $A(x) = \big(a_{ij}(x)\big)$ be a symmetric $n \times n$ matrix-valued function in $\bar{B}_1$ satisfying the following assumptions:

(i) there exists a $\lambda \in (0,1)$ such that

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \quad \text{for any } x \in \bar{B}_1 \text{ and } \xi \in \mathbb{R}^n; \tag{3.1.1}$$

(ii) there exists a $\Gamma > 0$ such that for any $i, j = 1, \cdots, n$

$$|a_{ij}(x) - a_{ij}(y)| \le \Gamma|x-y| \quad \text{for any } x, y \in B_1. \tag{3.1.2}$$

We consider solutions u of the equation

$$\mathcal{L}u \equiv \operatorname{div}\big(A(x)\nabla u(x)\big) = \partial_i\big(a_{ij}(x)\partial_j u\big) = 0 \quad \text{in } B_1. \tag{3.1.3}$$

Under the assumptions (3.1.1) and (3.1.2), it is well known that every H^1-solution of (3.1.3) is in $H^2_{\text{loc}}(B_1)$.

For $n \ge 3$, we define a Lipschitz metric $\bar{g} = \bar{g}_{ij}(x)dx_i \otimes dx_j$ on B_1 by setting

$$\bar{g}_{ij} = a^{ij}(\det A)^{\frac{1}{n-2}}, \tag{3.1.4}$$

where $a^{ij}(x)$ denote the entries of $A(x)^{-1}$. Letting $\big(\bar{g}^{ij}(x)\big) = \big(\bar{g}_{ij}(x)\big)^{-1}$ and setting

$$r(x)^2 = \bar{g}_{ij}(0)x_ix_j, \quad \eta(x) = \bar{g}^{kl}(x)\partial_{x_k}r(x)\partial_{x_l}r(x), \tag{3.1.5}$$

we can easily verify that η is a positive Lipschitz function in B_1, whose Lipschitz constant depends on n, λ and Γ. In fact, η can be written as

$$\eta(x) = \frac{1}{r^2(x)}\bar{g}^{kl}(x)\bar{g}_{ik}(0)\bar{g}_{jl}(0)x_ix_j = \frac{1}{r^2(x)}\big(\bar{g}^{kl}(x) - \bar{g}^{kl}(0)\big)\bar{g}_{ik}(0)\bar{g}_{jl}(0)x_ix_j + 1.$$

Next, we introduce a new metric $g = g_{ij}(x)dx_i \otimes dx_j$ in B_1 by defining

$$g_{ij}(x) = \eta(x)\bar{g}_{ij}(x). \tag{3.1.6}$$

In the intrinsic geodesic polar coordinates $(r, \theta_1, \cdots, \theta_{n-1})$ with the pole at zero, the metric tensor g takes the form

$$g = dr \otimes dr + r^2 b_{ij}(r,\theta)d\theta_i \otimes d\theta_j, \tag{3.1.7}$$

where b_{ij} satisfy

$$b_{ij}(0,0) = \delta_{ij} \quad \text{for } i, j = 1, \cdots, n-1, \tag{3.1.8}$$

and

$$\big|\partial_r b_{ij}(r,\theta)\big| \le \Lambda \quad \text{for } i, j = 1, \cdots, n-1, \tag{3.1.9}$$

with a positive constant Λ depending on n, λ and Γ. Such intrinsic geodesic polar coordinates were used by Aronszajn, Krzywicki and Szarski [**9**]. We denote by $g^{ij}(x)$ the elements of the inverse matrix of g_{ij} and set

$$|g(x)| = |\det\big(g_{ij}(x)\big)|.$$

In this new metric, we rewrite (3.1.3) as

$$\operatorname{div}_g\big(\mu(x)\nabla_g u(x)\big) = 0 \quad \text{in } B_1, \tag{3.1.10}$$

where μ is given by

$$\mu = \eta^{-\frac{n-2}{2}}.$$

Here we denote by $\nabla_g u$ and $\mathrm{div}_g X$ respectively the intrinsic gradient of a function u and the intrinsic divergence of a vector field X on B_1 in the metric g, i.e.,

$$\nabla_g u = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j},$$

$$\mathrm{div}_g X = \frac{1}{|g|} \sum_{i=1}^n \frac{\partial}{\partial x_i} (\sqrt{|g|} X_i).$$

Obviously, μ is a Lipschitz function in B_1 satisfying

$$C_1 \le \mu(x) \le C_2 \quad \text{for any } x \in \bar{B}_1, \tag{3.1.11}$$

and in polar coordinates in $\bar{B}_1$

$$\mu(0,0) = 1, \quad \big|\partial_r \mu(r,\theta)\big| \le \Lambda, \tag{3.1.12}$$

where C_1, C_2 and Λ are positive constants depending only on n, λ and Γ.

To proceed, we consider a solution $u \in H^1(B_1)$ of (3.1.10) and, for any $r \in (0,1)$, we define

$$D_g(r) = \int_{B_r} \mu |\nabla_g u|^2 dV_g, \tag{3.1.13}$$

$$H_g(r) = \int_{\partial B_r} \mu u^2 dV_{\partial B_r}. \tag{3.1.14}$$

Here B_r represents the geodesic ball in the metric g of radius r and centered at the origin. By (3.1.7), B_r coincides with the usual Euclidean ball. Now for $r \in (0,1)$, we define *the generalized frequency* of u by

$$N_g(r) = \frac{r D_g(r)}{H_g(r)}, \tag{3.1.15}$$

if $H_g(r) \ne 0$. The main result in this section is the following theorem, which is often referred to as the monotonicity formula.

THEOREM 3.1.1. *If $u \in H^1(B_1)$ is a nontrivial solution of (3.1.10), then there exists a positive constant $C = C(n,\Lambda)$ such that*

$$\bar{N}_g(r) = \exp(Cr) N_g(r) \tag{3.1.16}$$

is an increasing function of $r \in (0,1)$.

PROOF. We write $D(r), H(r)$ and $N(r)$ instead of $D_g(r), H_g(r)$ and $N_g(r)$. We start by considering $N(r)$ in (3.1.15). A simple differentiation yields

$$N'(r) = N(r) \left\{ \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \right\}. \tag{3.1.17}$$

Therefore, the theorem will be proved if we can show

$$\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \ge -C(n,\Lambda). \tag{3.1.18}$$

To this end, we compute the derivatives $H'(r)$ and $D'(r)$.

By setting $b(r,\theta) = |\det\big(b_{ij}(r,\theta)\big)|$ and noting $dV_{\partial B_r} = r^{n-1}\sqrt{b(r,\theta)}d\theta$, we rewrite (3.1.14) as

$$H(r) = r^{n-1}\int_{\partial B_1}\mu(r,\theta)u^2(r,\theta)\sqrt{b(r,\theta)}d\theta. \tag{3.1.19}$$

Then we obtain

$$H'(r) = \frac{n-1}{r}H(r) + \int_{\partial B_r}\frac{1}{\sqrt{b}}\partial_\rho(\mu\sqrt{b})u^2 dV_{\partial B_r} + 2\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r},$$

where u_ρ denotes the radial differentiation, i.e., $u_\rho = <\nabla_g u, \frac{x}{\rho}>$. Using (3.1.9), (3.1.11) and (3.1.12), we have

$$H'(r) = \left(\frac{n-1}{r} + O(1)\right)H(r) + 2\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r}, \tag{3.1.20}$$

where $O(1)$ denotes a function of (r,θ) which is bounded in absolute value by a constant $C = C(n,\Lambda)$.

Next, by (3.1.10) we have

$$\int_{B_r}\operatorname{div}_g\big(\mu\nabla_g u^2\big)dV_g = 2\int_{B_r}\mu|\nabla_g u|^2 dV_g.$$

On the other hand, the divergence theorem yields

$$\int_{B_r}\operatorname{div}_g\big(\mu\nabla_g u^2\big)dV_g = 2\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r}.$$

Hence, we have

$$D(r) = \int_{\partial B_r}\mu u u_\rho dV_{\partial B_r}. \tag{3.1.21}$$

We rewrite (3.1.20) as

$$H'(r) = \left(\frac{n-1}{r} + O(1)\right)H(r) + 2D(r). \tag{3.1.22}$$

We now turn to the computation of $D'(r)$. For any fixed $r, \triangle r \in (0,1)$ with $r + \triangle r < 1$ and a fixed $t \in \big(0, 1 + \frac{\triangle r}{r + \triangle r}\big)$, define a function $w_t : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$w_t(\rho) = \begin{cases} t & \text{for } \rho \le r, \\ 1 & \text{for } \rho \ge r + \triangle r, \\ t\dfrac{r + \triangle r - \rho}{\triangle r} + \dfrac{\rho - r}{\triangle r} & \text{for } r \le \rho \le r + \triangle r. \end{cases} \tag{3.1.23}$$

By (3.1.7), we have $\operatorname{dist}_g(x,0) = |x|$. Now we define $l_t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$l_t(x) = w_t(|x|)x. \tag{3.1.24}$$

It is easy to see that l_t is a bi-Lipschitz map from B_1 to B_1. Then we set in B_1

$$u^t = u \circ l_t^{-1}.$$

For $u \in H^1(B_1)$, we have $u^t \in H^1(B_1)$, and consider

$$I(u^t) = \int_{B_1}|\nabla u^t|^2.$$

Since u is a solution of (3.1.10), we have $u \in H^2_{\mathrm{loc}}(B_1)$ by the regularity of solutions of elliptic equations of the divergence form. Note that u^t is independent of t in $B_1 \setminus B_{r+\triangle r}$. Hence we have $\frac{d}{dt}\big|_{t=1} u^t \in H^1_0(B_1)$ and then

$$\frac{d}{dt}\Big|_{t=1} I(u^t) = 0. \tag{3.1.25}$$

To proceed, we write

$$\begin{aligned} I(u^t) =& \int_{B_{rt}} \mu |\nabla_g u^t|^2 dV_g + \int_{B_{r+\triangle r}\setminus B_{rt}} \mu |\nabla_g u^t|^2 dV_g \\ &+ \int_{B_1 \setminus B_{r+\triangle r}} \mu |\nabla_g u^t|^2 dV_g \\ =& I_1 + I_2 + I_3. \end{aligned} \tag{3.1.26}$$

In the following, we write

$$u^t(y) = u(x),$$

with $y = l_t(x)$. For I_3, we have $y = x$ with $r + \triangle r \le |y| \le 1$. This implies

$$I_3 = \int_{B_1\setminus B_{r+\triangle r}} \mu |\nabla_g u^t|^2 dV_g = \int_{B_1\setminus B_{r+\triangle r}} \mu |\nabla_g u|^2 dV_g,$$

and then

$$\frac{d}{dt}\Big|_{t=1} I_3 = 0. \tag{3.1.27}$$

For I_1, we have $y = l_t(x) = tx$ for $|y| < tr$ and $|x| < r$, and hence

$$u^t(y) = u\left(\frac{y}{t}\right).$$

Then we have

$$|\nabla_g u^t(y)|^2 = g^{ij}(y)\partial_{y_i} u^t(y)\partial_{y_j} u^t(y) = \frac{1}{t^2} g^{ij}(tx)\partial_{x_i} u(x)\partial_{x_j} u(x).$$

The volume element is given by $dV_g = \sqrt{|g(y)|}dy = t^n \sqrt{|g(tx)|}dx$. Hence, we get

$$I_1 = \int_{B_{rt}} |\nabla u^t|^2 dV_g = t^{n-2} \int_{B_r} \mu(tx) g^{ij}(tx)\partial_{x_i} u(x)\partial_{x_j} u(x)\sqrt{|g(tx)|}dx,$$

and

$$\frac{d}{dt}\Big|_{t=1} I_1 = (n-2)D(r) + O(r)D(r), \tag{3.1.28}$$

where $O(r)$ denotes a function of (r, θ) whose absolute value is bounded by Cr, where $C = C(n, \Lambda)$. Last, for I_2, we have

$$y = \left(t\frac{r + \triangle r - |x|}{\triangle r} + \frac{|x| - r}{\triangle r}\right) x,$$

for $tr \le |y| \le r + \triangle r$ and $r \le |x| \le r + \triangle r$. By setting $s = |y|$ and $\rho = |x|$ and using polar coordinates, we have

$$\begin{aligned} |\nabla u^t(y)|^2 =& |\partial_s u^t(s,\theta)|^2 + \frac{1}{s^2} b^{ij}(s,\theta)\partial_{\theta_i} u^t(s,\theta)\partial_{\theta_j} u^t(s,\theta) \\ =& |\partial_\rho u(\rho,\theta)|^2 \left(\frac{\partial\rho}{\partial s}\right)^2 + \frac{1}{s^2} b^{ij}(s,\theta)\partial_{\theta_i} u(\rho,\theta)\partial_{\theta_j} u(\rho,\theta), \end{aligned}$$

where we denote by b^{ij} the entries of the matrix $(b_{ij})^{-1}$. Note that the volume element is given by

$$dV_g = \sqrt{|g(s,\theta)|}dy = s^{n-1}\sqrt{|g(s,\theta)|}dsd\theta = s^{n-1}\sqrt{|g(s,\theta)|}\frac{\partial s}{\partial \rho}d\rho d\theta.$$

Then we have

$$\begin{aligned} I_2 = \int_{B_{r+\triangle r}\setminus B_{rt}} \mu|\nabla_g u^t|^2 dV_g = \int_r^{r+\triangle r}\int_{\mathbb{S}^{n-1}} \Big(s^{n-1}\frac{\partial\rho}{\partial s}\mu(s,\theta)u_\rho^2(\rho,\theta)\sqrt{|g(s,\theta)|} \\ + s^{n-3}\frac{\partial s}{\partial \rho}\mu(s,\theta)b^{ij}(s,\theta)\partial_{\theta_i}u(\rho,\theta)\partial_{\theta_j}u(\rho,\theta)\sqrt{|g(s,\theta)|}\Big)\, d\rho d\theta. \end{aligned}$$

Note

$$s = \rho\left(t\frac{r+\triangle r-\rho}{\triangle r} + \frac{\rho - r}{\triangle r}\right),$$

and

$$\frac{\partial s}{\partial \rho} = t\frac{r+\triangle r - 2\rho}{\triangle r} + \frac{2\rho - r}{\triangle r}.$$

Then a straightforward calculation yields

$$\begin{aligned} \frac{d}{dt}\Big|_{t=1} I_2 = \frac{1}{\triangle r}\int_{B_{r+\triangle r}\setminus B_r} \mu\big\{\big((n-1+O(\rho))(r+\triangle r-\rho) - (r+\triangle r-2\rho)\big)u_\rho^2 \\ + \big((n-3+O(\rho))(r+\triangle r-\rho) + (r+\triangle r - 2\rho)\big)(|\nabla_g u|^2 - u_\rho^2)\big\}dV_g. \end{aligned}$$

By letting $\triangle r \to 0+$, we obtain

$$\frac{d}{dt}\Big|_{t=1} I_2 = r\int_{\partial B_r}\mu\big(2u_\rho^2 - |\nabla_g u|^2\big)dV_{\partial B_r},$$

or

$$\frac{d}{dt}\Big|_{t=1} I_2 = 2r\int_{\partial B_r}\mu u_n^2 dV_{\partial B_r} - rD'(r). \tag{3.1.29}$$

With (3.1.25), (3.1.27)-(3.1.29), we get

$$rD'(r) - \big(n-2+O(r)\big)D(r) = 2r\int_{\partial B_r}\mu u_\rho^2 dV_{\partial B_r}. \tag{3.1.30}$$

With (3.1.22) and (3.1.30), we obtain by the Cauchy inequality

$$\begin{aligned} \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} &= O(1) + 2\frac{\int_{\partial B_r}\mu u_\rho^2 dV_{\partial B_r}}{\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r}} - \frac{\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r}}{\int_{\partial B_r}\mu u^2 dV_{\partial B_r}} \\ &\geq O(1) \geq -C(n,\Lambda). \end{aligned}$$

This finishes the proof of (3.1.18) and hence the theorem. □

Returning to solutions of (3.1.3), we then obtain the following result.

COROLLARY 3.1.2. *Let $\mathcal{L} = div\big(A(x)\nabla\big)$ be an elliptic operator in $B_1 \subset \mathbb{R}^n$, $n \geq 3$, with $A(x)$ satisfying (3.1.1) and (3.1.2). Then there exists a positive constant C, depending only on n, λ and Γ, such that*

$$\bar{N}(r) = \exp(Cr)N_g(r) \tag{3.1.31}$$

is an increasing function of $r \in (0,1)$.

It is easy to check that $D(r) = D_g(r)$ and $H(r) = H_g(r)$ in (3.1.13) and (3.1.14) are in fact given by

$$\begin{aligned} D(r) &= \int_{B_r} a_{ij}\partial_i u \partial_j u dx, \\ H(r) &= \int_{\partial B_r} \mu_0 u^2 dH^{n-1}, \end{aligned}$$

where μ_0 is a positive Lipschitz function in B_1 satisfying (3.1.11) and (3.1.12), with C_1, C_2 and Λ depending only on n, λ and Γ.

Next, we prove the doubling condition. We define

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}.$$

THEOREM 3.1.3. *Let $\mathcal{L} = div\big(A(x)\nabla\big)$ be an elliptic operator in $B_1 \subset \mathbb{R}^n$, $n \geq 3$, with $A(x)$ satisfying (3.1.1) and (3.1.2), and let $u \in H^1(B_1)$ be a solution of (3.1.3). Then there exist positive constants c_1 and c_2, depending only on n, λ and Γ, such that for any $r \in (0, 1/2]$ and $\eta \in (1, 2]$*

$$\begin{aligned} \fint_{\partial B_{\eta r}} u^2 &\leq c_1 \eta^{c_2(1+N)} \fint_{\partial B_r} u^2, \\ \fint_{B_{\eta r}} u^2 &\leq c_1 \eta^{c_2(1+N)} \fint_{B_r} u^2. \end{aligned} \tag{3.1.32}$$

PROOF. We use the same setting for the proof of Theorem 3.1.1. Let $H(r)$ be defined as in (3.1.14). By (3.1.22), we get

$$\left(\log \frac{H(r)}{r^{n-1}}\right)' = O(1) + 2\bar{N}(r)\frac{\exp(-Cr)}{r}, \tag{3.1.33}$$

where we used the definition (3.1.16) of the modified frequency, and C is the constant in (3.1.31). Now, for any $r < 1/2$, we integrate (3.1.33) between r and ηr. Using (3.1.31), we obtain

$$\log\left(\frac{H(\eta r)}{H(r)}\eta^{-n+1}\right) \leq C' r + 2\log \eta \bar{N}(1). \tag{3.1.34}$$

Exponentiating (3.1.34) yields

$$H(\eta r) \leq \eta^{n-1} H(r) \exp\big(C' + 2\log \eta \bar{N}(1)\big). \tag{3.1.35}$$

This implies the first part of (3.1.32) with the estimate on μ in (3.1.11). Integrating (3.1.35) in r gives

$$\int_{B_{\eta r}} \mu u^2 dV_g \leq \eta^{n-1} \exp\big(C' + 2\log\eta\bar{N}(1)\big) \int_{B_r} \mu u^2 dV_g. \tag{3.1.36}$$

This implies the second part of (3.1.32) with (3.1.11) again. □

REMARK 3.1.4. For $\eta = 2$, we simply write (3.1.32) as

$$\begin{aligned} \fint_{\partial B_{2r}} u^2 &\leq c_1 2^{c_2 N} \fint_{\partial B_r} u^2, \\ \fint_{B_{2r}} u^2 &\leq c_1 2^{c_2 N} \fint_{B_r} u^2, \end{aligned} \tag{3.1.37}$$

where c_1 and c_2 are positive constants depending only on n, λ and Γ. Note that (3.1.37) is referred to as *the doubling condition*.

Next, we introduce the notion of the vanishing order for nonsmooth functions. First, we note the following simple result. For any $u \in L^2(B_1)$ and any constants $k < l$, we have

$$\limsup_{r\to 0+} \frac{1}{r^l}\Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} < \infty \text{ implies } \limsup_{r\to 0+} \frac{1}{r^k}\Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} = 0,$$

and

$$\limsup_{r\to 0+} \frac{1}{r^k}\Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} > 0 \text{ implies } \limsup_{r\to 0+} \frac{1}{r^l}\Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} = \infty.$$

For any $u \in L^2(B_1)$ with $\limsup\limits_{r\to 0+} \Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} = 0$, we define the *vanishing order* d of u at 0 by

$$d = \sup\{k;\ \limsup_{r\to 0+} \frac{1}{r^k}\Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} = 0\}.$$

If d is finite, we obtain

$$\limsup_{r\to 0+} \frac{1}{r^l}\Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} = \infty \quad \text{for any } l > d.$$

If d is infinite, we say u vanishes up to infinite order at 0. In other words, u vanishes up to infinite order at 0 if

$$\fint_{B_r} u^2 = O(r^l), \tag{3.1.38}$$

for any positive l.

THEOREM 3.1.5. *Let $\mathcal{L} = div\big(A(x)\nabla\big)$ be an elliptic operator in $B_1 \subset \mathbb{R}^n$, $n \geq 3$, with $A(x)$ satisfying (3.1.1) and (3.1.2), and let $u \in H^1(B_1)$ be a solution of (3.1.3). If u vanishes up to infinite order at 0, then $u \equiv 0$ in B_1.*

PROOF. With $C = c_1 2^{c_2 N}$, we obtain by (3.1.37) for any i

$$\fint_{B_1} u^2 \leq C^i \fint_{2^{-i}} u^2 = C^i |B_{2^{-i}}|^l \frac{1}{|B_{2^{-i}}|^l} \fint_{2^{-i}} u^2,$$

with $l > 0$ to be chosen. Now we take l such that $C 2^{-nl} = 1$. This yields by (3.1.38)

$$\fint_{B_1} u^2 \leq (\omega_n)^l \frac{1}{|B_{2^{-i}}|^l} \fint_{2^{-i}} u^2 \to 0 \quad \text{as } i \to \infty.$$

This implies $u \equiv 0$ in B_1. □

REMARK 3.1.6. In fact, there holds for the vanishing order d of u at 0

$$d \leq c_1 + c_2 N \quad \text{for any } x \in B_{\frac{1}{2}},$$

where c_1 and c_2 are positive constants depending only on n, λ and Γ. This can be seen easily by the expression of C and the choice of l in the proof of Theorem 3.1.5.

Next, we prove u^2 satisfies a reverse Hölder inequality.

THEOREM 3.1.7. *Let $\mathcal{L} = div\big(A(x)\nabla\big)$ be an elliptic operator in $B_1 \subset \mathbb{R}^n$, $n \geq 3$, with $A(x)$ satisfying (3.1.1) and (3.1.2), and let $u \in H^1(B_1)$ be a solution of (3.1.3). Then for any $\delta > 0$*

$$\left(\fint_{B_r} u^{2(1+\delta)}\right)^{\frac{1}{1+\delta}} \leq c_1 2^{c_2 N} \fint_{B_r} u^2, \tag{3.1.39}$$

where c_1 and c_2 are constants depending only on n, λ and Γ.

PROOF. By Theorem 1.1.4, we have for any $r \in (0, 1/2)$

$$\sup_{B_r} u^2 \leq c \fint_{B_{2r}} u^2,$$

where c is a positive constant depending only on n and λ. On the other hand, we trivially have for any $\delta > 0$

$$\left(\fint_{B_r} u^{2(1+\delta)}\right)^{\frac{1}{1+\delta}} \leq \sup_{B_r} u^2.$$

We obtain (3.1.39) easily with (3.1.37). ☐

Next, we prove that $|\nabla u|^2$ also satisfies doubling conditions and a reverse Hölder inequality. For the next result, we set

$$u_r = \fint_{B_r} u.$$

THEOREM 3.1.8. *Let $\mathcal{L} = div\big(A(x)\nabla\big)$ be an elliptic operator in $B_1 \subset \mathbb{R}^n$, $n \geq 3$, with $A(x)$ satisfying (3.1.1) and (3.1.2), and let $u \in H^1(B_1)$ be a nonconstant solution of (3.1.3). Then for any $r \in (0, 1/4)$*

$$\fint_{B_{2r}} |\nabla u|^2 \leq c_1 2^{c_2 N_0} \fint_{B_r} |\nabla u|^2, \tag{3.1.40}$$

and

$$\left(\fint_{B_r} |\nabla u|^2\right)^{\frac{1}{2}} \leq c_1 2^{c_2 N_0} \left(\fint_{B_r} |\nabla u|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}}, \tag{3.1.41}$$

where N_0 is given by

$$N_0 = \sup_r \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} |u - u_r|^2},$$

and c_1 and c_2 are constants depending only on n, λ and Γ.

PROOF. If u is a solution of (3.1.3), so is $u - u_r$. The assumption that u is not constant and the maximum principle imply

$$\inf_r \int_{\partial B_1} |u - u_r|^2 > 0.$$

Hence N_0 is well-defined and finite. By applying (3.1.32) to $u - u_r$, we have for any $r \in (0, 1/4)$

$$\fint_{B_{4r}} |u - u_r|^2 \leq c_1 2^{c_2 N_0} \fint_{B_r} |u - u_r|^2,$$

where c_1 and c_2 have the stated dependence. Now Lemma 1.1.5 applied to $u - u_r$ yields

$$\fint_{B_{2r}} |\nabla u|^2 \leq \frac{c}{r^2} \fint_{B_{4r}} |u - u_r|^2,$$

where c is a positive constant depending only on n and λ. On the other hand, the Poincaré inequality yields

$$\fint_{B_r} |u - u_r|^2 \leq cr^2 \fint_{B_r} |\nabla u|^2.$$

We then obtain (3.1.40) easily.

The proof of (3.1.41) is similar. By applying (3.1.32) to $u - u_r$, we have for any $r \in (0, 1/4)$

$$\fint_{B_{2r}} |u - u_r|^2 \leq c_1 2^{c_2 N_0} \fint_{B_r} |u - u_r|^2,$$

where c_1 and c_2 have the stated dependence. Now Lemma 1.1.5 applied to $u - u_r$ yields

$$\fint_{B_r} |\nabla u|^2 \leq \frac{c}{r^2} \fint_{B_{2r}} |u - u_r|^2,$$

where c is a positive constant depending only on n and λ. On the other hand, the Poincaré inequality yields

$$\fint_{B_r} |u - u_r|^2 \leq cr^2 \left(\fint_{B_r} |\nabla u|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}}.$$

We then get (3.1.41) similarly. □

3.2. Doubling Conditions

In this section, we discuss the elliptic equation of the form

$$\mathcal{L}u = -\partial_i\big(a_{ij}(x)\partial_j u\big) + b_i(x)\partial_i u + c(x)u = 0 \quad \text{in } B_1 \subset \mathbb{R}^n. \tag{3.2.1}$$

We assume

(i) there exists a $\lambda \in (0, 1)$ such that

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \bar{B}_1 \text{ and } \xi \in \mathbb{R}^n; \tag{3.2.2}$$

(ii) there exists a $\Gamma > 0$ such that for any $i, j = 1, \cdots, n$

$$|a_{ij}(x) - a_{ij}(y)| \leq \Gamma|x - y| \quad \text{for any } x, y \in B_1; \tag{3.2.3}$$

(iii) there exists a $\kappa > 0$ such that

$$|a_{ij}(x)| + |b_i(x)| + |c(x)| \leq \kappa \quad \text{for any } x \in B_1. \tag{3.2.4}$$

In the following, we write $A(x) = \big(a_{ij}(x)\big)$ as a symmetric $n \times n$ matrix-valued function in $\bar{B}_1$ and write $\mathbf{b} = (b_1, \cdots, b_n)$.

By introducing a metric g as in (3.1.6), we write (3.2.1) as

$$-\mathrm{div}_g\big(\mu(x)\nabla_g u\big) + \mathbf{b}_g(x) \cdot \nabla_g u + c_g(x)u = 0,$$

where $\mathbf{b}_g$ is a vector field given by $\mathbf{b}_g = G\mathbf{b}/\sqrt{\det G}$ and $c_g = c/\sqrt{\det G}$, for $G = (g_{ij})$. Note that $(\sqrt{\det G})^{-1}$ is a bounded Lipschitz function whose bounds depend only on n, λ, Γ and κ. Therefore, $\mathbf{b}_g$ and c_g satisfy the same assumption (iii) as $\mathbf{b}$ and c. For the ease of notations, we simply write

$$-\mathrm{div}_g\big(\mu(x)\nabla_g u\big) + \mathbf{b}(x)\cdot\nabla_g u + c(x)u = 0. \tag{3.2.5}$$

To proceed, we consider a solution $u \in H^1(B_1)$ of (3.2.5) and, for any $r \in (0,1)$, we introduce

$$H_g(r) = \int_{\partial B_r} \mu u^2 dV_{\partial B_r}, \tag{3.2.6}$$

$$I_g(r) = \int_{B_r} \big(\mu|\nabla_g u|^2 + u\mathbf{b}\cdot\nabla_g u + cu^2\big)dV_g, \tag{3.2.7}$$

$$D_g(r) = \int_{B_r} \mu|\nabla_g u|^2 dV_g. \tag{3.2.8}$$

Here, B_r represents the geodesic ball in the metric g of radius r and centered at the origin. By (3.1.7), B_r coincides with the usual Euclidean ball. Now for $r \in (0,1)$, we define *the generalized frequency* of u by

$$N_g(r) = \frac{rI_g(r)}{H_g(r)}, \tag{3.2.9}$$

if $H_g(r) \neq 0$. The main result in this section is the following theorem, which was proved by Garofalo and Lin [**36**].

THEOREM 3.2.1. *Let $u \in H^1(B_1)$ be a nonzero solution of (3.2.5). Then there exist constants r_0, c_1 and c_2, depending only on n, κ and Γ, such that*

$$N_g(r) \leq c_1 + c_2 N_g(r_0) \quad \textit{for any } r \in (0, r_0). \tag{3.2.10}$$

Before proving Theorem 3.2.1, we establish some lemmas.

LEMMA 3.2.2. *For any $u \in H^1(B_r)$ with $r > 0$, there holds*

$$\int_{B_r} u^2 \leq \frac{2r}{n}\int_{\partial B_r} u^2 + \frac{4r^2}{n^2}\int_{B_r} |\nabla u|^2. \tag{3.2.11}$$

PROOF. A simple calculation shows

$$\begin{aligned}\int_{B_r} u^2 &= \int_0^r \left(\int_{\partial B_1} u^2(\rho\omega)d\omega\right)\rho^{n-1}d\rho\\ &= \frac{r^n}{n}\int_{\partial B_1} u^2(r\omega)d\omega - \frac{2}{n}\int_0^r\left(\int_{\partial B_1} u(\rho\omega)u_\rho(\rho\omega)d\omega\right)\rho^n d\rho\\ &= \frac{r}{n}\int_{\partial B_r} u^2 - \frac{2}{n}\int_{B_r}|x|uu_\rho.\end{aligned}$$

Then we have for any $\varepsilon > 0$

$$\int_{B_r} u^2 \leq \frac{r}{n}\int_{\partial B_r} u^2 + \frac{1}{n}\left(\varepsilon\int_{B_r} u^2 + \frac{1}{\varepsilon}\int_{B_r}|x|^2|\nabla u|^2\right).$$

We get (3.2.11) by taking $\varepsilon = n/2$. □

LEMMA 3.2.3. *Let $u \in H^1(B_1)$ be a nonzero solution of (3.2.5). Then there exist constants r_0 and C, depending only on n, κ and Γ, such that*

$$D_g(r) \leq 2I_g(r) + CH_g(r) \quad \textit{for any } r \in (0, r_0), \tag{3.2.12}$$

and

$$D_g(r) \geq \frac{1}{2} I_g(r) - CH_g(r) \quad \textit{for any } r \in (0, r_0). \tag{3.2.13}$$

Proof. First, we have by the definition of $I_g(r)$

$$\begin{aligned}\int_{B_r} \mu|\nabla_g u|^2 =& I_g(r) - \int_{B_r} \big(u\mathbf{b} \cdot \nabla_g u + cu^2\big) dV_g \\ \leq& I_g(r) + \int_{B_r} \big(|u||\mathbf{b}||\nabla_g u| + |c|u^2\big) dV_g.\end{aligned}$$

By the Cauchy inequality and (3.2.11), we obtain

$$\begin{aligned}\int_{B_r} \mu|\nabla_g u|^2 \leq& I_g(r) + \frac{1}{4}\int_{B_r} \mu|\nabla_g u|^2 dV_g + C\kappa \int_{B_r} \mu u^2 dV_g \\ \leq& I_g(r) + \big(\frac{1}{4} + C\kappa r^2\big) \int_{B_r} \mu|\nabla_g u|^2 dV_g + C\kappa r \int_{\partial B_r} \mu u^2 dV_{\partial B_r}.\end{aligned}$$

We get (3.2.12) by taking $C\kappa r^2 < 1/4$. We may prove (3.2.13) similarly. ☐

Remark 3.2.4. It is clear that we may take $r_0 = 1$ if κ is sufficiently small.

Corollary 3.2.5. *Let $u \in H^1(B_1)$ be a nonzero solution of (3.2.5). Then there exists a constant r_0, depending only on n, κ and Γ, such that*

$$H_g(r) \neq 0 \quad \textit{for any } r \in (0, r_0). \tag{3.2.14}$$

Proof. We argue by contradiction. Assume $H_g(r) = 0$ for some $r \in (0, r_0]$, where r_0 is as in Lemma 3.2.3. The definition of $H_g(r)$ implies $u = 0$ on ∂B_r. Equation (3.2.5) and the divergence theorem yield $I_g(r) = 0$. By (3.2.12), we get $D_g(r) = 0$ and hence $u = 0$ in B_r, which leads to a contradiction. ☐

Corollary 3.2.5 implies that $r \mapsto N_g(r)$ is absolutely continuous on $(0, r_0)$. Therefore, if we set

$$\Omega_{r_0} = \big\{r \in (0, r_0); N_g(r) > \max\{1, N_g(r_0)\}\big\},$$

then Ω_{r_0} is an open subset of $\mathbb{R}$. Hence there holds a decomposition

$$\Omega_{r_0} = \bigcup_{j=1}^{\infty} (a_j, b_j) \quad \text{with } a_j, b_j \notin \Omega_{r_0}. \tag{3.2.15}$$

Obviously in Ω_{r_0}, we have $N_g(r) > 1$, or

$$\frac{H_g(r)}{r} < I_g(r). \tag{3.2.16}$$

Corollary 3.2.6. *Let $u \in H^1(B_1)$ be a nonzero solution of (3.2.5). Then there exists a constant C, depending only on n, κ and Γ, such that*

$$D_g(r) \leq CI_g(r), \quad \int_{B_r} u^2 dV_g \leq Cr^2 I_g(r) \quad \textit{for any } r \in \Omega_{r_0}. \tag{3.2.17}$$

Proof. Note (3.2.17) follows readily from (3.2.12), (3.2.11) and (3.2.16). ☐

Now we are ready to prove Theorem 3.2.1.

PROOF OF THEOREM 3.2.1. We write $D(r), I(r), H(r)$ and $N(r)$ instead of $D_g(r)$, $I_g(r)$, $H_g(r)$ and $N_g(r)$. As in (3.1.17), we have

$$N'(r) = N(r)\left\{\frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)}\right\}.$$

Therefore, the theorem will be proved if we can show

$$\frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} \geq -C \quad \text{for any } r \in \Omega_{r_0}, \tag{3.2.18}$$

where C is a positive constant depending only on n, κ and Γ.

As in (3.1.20), we have

$$H'(r) = \left(\frac{n-1}{r} + O(1)\right) H(r) + 2\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}, \tag{3.2.19}$$

or

$$\frac{H'(r)}{H(r)} = \frac{n-1}{r} + O(1) + 2\frac{\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}}{\int_{\partial B_r} \mu u^2 dV_{\partial B_r}}, \tag{3.2.20}$$

where $O(1)$ denotes a function of (r, θ) which is bounded in absolute value by a constant C.

Since u is a solution of (3.2.5), we have

$$\int_{B_r} \mathrm{div}_g\big(\mu \nabla_g u^2\big) dV_g = 2\int_{B_r} \big(\mu |\nabla_g u|^2 + u\mathbf{b}\cdot\nabla_g u + cu^2\big) dV_g = 2I(r).$$

On the other hand, the divergence theorem yields

$$\int_{B_r} \mathrm{div}_g\big(\mu \nabla_g u^2\big) dV_g = 2\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}.$$

Hence, we have

$$I(r) = \int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}. \tag{3.2.21}$$

To compute $I'(r)$, we employ a radial deformation, as used in the proof of Theorem 3.1.1, combined with first variation estimates. Hence, we have

$$\begin{aligned} I'(r) = {} & \big(\frac{n-2}{r} + O(1)\big) I(r) + 2\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} \\ & + \int_{\partial B_r} \big(u\mathbf{b}\cdot\nabla_g u + cu^2\big) dV_{\partial B_r} - \frac{n-2}{r}\int_{B_r} \big(u\mathbf{b}\cdot\nabla_g u + cu^2\big) dV_g \\ & - \frac{2}{r}\int_{B_r} \rho u_\rho \big(\mathbf{b}\cdot\nabla_g u + cu\big) dV_g. \end{aligned} \tag{3.2.22}$$

We omit details. By (3.2.16) and (3.2.17), we obtain for any $r \in \Omega_{r_0}$

$$\frac{1}{r}\int_{B_r} \big(|u\mathbf{b}\cdot\nabla_g u| + |c|u^2\big) dV_g + \frac{1}{r}\int_{B_r} \rho\big(|u_\rho \mathbf{b}\cdot\nabla_g u| + |cuu_\rho|\big) dV_g \leq CI(r).$$

In deriving this inequality, we have used $|\partial_\rho u| \leq |\nabla_g u|$. This implies for $r \in \Omega_{r_0}$

$$I'(r) = \big(\frac{n-2}{r} + O(1)\big) I(r) + 2\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} + \int_{\partial B_r} (u\mathbf{b}\cdot\nabla_g u + cu^2) dV_{\partial B_r}. \tag{3.2.23}$$

The Cauchy inequality now yields

$$\begin{aligned} I(r)^2 = &\left(\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}\right)^2 \le \int_{\partial B_r} \mu u^2 dV_{\partial B_r} \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} \\ =&H(r) \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r}. \end{aligned}$$

By (3.2.16), we have

$$I(r) \le r \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} \quad \text{for any } r \in \Omega_{r_0}. \tag{3.2.24}$$

On the other hand, (3.2.7) gives

$$I'(r) = \int_{\partial B_r} \mu |\nabla_g u|^2 dV_{\partial B_r} + \int_{\partial B_r} \big(u\mathbf{b} \cdot \nabla_g u + cu^2\big) dV_{\partial B_r}. \tag{3.2.25}$$

Comparing (3.2.25) with (3.2.23) and using (3.2.24), we obtain for any $r \in \Omega_{r_0}$

$$\begin{aligned} \int_{\partial B_r} \mu |\nabla_g u|^2 dV_{\partial B_r} &\le \big(\frac{n-2}{r} + O(1)\big) I(r) + 2\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} \\ &\le C \int_{\partial B_r} \mu u^2 dV_{\partial B_r}. \end{aligned} \tag{3.2.26}$$

Now, we discuss two cases.

Case 1. Let us assume

$$\int_{\partial B_r} \mu u^2 dV_{\partial B_r} \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} \le 2\left(\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}\right)^2. \tag{3.2.27}$$

Then, we have by (3.2.26) and (3.2.27)

$$\begin{aligned} \left|\int_{\partial B_r} u\mathbf{b} \cdot \nabla_g u dV_{\partial B_r}\right| &\le C\left(\int_{\partial B_r} \mu u^2 dV_{\partial B_r} \int_{\partial B_r} \mu |\nabla_g u|^2 dV_{\partial B_r}\right)^{\frac{1}{2}} \\ &\le C\left(\int_{\partial B_r} \mu u^2 dV_{\partial B_r} \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r}\right)^{\frac{1}{2}} \\ &\le C \int_{\partial B_r} \mu u u_\rho dV_{\partial B_r} = CI(r). \end{aligned} \tag{3.2.28}$$

Hence by (3.2.23) and (3.2.28), we obtain

$$\frac{I'(r)}{I(r)} = \frac{n-2}{r} + O(1) + 2\frac{\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r}}{\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}}. \tag{3.2.29}$$

Substituting (3.2.20) and (3.2.29) in (3.2), we get by the Cauchy inequality

$$\begin{aligned} \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} &= O(1) + 2\frac{\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r}}{\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}} - 2\frac{\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}}{\int_{\partial B_r} \mu u^2 dV_{\partial B_r}} \\ &\ge O(1) \ge -C(n, \Lambda). \end{aligned} \tag{3.2.30}$$

Case 2. Let us assume

$$\int_{\partial B_r} \mu u^2 dV_{\partial B_r} \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} > 2\left(\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}\right)^2. \tag{3.2.31}$$

By (3.2.16), (3.2.26) and the Cauchy inequality, we have for any $r \in \Omega_{r_0}$

$$\begin{aligned}
&|\int_{\partial B_r} (u\mathbf{b} \cdot \nabla_g u + cu^2) dV_{\partial B_r}| \\
&\leq C \left(\int_{\partial B_r} \mu u^2 dV_{\partial B_r} \int_{\partial B_r} \mu |\nabla u|^2 dV_{\partial B_r} \right)^{\frac{1}{2}} + CH(r) \\
&\leq C \left(H(r) \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} \right)^{\frac{1}{2}} + CH(r) \\
&\leq \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} + CH(r) \leq \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} + CI(r).
\end{aligned} \tag{3.2.32}$$

Substituting (3.2.32) in (3.2.23), we obtain

$$I'(r) \geq \left(\frac{n-2}{r} + O(1) \right) I(r) + \int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r},$$

or

$$\frac{I'(r)}{I(r)} \geq \frac{n-2}{r} + O(1) + \frac{\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r}}{\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}}. \tag{3.2.33}$$

Substituting (3.2.20) and (3.2.33) in (3.2) and using (3.2.31), we get

$$\begin{aligned}
\frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} &= O(1) + \frac{\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r}}{\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}} - 2\frac{\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}}{\int_{\partial B_r} \mu u^2 dV_{\partial B_r}} \\
&\geq O(1) \geq -C(n, \Lambda).
\end{aligned} \tag{3.2.34}$$

Now, (3.2.30) and (3.2.34) allow us to conclude the existence of a positive constant C, depending only on n, κ and Γ, such that

$$\frac{N'(r)}{N(r)} \geq -C \quad \text{for any } r \in \Omega_{r_0}. \tag{3.2.35}$$

For any component (a_j, b_j) in the decomposition (3.2.15), we obtain

$$N(r) \leq CN(b_j) \exp\{b_j - r\} \leq C \max\big(1, N(r_0)\big) \quad \text{for any } r \in (a_j, b_j).$$

This finishes the proof. □

Next, we derive the doubling condition.

THEOREM 3.2.7. *Let $u \in H^1(B_1)$ be a nonzero solution of (3.2.1). Then there exist constants r_0, c_0, c_1 and c_2 such that, for any $p \in B_{r_0/4}$ and any $r \in (0, r_0/4)$,*

$$\fint_{\partial B_{2r}} u^2 \leq c_0 2^{c_1 N_g(0, r_0) + c_2} \fint_{\partial B_r} u^2,$$

and

$$\fint_{B_{2r}} u^2 \leq c_0 2^{c_1 N_g(0, r_0) + c_2} \fint_{B_r} u^2,$$

where r_0, c_0, c_1 and c_2 are positive constants depending only on n, λ, κ and Γ.

PROOF. The proof is similar to that of Corollary 2.2.5. With (3.2.21), we rewrite (3.2.19) as

$$H_g'(r) = \left(\frac{n-1}{r} + O(1) \right) H_g(r) + 2I_g(r),$$

or

$$\left(\frac{H_g(r)}{r^{n-1}}\right)' = O(1) + 2\frac{N_g(r)}{r}. \tag{3.2.36}$$

We integrate with respect to r as before. □

We also have the following result.

Corollary 3.2.8. *Let $u \in H^1(B_1)$ be a nonzero solution of (3.2.1). Then there exist constants r_0 and C such that, for any $s, r \in (0, r_0)$ with $s < r$,*

$$\fint_{\partial B_s} u^2 \leq C \fint_{\partial B_r} u^2, \tag{3.2.37}$$

and

$$\fint_{B_r} u^2 \leq C \fint_{\partial B_r} u^2, \tag{3.2.38}$$

where r_0 and C are positive constants depending only on n, λ and Γ.

Proof. The proof is similar to that of Corollary 2.2.7. By (3.2.36) and (3.2.12), we obtain for any $r \in (0, r_0)$

$$\left(\frac{H_g(r)}{r^{n-1}}\right)' \geq -C,$$

where r_0 is as in Lemma 3.2.3 and C is a positive constant depending only on n, λ and Γ. Now a simple integration implies (3.2.37). Another integration of (3.2.37) in s yields (3.2.38). □

Remark 3.2.9. Lemma 3.2.3 implies that $\frac{I_g(r)}{H_g(r)}$ and $\frac{D_g(r)}{H_g(r)}$ are comparable to each other. Hence in Theorem 3.2.7, we may replace $N_g(r) = \frac{rI_g(r)}{H_g(r)}$ by

$$N_g^*(r) = \frac{rD_g(r)}{H_g(r)}.$$

In applications, it is convenient to use integrals without weights related to coefficients of equations. For the next result, we define for any $p \in B_1$ and $r \in (0, 1 - |p|)$

$$N(p; r) = \frac{r\int_{B_r(p)} |\nabla u|^2}{\int_{\partial B_r(p)} u^2}.$$

Theorem 3.2.10. *Let $u \in H^1(B_1)$ be a nonzero solution of (3.2.1). Then there exist constants r_0, c_0, c_1 and c_2 such that, for any $p \in B_{r_0/4}$ and any $r \in (0, r_0/4)$,*

$$N(p, \frac{1}{4}r_0) \leq c_1 + c_2 N(0, r_0), \tag{3.2.39}$$

and

$$\fint_{B_{2r}(p)} u^2 \leq c_0 2^{c_1 N(0,r_0)+c_2} \fint_{B_r(p)} u^2, \tag{3.2.40}$$

where r_0, c_0, c_1 and c_2 are positive constants depending only on n, λ, κ and Γ.

Proof. The proof is similar to that of Theorem 2.2.8 and is omitted. □

Now, we consider a general elliptic differential equation of non-divergence form

$$a_{ij}\partial_{ij}u + b_i\partial_i u + cu = 0 \quad \text{in } B_1^n \subset \mathbb{R}^n. \tag{3.2.41}$$

If $a_{ij} \in Lip$ and $b_i, c \in L^\infty$, we write (3.2.41) as

$$\partial_i\big(a_{ij}\partial_j u\big) + \big(b_i - \partial_j a_{ij}\big)\partial_i u + cu = 0 \quad \text{in } B_1.$$

Then results established in this section hold for solutions of (3.2.41).

In the following, we rewrite (3.2.41) as a differential equation of divergence form with no lower order terms. For this, we require higher regularities of coefficients. For $(x, x_{n+1}) \in B_1^{n+1} \subset \mathbb{R}^{n+1}$, we define

$$v(x, x_{n+1}) = (2 - x_{n+1})u(x).$$

Then v satisfies

$$a_{ij}^{(1)}\partial_{ij}v + b_i^{(1)}\partial_i v = 0 \quad \text{in } B_1^{n+1} \subset \mathbb{R}^{n+1}, \tag{3.2.42}$$

where

$$(a_{ij}^{(1)}) = \begin{pmatrix} (a_{ij}) & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$b_i^{(1)} = \begin{cases} b_i & \text{for } i = 1, \cdots, n, \\ -(2 - x_{n+1})c(x) & \text{for } i = n+1. \end{cases}$$

Next, we let

$$w(x, x_{n+1}, x_{n+2}) = (2 - x_{n+2})v(x, v_{n+1}) \quad \text{for } (x, x_{n+1}, x_{n+2}) \in B_1^{n+2} \subset \mathbb{R}^{n+2},$$

and let M be a large number. Then w satisfies

$$a_{ij}^{(2)}\partial_{ij}w = 0 \quad \text{in } B_1^{n+2} \subset \mathbb{R}^{n+2}, \tag{3.2.43}$$

where

$$(a_{ij}^{(2)}) = \begin{pmatrix} (a_{ij}^{(1)}) & -\frac{b^{(1)}}{2}(2 - x_{n+2}) \\ -\left(\frac{b^{(1)}}{2}\right)^T (2 - x_{n+2}) & M^2 \end{pmatrix}.$$

Obviously, (3.2.43) is uniformly elliptic if M is large. We also note that

$$w(x, x_{n+1}, x_{n+2}) = 0 \quad \text{for } (x, x_{n+1}, x_{n+2}) \in B_1^{n+2}$$

if and only if $u(x) = 0$ for $x \in B_1^n$. Moreover, we have for any $p \in B_1^n \subset \mathbb{R}^n$ and $r < 1/4$

$$\begin{aligned} \int_{B_r^{n+2}(p,0)} w^2 &\le \int_{B_r^n(p)} u^2, \\ \int_{B_{3r}^{n+2}(p,0)} w^2 &\ge \int_{B_{2r}^n(p)} u^2. \end{aligned} \tag{3.2.44}$$

Now, we consider

$$a_{ij}\partial_{ij}u = 0 \quad \text{in } B_1^n \subset \mathbb{R}^n. \tag{3.2.45}$$

By assuming $a_{ij} \in C^2(B_1)$, we rewrite (3.2.45) as

$$\partial_i(a_{ij}\partial_j u) + b_i\partial_i u = 0,$$

with

$$b_i = -\partial_j a_{ij}.$$

For $(x, x_{n+1}) \in B_1^{n+1} \subset \mathbb{R}^{n+1}$, we define

$$\tilde{u}(x, x_{n+1}) = u(x).$$

Then $\tilde{u}$ satisfies

$$\partial_i(\tilde{a}_{ij}\partial_j\tilde{u}) = 0 \quad \text{in } B_1^{n+1} \subset \mathbb{R}^{n+1}, \tag{3.2.46}$$

where

$$(\tilde{a}_{ij}) = \begin{pmatrix} (a_{ij}) & x_{n+1}b \\ x_{n+1}b^T & M^2 \end{pmatrix}.$$

Obviously, (3.2.46) is uniformly elliptic if M is large. Inequalities similar to (3.2.44) hold for w replaced by $\tilde{u}$ and balls in $\mathbb{R}^{n+2}$ replaced by those in $\mathbb{R}^{n+1}$. Note that $\tilde{u}$ satisfies the doubling condition by Theorem 3.1.3. Hence, a doubling condition of u can also be derived from that of $\tilde{u}$.

3.3. Compact Classes of Operators and Solutions

For two positive constants $\lambda \leq 1$ and $\kappa \geq 1$, and for a continuous monotone function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0^+}\omega(t) = 0$, we denote by $\mathcal{L}(\lambda, \kappa, \omega)$ the class of linear elliptic operators of non-divergence form

$$\mathcal{L} = a_{ij}(x)\partial_{x_ix_j} + b_i(x)\partial_{x_i} + c(x) \quad \text{in } B_1 \subset \mathbb{R}^n,$$

if $a_{ij}(x)$, $b_i(x)$ and $c(x)$, $i, j = 1, \cdots, n$, satisfy the following conditions:

(i) $\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2$, for any $x \in B_1$ and $\xi \in \mathbb{R}^n$;

(ii) $\sum_{i=1}^n |b_i(x)| + |c(x)| \leq \kappa$, for any $x \in B_1$;

(iii) $|a_{ij}(x) - a_{ij}(y)| \leq \omega(|x - y|)$, for any $x, y \in B_1$, $i, j = 1, 2, \cdots, n$.

For any $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \omega)$ and any u satisfying

$$\mathcal{L}u = 0 \quad \text{in } B_1 \text{ with } \int_{B_1} u^2 \leq 1, \tag{3.3.1}$$

there holds for any $r \in (0, 1)$

$$\|u\|_{W^{2,p}(B_r)} \leq C, \tag{3.3.2}$$

where C is a positive constant depending only on n, λ, κ, ω, p and $(1 - r)^{-1}$.

Suppose (a_{ij}) is a constant matrix satisfying the condition (i), and let $\mathcal{L}_0 = a_{ij}\partial_{x_ix_j}$. Then it is obvious that $\mathcal{L}_0 \in \mathcal{L}(\lambda, \kappa, \omega)$, for any $\kappa > 0$ and any continuous and monotone function ω.

Let $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \omega)$. For any $x_0 \in B$ and $\rho \in (0, 1 - |x_0|)$, we define $\mathcal{L}_{x_0,\rho}$ by

$$\mathcal{L}_{x_0,\rho} \equiv a_{ij}(x_0 + \rho x)\partial_{x_ix_j} + \rho b_i(x_0 + \rho x)\partial_{x_i} + \rho^2 c(x_0 + \rho x). \tag{3.3.3}$$

Obviously, $\mathcal{L}_{x_0,\rho} \in \mathcal{L}(\lambda, \kappa, \omega)$. This is the translation and scaling invariant property of $\mathcal{L}(\lambda, \kappa, \omega)$, analogous to that of minimal surfaces.

Finally, we note that the class $\mathcal{L}(\lambda, \kappa, \omega)$ possesses a certain compactness. More precisely, let $\mathcal{L}_k$, $k = 1, 2, \cdots$, be a sequence of operators in $\mathcal{L}(\lambda, \kappa, \omega)$. Then, there is a subsequence $\mathcal{L}_{k'}$, $\{k'\} \subset \{k\}$, such that $\mathcal{L}_{k'}$ converges to an elliptic operator $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \omega)$ in the following sense: for any sequence of solutions $u_{k'}$ of $\mathcal{L}_{k'}u_{k'} = 0$ in B_1, with $\int_{B_1} u_{k'}^2 \leq 1$, there is a subsequence of $\{u_{k'}\}$ which converges weakly in $L^2(B_1)$ to some $u \in L^2(B_1)$ satisfying $\mathcal{L}u = 0$ in B_1. By (3.3.2), it is then easy to conclude such a subsequence converges to u in $W^{2,p}_{\text{loc}}(B_1)$. We should point out that, in the above convergence, the limit function u may be identically zero. It is crucial for us to rule out such bad limits in the discussion of nodal sets of solutions. On the other hand, it is obvious by (3.3.3) that, if $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \omega)$, $x_0 \in B_1$ and $\rho \to 0$, then $\mathcal{L}_{x_0,\rho}$ converges to $\mathcal{L}_{x_0} \equiv a_{ij}(x_0)\partial_{x_ix_j} \in \mathcal{L}(\lambda, \kappa, \omega)$.

In order to find nontrivial limiting solutions of $\mathcal{L}_{x_0}$ for any $x_0 \in B_1$ under a sequence of translations and scalings, we need to control the local growth of solutions of operators in $\mathcal{L}(\lambda, \kappa, \omega)$. For this purpose, we introduce the following subclass $\mathcal{S}_N$ of solutions.

Let $u \in L^2(B_1)$ be a solution of $\mathcal{L}u = 0$ in B_1 for some $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \omega)$. Then we say $u \in \mathcal{S}_N$ if

$$\fint_{B_{2r}(x_0)} u^2 \le 4^N \fint_{B_r(x_0)} u^2(x), \tag{3.3.4}$$

for any $x_0 \in B_{2/3}$ and $0 < 2r < 1 - |x_0|$.

When $\omega(t) = \Gamma t$, then (3.3.4) is valid for a constant

$$N = C(n, \lambda, \kappa, \Gamma) \cdot N_0,$$

provided $x \in B_{r_0}$ and $r \in (0, r_0)$ for some positive constant $r_0 = r_0(n, \lambda, \kappa, \Gamma)$. Here N_0 is defined by

$$N_0 = \frac{r_0 \int_{B_{r_0}} |\nabla u|^2}{\int_{\partial B_{r_0}} u^2}.$$

This is a consequence of Theorem 3.2.7.

The class $\mathcal{S}_N$ has the following important compactness property. Consider any $\mathcal{L}_k \in \mathcal{L}(\lambda, \kappa, \omega)$ and $u_k \in \mathcal{S}_N$ satisfying $\mathcal{L}_k u_k = 0$ in B_1 with $\int_{B_1} u_k^2 = 1$. Then there are subsequences $\mathcal{L}_{k'}$ and $u_{k'}$ such that $\mathcal{L}_{k'}$ converges to some $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \omega)$ and $u_{k'}$ converges to u in $W^{2,p}_{\mathrm{loc}}(B_1)$ such that $\mathcal{L}u = 0$. Moreover, $u \not\equiv 0$ in B_1. This compactness result is very important in discussions later on.

The discussion in this section holds also for operators of divergence form. For constants $\alpha, \lambda \in (0,1)$ and $\kappa \ge 1$, we denote by $\mathcal{L}(\lambda, \kappa, \alpha)$ a class of elliptic operators of divergence form

$$\mathcal{L} = \partial_{x_i}(a_{ij}(x)\partial_{x_j}) + b_i(x)\partial_{x_i} + c(x)$$

whose coefficients satisfy (i), (ii) and

(iii)′ $|a_{ij}(x) - a_{ij}(y)| \le \kappa|x - y|^\alpha$, for any $x, y \in B_1$ and $i, j = 1, \cdots, n$.

For any $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \alpha)$ and any u satisfying $\mathcal{L}u = 0$ in B_1 with $\int_{B_1} u^2 = 1$, instead of (3.3.2) we have for any $r \in (0,1)$

$$|u|_{C^{1,\alpha}(B_r)} \le C,$$

where C is a positive constant depending only on n, λ, κ, α and $(1-r)^{-1}$. Instead of (3.3.3), we define for $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \alpha)$

$$\mathcal{L}_{x_0,\rho} \equiv \partial_{x_i}\big(a_{ij}(x_0 + \rho x)\partial_{x_j}\big) + \rho b_i(x_0 + \rho x)\partial_{x_i} + \rho^2 c(x_0 + \rho x),$$

for any $x_0 \in B_1$ and any $\rho \in (0, 1 - |x_0|)$.

CHAPTER 4

Structures of Singular Sets

For a harmonic function in an open set in $\mathbb{R}^2$, the subset of critical points in the nodal set is exactly the singular part of the nodal set. For this reason, this subset of critical points is called the singular set. It is well known that the singular set of a 2-dimensional harmonic function is isolated. Around each point in the singular set, the nodal set consists of finitely many analytic curves intersecting at this point, forming equal angles. In fact, the number of singular points can be estimated in terms of the growth of harmonic functions. For details, see Section 2.4.

In this chapter, we study the structure of the critical nodal sets, or the singular sets, of solutions of homogeneous elliptic differential equations of the second order. In Chapter 7, we will study the size of the critical nodal sets.

We consider a homogeneous elliptic differential equation of the form

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u = 0 \quad \text{in } B_1 \subset \mathbb{R}^n,$$

where the coefficients a_{ij} satisfy

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n,\ x \in B_1,$$

for some positive constant λ. We assume a_{ij} are Lipschitz and b_i and c are at least bounded. The Lipschitz condition for the leading coefficients is essential. It implies the unique continuation for the operator $\mathcal{L}$. In other words, if a solution u vanishes to an infinite order at a point in B_1, then u is identically zero. For details, see Chapter 3.

For any C^2 solution u in B_1, we define the nodal set and the singular set by

$$\begin{aligned} \mathcal{N}(u) &= \{p \in B_1; u(p) = 0\}, \\ \mathcal{S}(u) &= \{p \in B_1; u(p) = |\partial u(p)| = 0\}. \end{aligned}$$

By the implicit function theorem, $\mathcal{N}(u)\backslash\mathcal{S}(u)$ is an $(n-1)$-dimensional hypersurface, at least locally. In this chapter, we study $\mathcal{S}(u)$ and prove that $\mathcal{S}(u)$ is countably $(n-2)$-rectifiable.

In order to study the structure of singular sets, we need to study the local behavior of solutions at these points. First, we blow up solutions and study blow-up limits. We expect the blow-up limits to be simple to analyze. Second, we study relations between the blow-up limit at one point and those at nearby points. Geometrically, this translates to studying the property of tangent planes at each point first and then studying the relation between a tangent plane at one point and those at nearby points.

Bers [**13**] proved that a classical solution of an elliptic differential equation, with a finite vanishing order at some point, must be asymptotic to a nonzero homogeneous polynomial with degree equal to the vanishing order. Moreover, this polynomial satisfies a constant coefficient elliptic equation. This result is essential in our discussion. It implies the existence and the uniqueness of the homogeneous blow-up limit. We should emphasize that the uniqueness of blow-up limits is rarely enjoyed by most problems. Then a natural choice of some subspace of the singular set of the blow-up limit can be associated to that of our solution. We need to establish that this subspace serves as a tangent space in some weak sense. In order to do this, we need to study relations of blow-up limits at various points. In other words, we need a uniform version of Bers' result. Specifically, we need a priori estimates on the homogeneous polynomials and error terms in the Bers' asymptotic formulas. These estimates were proved by Han [**45**]. Such results, in their simplest form, state as follows. Suppose u is an H^1-solution of a linear elliptic differential equation in B_1. If for some integer d, u satisfies

$$\limsup_{x\to 0}\frac{|u(x)|}{|x|^d}<\infty,$$

then

$$\sup_{|x|<\frac{1}{2}}\frac{|u(x)|}{|x|^d}\leq C\|u\|_{L^2(B_1)},$$

where C is a constant depending only on coefficients of the equation and the integer d. A result of this type is of interest on its own. It, in some sense, generalizes a priori estimates on higher order derivatives for smooth solutions.

4.1. Singular Sets of Smooth Solutions

In this section, we discuss the structure of singular sets of solutions of homogeneous elliptic differential equations of the second order with smooth coefficients.

We consider the homogeneous linear elliptic differential equation of the form

$$\mathcal{L}u\equiv\sum_{i,j=1}^n a_{ij}(x)\partial_{ij}u+\sum_{i=1}^n b_i(x)\partial_i u+c(x)u=0\quad\text{in } B_1\subset\mathbb{R}^n,\tag{4.1.1}$$

where the coefficients a_{ij} satisfy

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\geq\lambda|\xi|^2\quad\text{for any } \xi\in\mathbb{R}^n,\ x\in B_1,\tag{4.1.2}$$

for some positive constant λ.

We first prove a result due to Caffarelli and Friedman [**17**].

LEMMA 4.1.1. *Let a_{ij}, b_i, c be smooth in $B_1\subset\mathbb{R}^n$ satisfying (4.1.2) and u be a smooth solution of (4.1.1) in B_1. Then $\mathcal{S}(u)$ is contained in a countable union of $(n-2)$-dimensional smooth manifolds.*

PROOF. For any $p\in B_1$, we define the vanishing order $\mathcal{O}(p)$ of u at p by

$$\begin{aligned}\mathcal{O}(p)=\mathcal{O}_u(p)=\{d;\ &\partial^\nu u(p)=0\text{ for any }|\nu|<d,\\ &\partial^{\nu_0}u(p)\neq 0\text{ for some }|\nu_0|=d\}.\end{aligned}$$

Obviously, $\mathcal{O}(p) \geq 2$ for $p \in \mathcal{S}(u)$. For any $d \geq 2$, we set

$$\mathcal{S}_d(u) = \{p \in B_1;\ \mathcal{O}(p) = d\}.$$

Then we have

$$\mathcal{S}(u) = \bigcup_{d \geq 2} \mathcal{S}_d(u). \tag{4.1.3}$$

This is a finite union by the unique continuation. We prove that $\mathcal{S}_d(u)$ is $(n-2)$-dimensional for each fixed $d \geq 2$.

For any $p \in \mathcal{S}_d(u)$, there exists a $|\beta| = d-2$ such that $\partial^2 v(p) \neq 0$ for $v = \partial^\beta u$. Now applying ∂^β to (4.1.1) and evaluating at p, we obtain

$$\sum_{i,j=1}^n a_{ij}(p)\partial_{ij}v(p) = 0.$$

First, the Hessian matrix $\big(\partial^2 v(p)\big)$ has a nonzero eigenvalue. Next, we may diagonalize

$$\big(\partial^2 v(p)\big) = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

Then we have

$$a_1(p)\lambda_1 + \cdots + a_n(p)\lambda_n = 0,$$

for some positive constants $a_1(p), \cdots, a_n(p)$. By assuming $\lambda_1 \neq 0$, we have another nonzero eigenvalue, and hence we may assume $\lambda_2 \neq 0$. Note

$$\partial\partial_1 v(p) = (\lambda_1, 0 \cdots, 0), \quad \partial\partial_2 v(p) = (0, \lambda_2, 0, \cdots, 0).$$

By applying the implicit function theorem to $\partial_1 v$ and $\partial_2 v$, we conclude that $\{\partial_1 v = 0, \partial_2 v = 0\}$ is an $(n-2)$-dimensional smooth manifold in a neighborhood of p. Obviously, this manifold contains $\mathcal{S}_d(u)$ in a neighborhood of p. $\square$

Lemma 4.1.1 illustrates that $\mathcal{S}(u)$ is indeed the singular part of $\{u = 0\}$. We usually call $\mathcal{S}(u)$ the singular set of u.

Now, we discuss fine structures of $\mathcal{S}(u)$. Suppose u is a smooth function in B_1. For any $p \in B_1$, we define *the vanishing order* $\mathcal{O}(p)$ of u at p, as in the proof of Lemma 4.1.1, by

$$\begin{aligned}\mathcal{O}(p) = \mathcal{O}_u(p) = \{d;\ &\partial^\nu u(p) = 0 \text{ for any } |\nu| < d,\\ &\partial^{\nu_0} u(p) \neq 0 \text{ for some } |\nu_0| = d\}.\end{aligned}$$

Alternatively, the vanishing order can be defined as an integer d satisfying

$$\limsup_{x \to p} \frac{|u(x)|}{|x-p|^d} < \infty,$$
$$\limsup_{x \to p} \frac{|u(x)|}{|x-p|^{d+1}} = \infty.$$

Moreover, there exists a nonzero homogeneous polynomial P of degree d such that

$$u(x) = P(x-p) + o(|x-p|^d).$$

For convenience, we call the nonzero homogeneous polynomial P *the leading polynomial* of u at p.

LEMMA 4.1.2. *Suppose that* $\{\mathcal{L}_k\}_{k=0}^{\infty}$ *is a family of elliptic operators in* B_1 *of the form (4.1.1), with smooth coefficients satisfying (4.1.2), and that* u_k *is a smooth solution of* $\mathcal{L}_k u_k = 0$ *in* B_1 *for* $k = 0, 1, 2, \cdots$. *Suppose that* $\mathcal{L}_k \to \mathcal{L}_0$ *in the sense that the corresponding coefficients converge in* $C^m(B_1)$ *and that* $u_k \to u_0$ *in* $C^m(B_1)$ *for any integer* $m \geq 1$. *Then*

$$\limsup_{k\to\infty} \mathcal{O}_{u_k}(0) \leq \mathcal{O}_{u_0}(0). \tag{4.1.4}$$

If, in addition, $\mathcal{O}_{u_k}(0) = d$ *and* P_k *is the leading polynomial of* u_k *at* 0 *for* $k = 1, 2, \cdots$, *then the following conclusions hold:*

(i) if $\mathcal{O}_{u_0}(0) > d$, *then*

$$P_k \to 0 \quad \textit{uniformly in } B_1 \textit{ as } \ k \to \infty;$$

(ii) if $\mathcal{O}_{u_0}(0) = d$, *then*

$$P_k \to P_0 \quad \textit{uniformly in } B_1 \textit{ as } k \to \infty,$$

where P_0 *is the leading polynomial of* u_0 *at* 0.

Lemma 4.1.2 follows easily from interior estimates. Now we state the main result in this section, which was proved by Han [**43**].

THEOREM 4.1.3. *Let* a_{ij}, b_i, c *be smooth in* $B_1 \subset \mathbb{R}^n$ *satisfying (4.1.2) and* u *be a smooth solution of (4.1.1) in* B_1. *Then there exists the following decomposition*

$$\mathcal{S}(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^j(u),$$

where each $\mathcal{S}^j(u)$ *is on a countable union of* j*-dimensional* C^1 *manifolds,* $j = 0, 1, \cdots, n-3$, *and* $\mathcal{S}^{n-2}(u)$ *is on a countable union of* $(n-2)$*-dimensional* $C^{1,\beta}$ *manifolds for some* $\beta \in (0,1)$.

PROOF. The proof consists of several steps. For each fixed integer $d \geq 2$, we study

$$\mathcal{S}_d(u) = \{p \in \mathcal{S}(u); \mathcal{O}(u) = d\}.$$

Step 1. We study local behaviors at each point. For any $y \in B_{1/2} \cap \mathcal{S}_d(u)$ and any $r \in \big(0, \frac{1}{2}(1-|y|)\big)$, we set

$$u_{y,r}(x) = \frac{u(y+rx)}{\big(⨍_{\partial B_r(y)} |u|^2\big)^{\frac{1}{2}}} \quad \text{for any } x \in B_2. \tag{4.1.5}$$

Then, we have for any integer $m \geq 1$

$$u_{y,r} \to P \quad \text{in } C^m(B_2) \quad \text{as } r \to 0, \tag{4.1.6}$$

where $P = P_y$ is a non-zero homogeneous polynomial of degree d satisfying

$$\sum_{i,j=1}^{n} a_{ij}(y)\partial_{ij} P = 0. \tag{4.1.7}$$

Moreover, $\|P\|_{L^2(\partial B_1)} = 1$. Note that P is the normalized leading polynomial of u at y.

Since P is a non-zero homogeneous polynomial of degree d, we have

$$\mathcal{S}_d(P) = \{x; \partial^\nu P(x) = 0 \text{ for any } |\nu| \le d-1\}.$$

Obviously, $0 \in \mathcal{S}_d(P)$ by the homogeneity of P. Now we claim that $\mathcal{S}_d(P)$ is a linear subspace and

$$P(x) = P(x+z) \quad \text{for any } x \in \mathbb{R}^n \text{ and } z \in \mathcal{S}_d(P). \tag{4.1.8}$$

To see this, we take any $z \in \mathcal{S}_d(P)$. Then we have

$$D^\nu P(z) = 0 \quad \text{for any } |\nu| \le d-1.$$

By assuming

$$P(x) = \sum_{|\nu|=d} a_\nu x^\nu,$$

we have

$$P(x) = \sum_{|\nu|=d} a_\nu (x-z)^\nu.$$

This implies

$$P(x+z) = P(x) \quad \text{for any } x \in \mathbb{R}^n.$$

Note that P is a homogeneous polynomial of degree d. Then, we have

$$P(x+\lambda z) = P(x) \quad \text{for any } x \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R}.$$

Therefore, we obtain

$$D^\nu P(\lambda z) = 0 \quad \text{for any } |\nu| \le d-1.$$

Hence $\lambda z \in \mathcal{S}_d(P)$ for any $\lambda \in \mathbb{R}$. Now it is not difficult to conclude that $\mathcal{S}_d(P)$ is a linear subspace and that (4.1.8) holds. Next, we observe that $\dim \mathcal{S}_d(P) \le n-2$ for $d \ge 2$. In fact, (4.1.8) implies P is a function of n-$\dim \mathcal{S}_d(P)$ variables. If $\dim \mathcal{S}_d(P) = n-1$, P would be a degree d monomial of one variable satisfying (4.1.7). Hence $d < 2$, which leads to a contradiction.

Step 2. We define for each $j = 0, 1, 2, \ldots, n-2$,

$$\mathcal{S}_d^j(u) = \{y \in \mathcal{S}_d(u);\ \dim \mathcal{S}_d(P_y) = j\}.$$

We claim that $\mathcal{S}_d^j(u)$ is on a countable union of j-dimensional C^1 graphs. In fact, we prove that, for any $y \in \mathcal{S}_d^j(u)$, there exists an $r = r(y)$ such that $\mathcal{S}_d^j(u) \cap B_r(y)$ is contained in a (single piece of) j-dimensional C^1 graph.

To show this, we let ℓ_y be the j-dimensional linear subspace $\mathcal{S}_d(P_y)$ for any $y \in \mathcal{S}_d^j(u)$. For any $\{y_k\} \subset \mathcal{S}_d^j(u)$ with $y_k \to y$, we first prove

$$\text{Angle } < \overline{yy_k},\ \ell_y > \to 0. \tag{4.1.9}$$

To prove (4.1.9), we assume $y = 0$ and $p_k = \dfrac{y_k}{|y_k|} \to \xi \in \mathbb{S}^{n-1}$. Note $p_k \in \mathcal{S}_d(u_{0,|y_k|})$ for

$$u_{0,|y_k|}(x) = \frac{u(|y_k|x)}{\left(⨍_{\partial B_{|y_k|}} u^2\right)^{\frac{1}{2}}}.$$

See (4.1.5) for notations. We may show by an elementary calculation that

$$\mathcal{L}_k u_{0,|y_k|} = 0,$$

where $\mathcal{L}_k$ is some second order elliptic operator with a similar structure as $\mathcal{L}$. Moreover, for $\mathcal{L}$ as in (4.1.1), we have

$$\mathcal{L}_k \to \mathcal{L}_0 \equiv \sum_{i,j=1}^{n} a_{ij}(0)\partial_{ij},$$

in the sense that corresponding coefficients converge in $C^m(B_1)$ for any m. Then by applying Lemma 4.1.2, we obtain that P_y vanishes at ξ with an order at least d, i.e.,

$$\mathcal{O}_{P_y}(\xi) \geq d.$$

Since P_y is a homogeneous polynomial of degree d, then $\mathcal{O}_{P_y}(\xi) = d$ and $\xi \in \ell_y$. This implies (4.1.9).

By (4.1.9), we obtain that, for any $y \in \mathcal{S}_d^j(u)$ and any small $\varepsilon > 0$, there exists an $r = r(y, \varepsilon)$ such that

$$\mathcal{S}_d^j(u) \cap B_r(y) \subset B_r(y) \cap C_\varepsilon(\ell_y), \tag{4.1.10}$$

where

$$C_\varepsilon(\ell_y) = \{z \in \mathbb{R}^n; \operatorname{dist}(z, \ell_y) \leq \varepsilon|z|\}.$$

Let P_k and P be leading polynomials of u at y_k and $y = 0$, respectively. By Lemma 4.1.2, we have

$$P_k \to P \quad \text{uniformly in } C^d(B_1).$$

This implies

$$\ell_{y_k} \to \ell_y \quad \text{as } k \to \infty,$$

as subspaces in $\mathbb{R}^n$. By an argument similar to proving (4.1.9), we may prove that the constant r in (4.1.10) can be chosen uniformly for any point $z \in \mathcal{L}_d^j(u)$ in a neighborhood of y. In other words, for any $y \in \mathcal{S}_d^j(u)$ and any small $\varepsilon > 0$, there exists an $r = r(\varepsilon, y)$ such that

$$\mathcal{S}_d^j(u) \cap B_r(z) \subset B_r(z) \cap C_\varepsilon(\ell_z) \quad \text{for any } z \in \mathcal{S}_d^j(u) \cap B_r(y).$$

For $\varepsilon > 0$ small enough, this clearly implies that $\mathcal{S}_d^j(u) \cap B_r(y)$ is contained in a j-dimensional Lipschitz graph. By (4.1.9) this graph is C^1.

Step 3. $\mathcal{S}_d^{n-2}(u)$ is on a countable union of $(n-2)$-dimensional $C^{1,\beta}$ manifolds, for some $0 < \beta < 1$.

Take any $p \in \mathcal{S}_d^{n-2}(u)$, say $p = 0$, and let P be the leading polynomial of u at 0. Then $\dim \mathcal{S}_d(P) = n - 2$. By denoting $\mathbb{R}^n = \mathbb{R}^2 \times \mathcal{S}_d(P_p)$, we know P is a homogeneous harmonic polynomial of degree d in $\mathbb{R}^2$. Using polar coordinates (r, θ) in $\mathbb{R}^2$, we have

$$P = ar^d \cos d\theta$$

by some rotation, where a is a nonzero constant.

For any $x \in \mathcal{S}_d^{n-2}(u)$ which is close to 0, write $x = (x^1, x^2)$ with $x^1 \in \mathbb{R}^2, x^2 \in \mathbb{R}^{n-2}$. Since $Du(x) = 0$, then

$$DP(x) = -D(u - P)(x).$$

Obviously, we have

$$|x^1|^{d-1} \leq C|x|^d, \tag{4.1.11}$$

or

$$|x^1| \le C|x|^{1+\frac{1}{d-1}}, \tag{4.1.12}$$

for some constant $C > 0$. Hence the local $(n-2)$-dimensional C^1 manifold containing $\mathcal{S}_d^{n-2}(u)$ in a neighborhood of 0 is in fact $C^{1,\frac{1}{d}}$ at 0. $\square$

By setting

$$\mathcal{S}_g(u) = \bigcup_{d\ge 2} \mathcal{S}_d^{n-2}(u) \quad \text{and} \quad \mathcal{S}_b(u) = \bigcup_{j=0}^{n-3} \bigcup_{d\ge 2} \mathcal{S}_d^{j}(u),$$

we have the following corollary of Theorem 4.1.3.

Corollary 4.1.4. *Let u be a solution as in Theorem 4.1.3. Then*

$$\mathcal{S}(u) = \mathcal{S}_g(u) \bigcup \mathcal{S}_b(u),$$

where the Hausdorff dimension of $\mathcal{S}_b(u)$ is at most $n-3$ and $\mathcal{S}_g(u)$ is on a countable union of $(n-2)$-dimensional $C^{1,\beta}$ manifolds for some $\beta \in (0,1)$. Moreover, for any $p \in \mathcal{S}_g(u)$ the leading polynomial of u at p is a polynomial of two variables after some rotation of coordinates.

We often call $\mathcal{S}_g(u)$ and $\mathcal{S}_b(u)$ the good part and the bad part of $\mathcal{S}(u)$ respectively.

To conclude this section, we illustrate by an example that in $\mathbb{R}^3$ the singular set can be any closed subset in a line segment.

Example 4.1.5. For any closed subset $K \subset \mathbb{R}$, let f be a nonnegative smooth function vanishing exactly on K with $|ff''| + |f'^2| < 1/4$. Then $u(x, y, z) = xy + f^2(z)$ satisfies the elliptic differential equation

$$u_{xx} + u_{yy} + u_{zz} - (f^2)''(z)u_{xy} = 0,$$

and its singular set is $\{(0, 0)\} \times K$.

4.2. Asymptotic Expansions

In this section, we derive an asymptotic expansion for solutions of elliptic differential equations with an estimate on the leading polynomial and the error term. Such a uniform expansion plays an essential role in the discussion of the structure of singular sets later on. The entire section follows Han [**45**].

For homogeneous equations, such an asymptotic expansion was proved by Bers [**13**]. Specifically, he proved that a solution of a homogeneous linear elliptic differential equation, which vanishes at 0 with order d, is asymptotic to a nonzero homogeneous polynomial of degree d. It is reasonable to think that derivatives of order less than d of such a solution at 0 are zero, that derivatives of order d are given by the coefficients of such a polynomial and that whether the derivatives of order d are Hölder continuous is determined by the error term. Necessary a priori estimates on the leading polynomials and the error terms were provided by Han [**45**].

A consequence of such an expansion with an estimate on leading polynomials and error terms is the pointwise Schauder estimate. As we know, Schauder estimates play an important role in the theory of elliptic differential equations, and are the basis for the existence and regularity of solutions. In general, a priori estimates, or the regularity, of solutions in a set depend on properties of the coefficients and the nonhomogeneous terms in the same set. As a corollary of the uniform asymptotic expansion, we obtain Schauder estimates on higher order derivatives by comparing solutions with polynomials. One advantage of such a method is that we do not need to differentiate equations to get equations for derivatives. Hence, assumptions on coefficients and nonhomogeneous terms can be weakened. In order to discuss the regularity of solutions at one point, we only need appropriate assumptions on coefficients and nonhomogeneous terms at this particular point.

As the first step, we need to obtain an a priori estimate on solutions themselves with respect to their vanishing orders. Such a result, in its simplest form, is as follows. Suppose u is a solution of some homogeneous linear elliptic differential equation. If for some integer d,

$$\limsup_{x\to 0}\frac{|u(x)|}{|x|^d}<\infty,$$

then

$$\sup_{|x|<1}\frac{|u(x)|}{|x|^d}\le C|u|_{L^\infty(B_1)},$$

where C is a constant depending only on the coefficients of the equation and the integer d.

We first discuss solutions of Poisson equations with a prescribed asymptotic.

Lemma 4.2.1. *Let $p>n/2$, $\gamma>0$ and $\alpha\in(0,1)$ be constants and $d\ge 2$ be an integer. For any $f\in L^p(B_1)$ with*

$$\|f\|_{L^p(B_r)}\le\gamma r^{d-2+\alpha+\frac{n}{p}}\quad\text{for any } r\le 1, \tag{4.2.1}$$

there exists a function $u\in W^{2,p}(B_1)$ such that

$$\Delta u=f\quad\text{in } B_1, \tag{4.2.2}$$

and

$$\begin{aligned}|u(x)|&\le C\gamma|x|^{d+\alpha}\quad\text{for any } x\in B_{\frac12},\\ \sum_{i=1}^{2}r^i\|D^iu\|_{L^p(B_r)}&\le C\gamma r^{d+\alpha+\frac{n}{p}}\quad\text{for any } r\le 1,\end{aligned} \tag{4.2.3}$$

where C is a positive constant depending only on n,p,d and α.

Remark 4.2.2. In general, Lemma 4.2.1 does not hold if $\alpha=0$ or $\alpha=1$. The following example shows that α cannot be 0. In $B_R\subset\mathbb{R}^n$, with $R<1$, we consider

$$\triangle u=\frac{x_2^2-x_1^2}{2|x|^2}\left\{\frac{n+2}{(-\log|x|)^{1/2}}+\frac{1}{2(-\log|x|)^{3/2}}\right\},$$

where the right side is continuous in $\bar B_R$ if we set it equal to zero at the origin. Hence it satisfies (4.2.1) for $d=2$, $\alpha=0$ and any $p>1$. The function

$$u(x)=(x_1^2-x_2^2)(-\log|x|)^{1/2}\in C(\bar B_R)\cap C^\infty(\bar B_R\setminus\{0\})$$

satisfies the above equation in $B_R \setminus \{0\}$. It is easy to check that $D^2u \in L^p(B_R)$ for any $p > 1$. A similar calculation shows that

$$\|D^2u\|_{L^p(B_r)} = \left(\int_0^r (-\log\rho)^{\frac{p}{2}} \rho^{n-1} d\rho\right)^{\frac{1}{p}} + O(r^{\frac{n}{p}}) \quad \text{for } r \le R.$$

Therefore, we have

$$\frac{1}{r^{\frac{n}{p}}} \|D^2u\|_{L^p(B_r)} \to \infty \quad \text{as} \ \ r \to 0.$$

PROOF OF LEMMA 4.2.1. We prove for $n \ge 3$. The case $n = 2$ can be discussed similarly.

We denote by Γ the fundamental solution of the Laplacian operator and set

$$w(x) = \int_{|y|<1} \Gamma(x-y) f(y) dy.$$

Then w satisfies

$$\Delta w = f \quad \text{in } B_1.$$

Global $W^{2,p}$ estimates and the explicit expression for w imply

$$\|w\|_{W^{2,p}(B_1)} \le C(\|w\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}) \le C\gamma. \tag{4.2.4}$$

For each $y \ne 0$, consider the Taylor expansion of $\Gamma(x-y)$ at $x = 0$. For each nonnegative integer k, denote by Γ_k the k-th order terms, i.e.,

$$\Gamma_k(x, y) = \sum_{|\beta|=k} D_x^\beta \Gamma(-y) \frac{x^\beta}{\beta!} \quad \text{for } y \ne 0.$$

It is easy to see that for each $y \ne 0$

$$\Delta \Gamma_k(\cdot, y) = 0 \quad \text{in } B_1,$$

and

$$|D_x^\beta \Gamma(x)| \le \frac{C}{|x|^{n+|\beta|-2}} \quad \text{for any } x \in B_1, \tag{4.2.5}$$

where C is a positive constant depending only on n and $|\beta|$. Set

$$v(x) = \int_{|y|<1} \sum_{k=0}^{d} \Gamma_k(x, y) f(y) dy. \tag{4.2.6}$$

We first show that v is well-defined in B_1. Then v is a harmonic polynomial of degree $\le d$. In fact, we show by (4.2.1)

$$|v(x)| \le C\gamma \quad \text{in } B_1. \tag{4.2.7}$$

The proof is based on a direct calculation. By setting $1/p' = 1 - 1/p$, we have for any $x \in B_1$

$$\begin{aligned}
|v(x)| &\le C \sum_{k=0}^{d} \int_{|y|<1} \frac{|f(y)|}{|y|^{n+k-2}} dy \\
&\le C \sum_{k=0}^{d} \sum_{i=1}^{\infty} \int_{\frac{1}{2^i}<|y|<\frac{1}{2^{i-1}}} \frac{|f(y)|}{|y|^{n+k-2}} dy \\
&\le C \sum_{k=0}^{d} \sum_{i=1}^{\infty} \left(\int_{\frac{1}{2^i}<|y|<\frac{1}{2^{i-1}}} \frac{1}{|y|^{(n+k-2)p'}} dy \right)^{\frac{1}{p'}} \cdot \left(\int_{|y|<\frac{1}{2^{i-1}}} |f(y)|^p dy \right)^{\frac{1}{p}} \\
&\le C\gamma \sum_{k=0}^{d} \sum_{i=0}^{\infty} \left(\frac{1}{2^i} \right)^{\frac{n}{p'}-(n+k-2)} \left(\frac{1}{2^i} \right)^{d-2+\alpha+\frac{n}{p}} \\
&= C\gamma \sum_{k=0}^{d} \sum_{i=0}^{\infty} \left(\frac{1}{2^i} \right)^{d-k+\alpha} = C\gamma \sum_{k=0}^{d} \sum_{i=0}^{\infty} \left(\frac{1}{2^{d-k+\alpha}} \right)^{i} \\
&\le C\gamma.
\end{aligned}$$

Now we set

$$u(x) = w(x) - v(x) = \int_{|y|<1} \left\{ \Gamma(x-y) - \sum_{k=0}^{d} \Gamma_k(x,y) \right\} f(y) dy. \tag{4.2.8}$$

Obviously, we have $\Delta u = f$. We show that u vanishes with the order at least $d+\alpha$ at $x=0$.

Fix $0 < |x| < \frac{1}{2}$ and split this integral into three parts as follows

$$\begin{aligned}
I_1 &= \int_{|y|<2|x|} \Gamma(x-y) f(y) dy, \\
I_2 &= -\int_{|y|<2|x|} \sum_{k=0}^{d} \Gamma_k(x,y) f(y) dy, \\
I_3 &= \int_{2|x|<|y|<1} \left[\Gamma(x-y) - \sum_{k=0}^{d} \Gamma_k(x,y) \right] f(y) dy.
\end{aligned}$$

Again denote $p' = \dfrac{p}{p-1}$. Then by the Hölder inequality, we have

$$\begin{aligned}
|I_1| &\le \left\{ \int_{|y|<2|x|} \left(\frac{C}{|x-y|^{n-2}} \right)^{p'} dy \right\}^{\frac{1}{p'}} \cdot \left(\int_{|y|<2|x|} |f(y)|^p dy \right)^{\frac{1}{p}} \\
&\le \left(\int_{|z|<3|x|} \frac{C}{|z|^{(n-2)p'}} dz \right)^{\frac{1}{p'}} \cdot \left(\int_{|y|<2|x|} |f(y)|^p dy \right)^{\frac{1}{p}} \\
&\le C\gamma |x|^{\frac{n}{p'}-(n-2)} \cdot |x|^{d-2+\alpha+\frac{n}{p}} = C\gamma |x|^{d+\alpha},
\end{aligned}$$

provided $p > n/2$. For I_2, we use a similar method as that for (4.2.7). By (4.2.5), we have

$$\begin{aligned}
|I_2| &\leq C\sum_{k=0}^{d}|x|^k\int_{|y|<2|x|}\frac{|f(y)|}{|y|^{n+k-2}}dy\\
&\leq C\sum_{k=0}^{d}|x|^k\sum_{i=0}^{\infty}\int_{\frac{|x|}{2^i}<|y|<\frac{2|x|}{2^i}}\frac{|f(y)|}{|y|^{n+k-2}}dy\\
&\leq C\sum_{k=0}^{d}|x|^k\sum_{i=0}^{\infty}\left(\int_{\frac{|x|}{2^i}<|y|<\frac{2|x|}{2^i}}\frac{1}{|y|^{(n+k-2)p'}}dy\right)^{\frac{1}{p'}}\cdot\left(\int_{|y|<\frac{2|x|}{2^i}}|f(y)|^p dy\right)^{\frac{1}{p}}\\
&\leq C\gamma\sum_{k=0}^{d}|x|^k\sum_{i=0}^{\infty}\left(\frac{|x|}{2^i}\right)^{\frac{n}{p'}-(n+k-2)}\left(\frac{|x|}{2^i}\right)^{d-2+\alpha+\frac{n}{p}}\\
&= C\gamma\sum_{k=0}^{d}|x|^k\sum_{i=0}^{\infty}\left(\frac{|x|}{2^i}\right)^{d-k+\alpha}\\
&= C\gamma|x|^{d+\alpha}\sum_{k=0}^{d}\sum_{i=0}^{\infty}\left(\frac{1}{2^{d-k+\alpha}}\right)^i\\
&\leq C\gamma|x|^{d+\alpha}.
\end{aligned}$$

For I_3, we proceed as follows. For x, y satisfying $2|x| < |y|$, by the Taylor expansion, we have

$$\Gamma(x-y)-\sum_{k=0}^{d}\Gamma_k(x,y)=\sum_{|\beta|=d+1}D_x^\beta\Gamma(\theta x,y)\frac{x^\beta}{\beta!},$$

where $\theta = \theta(x, y) \in (0, 1)$. By (4.2.5), this is bounded by

$$C\frac{|x|^{d+1}}{|y-\theta x|^{n+d+1-2}}\leq C\frac{|x|^{d+1}}{|y|^{n+d-1}},$$

since $|x| < \dfrac{|y|}{2}$. For any $x \in B_{1/2}$, there exists an integer M such that $2^{M-1}|x| \leq 1 < 2^M|x|$. Then we have by extending $f = 0$ outside B_1

$$\begin{aligned}
|I_3| &\leq |x|^{d+1} \int_{2|x|<|y|<1} \frac{|f(y)|}{|y|^{n+d-1}} dy \\
&\leq |x|^{d+1} \sum_{i=1}^{M-1} \int_{2^i|x|<|y|<2^{i+1}|x|} \frac{|f(y)|}{|y|^{n+d-1}} dy \\
&\leq |x|^{d+1} \sum_{i=1}^{M-1} \left(\int_{2^i|x|<|y|<2^{i+1}|x|} \frac{1}{|y|^{(n+d-1)p'}} dy \right)^{\frac{1}{p'}} \\
&\qquad\qquad \cdot \left(\int_{|y|<2^{i+1}|x|} |f(y)|^p dy \right)^{\frac{1}{p}} \\
&\leq C\gamma |x|^{d+1} \sum_{i=1}^{M-1} (2^i|x|)^{\frac{n}{p'}-(n+d-1)} \cdot (2^i|x|)^{d-2+\alpha+\frac{n}{p}} \\
&= C\gamma |x|^{d+1} \sum_{i=1}^{M-1} \frac{1}{(2^i|x|)^{1-\alpha}} \\
&= C\gamma |x|^{d+\alpha} \sum_{i=1}^{M-1} \frac{1}{(2^{1-\alpha})^i} \\
&\leq C\gamma |x|^{d+\alpha}.
\end{aligned}$$

This finishes the proof of the first part of (4.2.3). Since $\Delta u = f$ in B_1, $W^{2,p}$ estimates imply

$$\sum_{i=1}^{2} r^i \|D^i u\|_{L^p(B_r)} \leq C\gamma r^{d+\alpha+\frac{n}{p}} \quad \text{for any } r \leq \frac{1}{4}. \tag{4.2.9}$$

By (4.2.4) and (4.2.7), the estimate (4.2.9) can be extended to B_1. □

Corollary 4.2.3. *Let $p > n/2$, $\gamma > 0$ and $\alpha \in (0,1)$ be constants and $d \geq 2$ be an integer. For any $f \in L^p(B_1)$ with*

$$\|f\|_{L^p(B_r)} \leq \gamma r^{d-2+\alpha+\frac{n}{p}} \quad \textit{for any } r \leq 1,$$

and any solution $u \in W^{2,p}(B_1)$ of $\Delta u = f$ in B_1, there exists a harmonic polynomial P_d of degree $\leq d$ such that

$$|u(x) - P(x)| \leq \big(\gamma + \|u\|_{L^p(B_1)}\big)|x|^{d+\alpha} \quad \textit{for any } x \in B_{\frac{1}{2}},$$

where C is a positive constant depending only on n, p, d and α.

Proof. By Lemma 4.2.1, there exists a $v \in W^{2,p}(B_1)$ satisfying $\Delta v = f$ in B_1 such that

$$|v(x)| \leq C\gamma |x|^{d+\alpha} \quad \text{for any } x \in B_{\frac{1}{2}},$$

and

$$\|v\|_{L^p(B_1)} \leq C\gamma.$$

Note that $\Delta(u-v)=0$ in B_1. We write

$$u(x)-v(x)=P_d(x)+R_d(x),$$

where P_d is a polynomial of degree $\le d$ and R_d satisfies

$$|R_d(x)|\le C\|u-v\|_{L^p(B_1)}|x|^{d+1}\le C\left(\gamma+\|u\|_{L^p(B_1)}\right)|x|^{d+1},$$

for any $x\in B_{\frac12}$. Now we have $u=P_d+v+R_d$, and $u-P_d$ has the required estimate. $\square$

Next, we control the leading polynomials and error terms in a uniform way. Roughly speaking, if a solution u vanishes with order d at 0, then "derivatives" of order up to $d-1$ vanish at 0 and "derivatives" of order d and the error term can be estimated uniformly.

Suppose that $\mathcal{L}$ is a second order homogeneous linear elliptic operator in $B_1\subset\mathbb{R}^n$ given by

$$\mathcal{L}=a_{ij}(x)\partial_{x_ix_j}+b_i(x)\partial_{x_i}+c(x)\quad\text{in } B_1\subset\mathbb{R}^n, \tag{4.2.10}$$

where the coefficients a_{ij}, b_i and c verify the following assumptions:

$$\lambda|\xi|^2\le a_{ij}(0)\xi_i\xi_j\le\lambda^{-1}|\xi|^2\quad\text{for any } \xi\in\mathbb{R}^n; \tag{4.2.11}$$

$$\sum_{i,j=1}^n|a_{ij}(x)|+\sum_{i=1}^n|b_i(x)|+|c(x)|\le\kappa\quad\text{for any } x\in B_1; \tag{4.2.12}$$

$$\sum_{i,j=1}^n|a_{ij}(x)-a_{ij}(0)|\le\omega(|x|)\quad\text{for any } x\in B_1, \tag{4.2.13}$$

for some positive constants λ,κ, and some continuous and increasing function $\omega:\mathbb{R}^+\to\mathbb{R}^+$ with $\omega(r)\to0$ as $r\to0+$.

We should emphasize that assumptions (4.2.11) and (4.2.13) are made only at the origin.

We first prove an interior estimate with balls centered only at origin.

Lemma 4.2.4. *Let $\mathcal{L}$ be a second order elliptic operator in B_1 with the form (4.2.10) satisfying (4.2.11)-(4.2.13) and u a $W^{2,p}$ solution of $\mathcal{L}u=f$ in B_1 for some $f\in L^p(B_1)$ with $p>n/2$. Then there holds for any $r\le R$*

$$\sum_{i=1}^2 r^i\|D^iu\|_{L^p(B_r)}\le C\big(\|u\|_{L^p(B_{2r})}+r^2\|f\|_{L^p(B_{2r})}\big),$$

where C and R are positive constants depending only on λ,κ and ω.

Proof. Set $\mathcal{L}_0=a_{ij}(0)\partial_{ij}$. Then we write $\mathcal{L}u=f$ as

$$\mathcal{L}_0u=-\big(a_{ij}(x)-a_{ij}(0)\big)\partial_{ij}u-b_i(x)\partial_iu-c(x)u+f.$$

By introducing cut-off functions, we have for any $0<r<R\le1$

$$\begin{aligned}\|D^2u\|_{L^p(B_r)}\le C\bigg\{&(\omega(R)+\varepsilon)\|D^2u\|_{L^p(B_R)}\\&+\frac{C(\varepsilon)}{(R-r)^2}\|u\|_{L^p(B_R)}+\|f\|_{L^p(B_R)}\bigg\},\end{aligned}$$

where ε is an arbitrary positive constant and the constant C depends only on λ, κ and ω. Choose ε such that $C\varepsilon = 1/4$. Then for any R such that $C\omega(R) \leq 1/4$, we obtain for any $0 < r \leq R$

$$\|D^2u\|_{L^p(B_r)} \leq \frac{1}{2}\|D^2u\|_{L^p(B_R)} + C\left\{\frac{1}{(R-r)^2}\|u\|_{L^p(B_R)} + \|f\|_{L^p(B_R)}\right\}.$$

By a standard iteration, we get for any $r \leq R$

$$\|D^2u\|_{L^p(B_r)} \leq C\left\{\frac{1}{(R-r)^2}\|u\|_{L^p(B_R)} + \|f\|_{L^p(B_R)}\right\}.$$

Hence for any $r > 0$ with $C\omega(2r) < 1/4$, we obtain

$$r^2\|D^2u\|_{L^p(B_r)} \leq C(\|u\|_{L^p(B_{2r})} + r^2\|f\|_{L^p(B_{2r})}).$$

We finish the proof by using interpolation inequalities. $\square$

For subsequent results, we need more assumptions on leading coefficients. We assume, in addition,

$$\sum_{i,j=1}^{n} \|a_{ij} - a_{ij}(0)\|_{L^r(B_r)} \leq Kr^{\alpha+\frac{n}{p}} \quad \text{for any } r \leq 1, \tag{4.2.14}$$

for some constants $K > 1$ and $\alpha \in (0,1)$.

Theorem 4.2.5. *Let $\mathcal{L}$ be a second order elliptic operator in B_1 with the form (4.2.10) satisfying (4.2.11)-(4.2.14) and u a $W^{2,p}$ solution of $\mathcal{L}u = f$ in B_1 for some $f \in L^p(B_1)$ with $p > n/2$. Suppose, for some homogeneous polynomial Q of degree $d-2$, f satisfies*

$$\|f - Q\|_{L^p(B_r)} \leq \gamma r^{d-2+\alpha+\frac{n}{p}} \quad \textit{for any } r \leq 1, \tag{4.2.15}$$

for some integer $d \geq 2$ and constant $\gamma > 0$. Then if

$$\limsup_{r\to 0} \frac{1}{r^{d-1+\beta+\frac{n}{p}}}\|u\|_{L^p(B_r)} < \infty, \tag{4.2.16}$$

for some $\beta \in (0,1]$, there holds

$$\|u\|_{L^p(B_r)} \leq C(\|u\|_{L^p(B_1)} + \gamma + \|Q\|_{L^p(B_1)})r^{d+\frac{n+m}{p}} \quad \textit{for } r \leq 1, \tag{4.2.17}$$

where C is a constant depending only on n, p, d, λ, κ, α and K. Moreover, there exists a homogeneous polynomial P of degree d such that

$$a_{ij}(0)\partial_{ij}P = Q \quad \textit{in } \mathbb{R}^n, \tag{4.2.18}$$

$$|P(x)| \leq C\big(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)}\big)|x|^d \quad \textit{in } B_1, \tag{4.2.19}$$

$$|u(x) - P(x)| \leq C\big(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)}\big)|x|^{d+\alpha} \quad \textit{in } B_R \tag{4.2.20}$$

and

$$\begin{aligned}&\sum_{i=1}^{2} r^i\|D^i(u-P)\|_{L^p(B_r)}\\ \leq &C\big(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)}\big)r^{d+\alpha+\frac{n}{p}} \quad \textit{for any } r \leq R,\end{aligned} \tag{4.2.21}$$

where C and R are constants depending only on n, p, d, λ, α, κ, K and ω.

PROOF. We prove Theorem 4.2.5 in two steps.

Step 1. The existence of the homogeneous polynomial P.

We set for nonnegative integer k

$$c_k = \sup_{r\le 1} \frac{\|u\|_{L^p(B_r)}}{r^{d-1+\beta+k\alpha_1+\frac{n}{p}}}, \tag{4.2.22}$$

for some $0 < \alpha_1 \le \alpha$. We prove $c_k < \infty$ as long as $\beta + k\alpha_1 \le 1$. By (4.2.16), we know $c_0 < \infty$.

Lemma 4.2.4 implies for any $r \le R$

$$\begin{aligned} \sum_{i=1}^{2} r^i\|D^i u\|_{L^p(B_r)} &\le C(\|u\|_{L^p(B_{2r})} + r^2\|f\|_{L^p(B_{2r})}) \\ &\le C\{c_0 r^{d-1+\beta+\frac{n}{p}} + (\gamma + \|Q\|_{L^p(B_1)})r^{d+\frac{n}{p}}\} \\ &\le C(c_0+\gamma+\|Q\|_{L^p(B_1)})r^{d-1+\beta+\frac{n}{p}}. \end{aligned} \tag{4.2.23}$$

Set $\tilde{\mathcal{L}} = a_{ij}\partial_{ij}$ and $\mathcal{L}_0 = a_{ij}(0)\partial_{ij}$. Without loss of generality, we assume $a_{ij}(0) = \delta_{ij}$ and $\mathcal{L}_0 = \Delta$. We write $\mathcal{L}u = f$ as

$$\tilde{\mathcal{L}}u = Q - b_i\partial_i u - cu + (f - Q) \equiv Q + \phi.$$

By taking L^p-norms in B_r and combining with (4.2.23), we get for any $r \le R$

$$\begin{aligned} \|\phi\|_{L^p(B_r)} &\le C(r^{-1}\|u\|_{L^p(B_{2r})} + r\|f\|_{L^p(B_r)} + \|f-Q\|_{L^p(B_{2r})}) \\ &\le C(c_0+\gamma+\|Q\|_{L^p(B_1)})r^{d-2+\min(\beta,\alpha)+\frac{n}{p}}. \end{aligned} \tag{4.2.24}$$

Take a homogeneous polynomial P_1 of degree d such that $\mathcal{L}_0 P_1 = Q$. Then, we have

$$\begin{aligned} \tilde{\mathcal{L}}(u-P_1) &= (\mathcal{L}_0 - \tilde{\mathcal{L}})P_1 + \phi \\ &= -\sum_{i,j=1}^{n}(a_{ij}-a_{ij}(0))\partial_{ij}P_1 + \phi. \end{aligned}$$

By (4.2.14), we get for any $r \le 1$

$$\|\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}P_1\|_{L^p(B_r)} \le C\|P_1\|_{L^p(B_1)}r^{d-2+\alpha+\frac{n}{p}},$$

where C is a positive constant depending only on n, d, p and K. Therefore, $u - P_1$ satisfies

$$\tilde{\mathcal{L}}(u-P_1) = \tilde{\phi} \quad \text{in } B_1,$$

with

$$\|\tilde{\phi}\|_{L^p(B_r)} \le C\big(c_0+\gamma+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)}\big)r^{d-2+\min(\beta,\alpha)+\frac{n}{p}}, \tag{4.2.25}$$

for any $r \le R$. It is obvious that (4.2.25) also holds for any q to replace p with $1 < q < p$. By (4.2.23), we get for any $r \le R$

$$\begin{aligned} &\sum_{i=1}^{2} r^i\|D^i(u-P_1)\|_{L^p(B_r)} \\ &\le C(c_0+\gamma+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)})r^{d-1+\beta+\frac{n}{p}}. \end{aligned}$$

We write $\tilde{\mathcal{L}}(u-P_1)=\tilde{\phi}$ as

$$\begin{aligned}\mathcal{L}_0(u-P_1) &= (\mathcal{L}_0-\tilde{\mathcal{L}})(u-P_1)+\tilde{\phi}\\ &= -\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}(u-P_1)+\tilde{\phi}.\end{aligned}\tag{4.2.26}$$

For any q with $n/2<q<p$, we have

$$\begin{aligned}&\|\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}(u-P_1)\|_{L^q(B_r)}\\ &\le \|a_{ij}-a_{ij}(0)\|_{L^{\frac{pq}{p-q}}(B_r)}\|D^2(u-P_1)\|_{L^p(B_r)}\\ &\le C\|a_{ij}-a_{ij}(0)\|_{L^{\frac{pq}{p-q}}(B_r)}\cdot(c_0+\gamma+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)})r^{d-3+\beta+\frac{n}{p}}.\end{aligned}$$

If $p>n$, we take $q=p/2>n/2$. Then, we have $\dfrac{pq}{p-q}=p$ and hence

$$\begin{aligned}&\|\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}(u-P_1)\|_{L^q(B_r)}\\ &\le C(c_0+\gamma+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)})r^{d-3+\beta+\alpha+\frac{n}{q}}.\end{aligned}$$

If $p\le n$, we may take any q with $n/2<q<p$. Then we have $\dfrac{pq}{p-q}>p$ and hence

$$\|a_{ij}-a_{ij}(0)\|_{L^{\frac{pq}{p-q}}(B_r)}\le C\|a_{ij}-a_{ij}(0)\|_{L^p(B_r)}^{\frac{p-q}{q}}\le Cr^{(\alpha+\frac{n}{p})(\frac{p}{q}-1)}.$$

This implies

$$\begin{aligned}&\|\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}(u-P_1)\|_{L^q(B_r)}\\ &\le C(c_0+\gamma+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)})r^{d-3+\beta+\frac{\alpha(p-q)}{q}+\frac{n}{q}}.\end{aligned}$$

In both cases, we conclude for some $\alpha_1\le\alpha$ and some q with $n/2<q<p$

$$\begin{aligned}&\|\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}\psi\|_{L^q(B_r)}\\ &\le C(c_0+\gamma+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B1)})r^{d-3+\beta+\alpha_1+\frac{n}{q}},\end{aligned}\tag{4.2.27}$$

for any $r\le R$.

If $\alpha_1+\beta<1$, we apply Corollary 4.2.3 to $u-P_1$ with p replaced by q. By (4.2.25)-(4.2.27), there exists a polynomial P_0 with degree $\le d-1$ such that

$$\begin{aligned}&|u(x)-P_1(x)-P_0(x)|\\ &\le C\left\{c_0+\gamma+\|Q\|_{L^p(B_1)}+\|u\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)}\right\}|x|^{d-1+\beta+\alpha_1},\end{aligned}$$

or

$$\begin{aligned}&|u(x)-P_0(x)|\\ &\le C\left\{c_0+\gamma+\|u\|_{L^p(B_1)}+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)}\right\}|x|^{d-1+\beta+\alpha_1},\end{aligned}$$

for any $x\in B_R$. By (4.2.16), this implies $P_0\equiv 0$ and

$$\limsup_{r\to 0}\frac{1}{r^{d-1+\beta+\alpha_1+\frac{n}{p}}}\|u\|_{L^p(B_r)}<\infty.$$

Hence $c_1<\infty$. This is an improvement, compared with (4.2.16). We can repeat the above argument. We assume, by choosing a smaller β if necessary, that

$$k\alpha_1+\beta<1<(k+1)\alpha_1+\beta=\alpha_0$$

for some nonnegative integer k. By repeating the above argument k times, we obtain

$$\limsup_{r\to 0}\frac{1}{r^{d-1+k\beta+\alpha_1+\frac{n}{p}}}\|u\|_{L^p(B_r)}<\infty.$$

Then we get instead of (4.2.27)

$$\begin{aligned}&\|\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}\psi\|_{L^q(B_r)}\\ &\le C(c_k+\gamma+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)})r^{d-m+\alpha_0+\frac{n}{q}},\end{aligned}$$

for any $r\le R$. By Corollary 4.2.3 again, there exists a polynomial P_2 of degree $\le d$ with $\mathcal{L}_0P_2=0$ such that

$$\begin{aligned}&|u(x)-P_1(x)-P_2(x)|\\ &\le C\left\{c_0+\gamma+\|u\|_{L^p(B_1)}+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)}\right\}|x|^{d+\alpha_0},\end{aligned}$$

for any $x\in B_R$. Set $P=P_1+P_2$. Then $\mathcal{L}_0P=Q$ and (4.2.16) implies that P is homogeneous of degree d. In particular, we have

$$\limsup_{r\to 0}\frac{1}{r^{d+\frac{n}{p}}}\|u\|_{L^p(B_r)}<\infty. \tag{4.2.28}$$

Now set

$$\tilde c=\sup_{r\le 0}\frac{1}{r^{d+\frac{n}{p}}}\|u\|_{L^p(B_r)}<\infty.$$

By essentially the same argument, we obtain for any $x\in B_R$

$$\begin{aligned}&|u(x)-P(x)|\\ &\le C\left\{\tilde c+\gamma+\|u\|_{L^p(B_1)}+\|Q\|_{L^p(B_1)}+\|P_1\|_{L^p(B_1)}\right\}|x|^{d+\alpha}.\end{aligned} \tag{4.2.29}$$

Step 2. Estimates of P and $u-P$.

We prove the required estimates under the additional assumption

$$|a_{ij}(x)-a_{ij}(0)|+|b_i(x)|+|c(x)|\le\eta\quad\text{for any } x\in B_1, \tag{4.2.30}$$

for some small $\eta>0$, depending only on $n,p,d,\ \lambda,\ \kappa,\alpha,\ K$ and ω. The general case can be recovered by a simple transformation $x\mapsto Rx$ for an appropriate $R\in(0,1)$.

Set $\psi=u-P$ and

$$\delta=\sup_{r\le 1}\frac{\|\psi\|_{L^p(B_r)}}{r^{d+\alpha+\frac{n}{p}}}. \tag{4.2.31}$$

This is finite by (4.2.29). Then we write $\mathcal{L}u=f$ as

$$\mathcal{L}\psi=f-\mathcal{L}P=(f-Q)+(\mathcal{L}_0-\mathcal{L})P,$$

since $\mathcal{L}_0P=Q$. By Lemma 4.2.4, we have for any $r\le R$

$$\begin{aligned}&\sum_{i=1}^{2}r^i\|D^i\psi\|_{L^p(B_r)}\\ &\le C(\|\psi\|_{L^p(B_{2r})}+r^m\|f-Q\|_{L^p(B_{2r})}+r^m\|(\mathcal{L}_0-\mathcal{L})P\|_{L^p(B_{2r})})\\ &\le C(\delta+\gamma+\|P\|_{L^p(B_1)})r^{d+\alpha+\frac{n}{p}}.\end{aligned} \tag{4.2.32}$$

We write $\mathcal{L}u=f$ as

$$\mathcal{L}_0\psi=-\big(a_{ij}-a_{ij}(0)\big)\partial_{ij}\psi-b_i\partial_i\psi-c\psi+(f-Q)+(\mathcal{L}_0-\mathcal{L})P\equiv F.$$

Then we have by (4.2.30) for any $r \le R$

$$\|F\|_{L^p(B_r)} \le C(\eta\delta + \gamma + \|P\|_{L^p(B_1)})r^{d-2+\alpha+\frac{n}{p}}. \tag{4.2.33}$$

Now we apply Corollary 4.2.3 to obtain a polynomial $\tilde{P}$ of degree $\le d$ such that

$$\|\psi - \tilde{P}\|_{L^p(B_r)} \le C(\eta\delta + \gamma + \|\psi\|_{L^p(B_1)} + \|P\|_{L^p(B_1)})r^{d+\alpha+\frac{n}{p}} \quad \text{for } r \le R.$$

Condition (4.2.31) implies $\tilde{P} \equiv 0$. Hence we have

$$\|\psi\|_{L^p(B_r)} \le C(\eta\delta + \gamma + \|\psi\|_{L^p(B_1)} + \|P\|_{L^p(B_1)})r^{d+\alpha+\frac{n}{p}} \quad \text{for } r \le R.$$

It is obviously true for $R \le r \le 1$. By taking the supremum over $r \in (0, 1]$, we get

$$\delta \le C(\eta\delta + \gamma + \|\psi\|_{L^p(B_1)} + \|P\|_{L^p(B_1)}).$$

If η is small such that $C\eta \le 1/2$, we have

$$\delta \le C(\gamma + \|P\|_{L^p(B_1)} + \|\psi\|_{L^p(B_1)}),$$

or

$$\|\psi\|_{L^p(B_r)} \le C(\gamma + \|P\|_{L^p(B_1)} + \|\psi\|_{L^p(B_1)})r^{d+\alpha+\frac{n}{p}} \quad \text{for } r \le 1.$$

By (4.2.33), we get

$$\|F\|_{L^p(B_r)} \le C(\gamma + \|P\|_{L^p(B_1)} + \|\psi\|_{L^p(B_1)})r^{d-2+\alpha+\frac{n}{p}}.$$

Hence Corollary 4.2.3 implies

$$|\psi(x)| \le C(\gamma + \|P\|_{L^p(B_1)} + \|\psi\|_{L^p(B_1)})|x|^{d+\alpha} \quad \text{in } B_R. \tag{4.2.34}$$

By the definition of ψ, we have

$$|P(x)| \le |u(x)| + C(\gamma + \|u\|_{L^p(B_1)} + |P|_{L^\infty(B_1)})|x|^{d+\alpha} \quad \text{in } B_R.$$

Again interior estimates imply

$$|u(x)| \le C(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)}) \quad \text{in } B_R.$$

We then obtain

$$|P(x)| \le C(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)}) + C|P|_{L^\infty(B_1)}|x|^{d+\alpha} \quad \text{in } B_R.$$

Suppose P restricted to $\mathbb{S}^{n-1}$ attains its maximum at e. Choose $x = re$. Then we get by the homogeneity of P

$$|P|_{L^\infty(B_1)}r^d \le C(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)}) + C|P|_{L^\infty(B_1)}r^{d+\alpha}.$$

Choosing r small, we obtain

$$|P|_{L^\infty(B_1)} \le C(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)}),$$

or

$$|P(x)| \le C(\gamma + \|u\|_{L^p(B_1)} + \|Q\|_{L^p(B_1)})|x|^d.$$

This is (4.2.19). With (4.2.34), we get (4.2.20). Then (4.2.21) follows from interior estimates. □

REMARK 4.2.6. We can prove a similar result for integer $d < 2$. Except $f \in L^p(B_1)$, we do not need any extra assumptions on f. Then (4.2.18) holds with $Q \equiv 0$ and (4.2.19)-(4.2.21) hold with $\gamma = \|f\|_{L^p(B_1)}$.

Not only do we have a priori estimates on higher order derivatives, we can also prove the continuity of these derivatives. Since "derivatives"of solutions are given by coefficients of polynomials, we just need to prove the convergence of such polynomials. In the following, we compare leading polynomials of two solutions. For $i = 1, 2$, we suppose that $\mathcal{L}_i$ is a second order elliptic operator as in Theorem 4.2.5 and that u_i is a $W^{2,p}$ solution of $\mathcal{L}_i u_i = f_i$ in B_1 for some $f_i \in L^p(B_1)$ with $p > n/2$. Suppose, for some homogeneous polynomial Q_i of degree $d-2$, f_i satisfies

$$\|f_i - Q_i\|_{L^p(B_r)} \le \gamma_i r^{d-2+\alpha+\frac{n}{p}} \quad \text{for any } r \le 1,$$

for some positive constants $\gamma_i > 0$. We assume

$$\limsup_{r\to 0} \frac{1}{r^{d+\frac{n}{p}}} \|u_i\|_{L^p(B_r)} < \infty,$$

and P_i is the homogeneous polynomial of degree d given by Theorem 4.2.5.

We subtract two equations to get

$$\mathcal{L}_2(u_2 - u_1) = (\mathcal{L}_1 - \mathcal{L}_2)u_1 + (f_2 - f_1) \equiv F. \tag{4.2.35}$$

We write $u_1 = P_1 + \psi_1$. By (4.2.19)-(4.2.21), we have

$$|P_1(x)| \le C(\gamma_1 + \|u_1\|_{L^p(B_1)} + \|Q_1\|_{L^p(B_1)})|x|^d \quad \text{in } B_1,$$

and

$$\begin{aligned}&\sum_{i=0}^{2} r^i \|D^i \psi_1\|_{L^p(B_r(0))}\\ \le& C(\gamma_1 + \|u_1\|_{L^p(B_1)} + \|Q_1\|_{L^p(B_1)}) r^{d+\alpha+\frac{n}{p}} \quad \text{for any } r \le R.\end{aligned}$$

The function F begins with the homogeneous polynomial of degree $d-2$

$$Q = \big(a_{2,ij}(0) - a_{1,ij}(0)\big)\partial_{ij} P_1 + (Q_2 - Q_1),$$

and the error term has the estimate

$$\begin{aligned}\|F - Q\|_{L^p(B_r)} \le |\mathcal{L}_1 - \mathcal{L}_2| \bigg(\sum_{i=0}^{2} r^i \|D^i \psi_1\|_{L^p(B_r)} + \sum_{i=0}^{1} r^i \|D^i P_1\|_{L^p(B_r)} \bigg)\\ + \|(f_1 - Q_1) - (f_2 - Q_2)\|_{L^p(B_r)},\end{aligned}$$

where $|\mathcal{L}_1 - \mathcal{L}_2|$ denotes the maximal difference of corresponding coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$. Set γ such that

$$\|(f_1 - Q_1) - (f_2 - Q_2)\|_{L^p(B_r)} \le \gamma r^{d-2+\alpha+\frac{n}{p}} \quad \text{for any } r \le 1.$$

Hence, we have

$$\begin{aligned}\|F - Q\|_{L^p(B_r)} \le& \Big\{ \gamma + C|\mathcal{L}_1 - \mathcal{L}_2|(\gamma_1 + \|u_1\|_{L^p(B_1)}\\ &+ \|Q_1\|_{L^p(B_1)}) \Big\} r^{d-2+\alpha+\frac{n}{p}} \quad \text{for any } r \le R.\end{aligned}$$

We apply Theorem 4.2.5 to (4.2.35) to obtain

$$|P_1(x) - P_2(x)| \le C_* |x|^d \quad \text{in } B_1,$$

and

$$\big|\big(u_1(x) - P_1(x)\big) - \big(u_2(x) - P_2(x)\big)\big| \le C_* |x|^{d+\alpha} \quad \text{in } B_{\frac{R}{2}},$$

where C_* is a positive constant satisfying

$$C_* \le C\Big\{\gamma + \|u_1 - u_2\|_{L^p(B_1)} + \|Q_1 - Q_2\|_{L^p(B_1)} \\ + |\mathcal{L}_1 - \mathcal{L}_2|(\gamma_1 + \|u_1\|_{L^p(B_1)} + \|Q_1\|_{L^p(B_1)})\Big\},$$

for a constant C depending only on n, p, d, λ, α, κ, K and ω.

Hence we have the following result.

Corollary 4.2.7. *Let $\{\mathcal{L}_i\}_{i=0}^\infty$ be a sequence of second order elliptic operators in B_1 with the form (4.2.10) satisfying (4.2.11)-(4.2.14) and $\{u_i\}_{i=0}^\infty$ a sequence of $W^{2,p}$ functions such that each u_i is a solution of $\mathcal{L}_i u_i = f_i$ in B_1 for some $f_i \in L^p(B_1)$ with $p > n/2$. Suppose that $\mathcal{L}_i \to \mathcal{L}_0$ as $i \to \infty$ in the sense that the corresponding coefficients converge in the sup-norm and that, for a sequence of homogeneous polynomials $\{Q_i\}_{i=0}^\infty$ of degree $d-2$, f_i satisfies*

$$\sup_{r\ge 1} \frac{1}{r^{d-2+\alpha+\frac{n}{p}}} \|f_0 - Q_0\|_{L^p(B_r)} < \infty,$$

and

$$\|Q_i - Q_0\|_{L^p(B_1)} + \sup_{r\ge 1} \frac{1}{r^{d-2+\alpha+\frac{n}{p}}} \|(f_i - Q_i) - (f_0 - Q_0)\|_{L^p(B_r)} \to 0 \\ \text{as } i \to \infty,$$

for some integer $d \ge 2$. If

$$\limsup_{r\to 0} \frac{1}{r^{d+\frac{n}{p}}} \|u_i\|_{L^p(B_r)} < \infty \quad \text{for any } i = 1, 2, \cdots,$$

and

$$u_i \to u_0 \quad \text{in } L^p(B_1) \text{ as } i \to \infty,$$

then

$$\sup_{|x|<\frac{1}{2}} \frac{1}{|x|^d} |u_i(x) - u_0(x)| \to 0 \quad \text{as } i \to \infty.$$

Moreover, if $\{P_i\}_{i=0}^\infty$ is the sequence of homogeneous polynomials of degree d for $\{u_i\}_{i=0}^\infty$ as in Theorem 4.2.5, then

$$|P_i - P_0|_{L^\infty(B_1)} + \sup_{|x|<\frac{1}{2}} \frac{1}{|x|^{d+\alpha}} \big|\big(u_i(x) - P_i(x)\big) - \big(u_0(x) - P_0(x)\big)\big| \\ \to 0 \quad \text{as } i \to \infty.$$

In the rest of this section, we prove a pointwise Schauder estimate. The result to be established is not employed directly in this book. We first introduce some terminology.

Definition 4.2.8. Let u be an L^p function in B_1 for $1 \le p \le \infty$. For $\alpha \in (0,1)$ and d a nonnegative integer,

(i) u is C^d at 0 in the L^p sense, $u \in C^d_{L^p}(0)$, for $d \ge 1$, if for some polynomial P of degree not exceeding $d-1$,

$$\limsup_{r\to 0} \frac{1}{r^{d+\frac{n}{p}}} \|u - P\|_{L^p(B_r)} < \infty;$$

(ii) u is $C^{d,\alpha}$ at 0 in the L^p sense, $u \in C^{d,\alpha}_{L^p}(0)$, if for some polynomial Q of degree not exceeding d,

$$\limsup_{r\to 0} \frac{1}{r^{d+\alpha+\frac{n}{p}}}\|u-Q\|_{L^p(B_r)} < \infty.$$

In Definition 4.2.8, the polynomials P and Q are unique if they exist. It is easy to check that $u \in C^{d,\alpha}_{L^p}(0)$ implies $u \in C^{d',\alpha'}_{L^p}(0)$ for any $\alpha, \alpha' \in [0,1)$ and any integers d, d' with $d'+\alpha' < d+\alpha$.

We also define the corresponding semi-norms as follows:

$$[u]_{C^d_{L^p}}(0) = \inf_P \sup_{0<r\le 1} \frac{1}{r^{d+\frac{n}{p}}}\|u-P\|_{L^p(B_r)},$$
$$[u]_{C^{d,\alpha}_{L^p}}(0) = \inf_Q \sup_{0<r\le 1} \frac{1}{r^{d+\alpha+\frac{n}{p}}}\|u-Q\|_{L^p(B_r)},$$

where the infimums are taken in the space of polynomials P of degree at most $d-1$ and the space of polynomials Q of degree at most d, respectively. For the completeness, we define

$$[u]_{C^0_{L^p}}(0) = \sup_{0<r\le 1} \frac{1}{r^{\frac{n}{p}}}\|u\|_{L^p(B_r)}.$$

We check easily that, for any integers $d \ge 0$, $l \ge 1$ and any constants $\alpha, \beta \in (0,1)$, if $u \in C^{d+l,\alpha}(0)$, then

$$\sum_{k=1}^{l}[u]_{C^{d+k}_{L^p}}(0) + \sum_{k=0}^{l-1}[u]_{C^{d+k,\beta}_{L^p}}(0) \le C\{[u]_{C^d_{L^p}}(0) + [u]_{C^{d+l,\alpha}_{L^p}}(0)\},$$

where C is a positive constant depending only on n, p, d, l, α and β.

We first discuss a special case. For the following result, it is convenient to write

$$\sum_{|\nu|=0}^{2} a_\nu \partial^\nu = a_{ij}\partial_{ij} + b_i\partial_i + c.$$

THEOREM 4.2.9. *Let $\mathcal{L}$ be a second order ellptic operator in B_1 with the form (4.2.10) satisfying (4.2.11)-(4.2.13) and u a $W^{2,p}$ solution of $\mathcal{L}u = f$ in B_1 for some $f \in L^p(B_1)$ with $p > n/2$. Suppose that $d \ge 2$ is a nonnegative integer such that*

$$\limsup_{r\to 0} \frac{1}{r^{d+\frac{n}{p}}}\|u\|_{L^p(B_r)} < \infty,$$
$$\limsup_{r\to 0} \frac{1}{r^{d-2+\frac{n}{p}}}\|f\|_{L^p(B_r)} < \infty.$$

If, for some $\alpha \in (0,1)$ and some integer $l \ge 0$, $f \in C^{d-2+l+\alpha}_{L^p}(0)$ and $a_\nu \in C^{l-2+|\nu|+\alpha}_{L^p}(0)$ for any $|\nu| \ge \max\{2-l, 0\}$, then $u \in C^{d+l+\alpha}_{L^\infty}(0)$. Moreover,

$$\begin{aligned}&\sum_{k=d}^{d+l}[u]_{C^k_{L^\infty}}(0) + [u]_{C^{d+l,\alpha}_{L^\infty}}(0)\\ \le& C\left\{\|u\|_{L^p(B_1)} + \sum_{k=d-2}^{d-2+l}[f]_{C^k_{L^p}}(0) + [f]_{C^{d-2+l+\alpha}_{L^p}}(0)\right\},\end{aligned}$$

where C *is a positive constant depending only on* $n, p, d, l, \lambda, \kappa, \alpha, \omega$, $[a_\nu]_{C^k_{L^p}}(0)$ *for* $k = d-2, \cdots, d-2+l$ *and* $[a_\nu]_{C^{l-2+|\nu|+\alpha}_{L^p}}(0)$ *for* $|\nu| \geq 2-l$.

The assumptions on coefficients can be put in the following form

$$\begin{aligned} &\text{for } l = 0: \quad a_{ij} \in C^{\alpha}_{L^p}(0);\\ &\text{for } l = 1: \quad a_{ij} \in C^{1+\alpha}_{L^p}(0), b_i \in C^{\alpha}_{L^p}(0);\\ &\text{for } l \geq 2: \quad a_{ij} \in C^{l+\alpha}_{L^p}(0), b_i \in C^{l-1+\alpha}_{L^p}(0), c \in C^{l-2+\alpha}_{L^p}(0).\end{aligned}$$

Proof. We prove Theorem 4.2.9 by an induction on l. For $l = 0$, it is Theorem 4.2.5. Note that (4.2.14) is equivalent to $a_{ij} \in C^{\alpha}_{L^p}(0)$. For an illustration, we prove for $l = 1$. By assumptions, there exist homogeneous polynomials Q_{d-2} and Q_{d-1} of degrees $d-2$ and $d-1$ respectively, homogeneous polynomials $a^{(0)}_{ij}, b^{(0)}_i$ and $a^{(1)}_{ij}$ of degree 0 and 1, such that

$$\begin{aligned} \|f - Q_{d-2} - Q_{d-1}\|_{L^p(B_r)} &\leq [f]_{C^{d-1+\alpha}_{L^p}}(0) r^{d-1+\alpha+\frac{n}{p}},\\ \|a_{ij} - a^{(0)}_{ij} - a^{(1)}_{ij}\|_{L^p(B_r)} &\leq [a_{ij}]_{C^{1+\alpha}_{L^p}}(0) r^{1+\alpha+\frac{n}{p}},\\ \|b_i - b^{(0)}_i\|_{L^p(B_r)} &\leq [b_i]_{C^{\alpha}_{L^p}}(0) r^{\alpha+\frac{n}{p}},\end{aligned}$$

for any $r \leq 1$. Write $\phi = f - Q_{d-2} - Q_{d-1}$. By Theorem 4.2.11, there exists a homogeneous polynomial P_d satisfying (4.2.18)-(4.2.21). Set $\psi = u - P_d$. Then ψ satisfies

$$\mathcal{L}\psi = Q_{d-1} + \phi + \big(a_{ij} - a_{ij}(0)\big)\partial_{ij}P_d + b_i\partial_i P_d + cP_d \equiv \tilde{f}.$$

It is easy to see that the polynomial

$$\tilde{Q} = Q_{d-1} + a^{(1)}_{ij}\partial_{ij}P_d + b^{(0)}_i\partial_i P_d$$

is homogeneous with degree $d-1$ satisfying

$$\|\tilde{f} - \tilde{Q}\|_{L^p(B_r)} \leq \Big\{[f]_{C^{d-1+\alpha}_{L^p}}(0) + C\|P_d\|_{L^p(B_1)}\Big\} r^{d-1+\alpha+\frac{n}{p}},$$

for any $r \leq 1$. By (4.2.21), we have

$$\limsup_{r\to 0} \frac{1}{r^{d+\alpha+\frac{n}{p}}}\|\psi\|_{L^p(B_r)} < \infty.$$

We apply Theorem 4.2.5 to ψ with d replaced by $d+1$. Hence, there exists a homogeneous polynomial P_{d+1} of degree $d+1$ satisfying

$$a_{ij}(0)\partial_{ij}P_{d+1} = \tilde{Q},$$

and

$$\begin{aligned} |P_{d+1}(x)| \leq& C_*|(x)|^{d+1} \quad \text{for } x \in B_{R/2},\\ |\psi(x) - P_{d+1}(x)| \leq& C_*|x|^{d+1+\alpha} \quad \text{for } x \in B_{R/2},\end{aligned}$$

where

$$C_* \leq C([f]_{C^{d-1+\alpha}_{L^p}}(0) + \|\psi\|_{L^p(B_R)} + \|\tilde{P}_d\|_{L^p(B_1)} + \|\tilde{Q}\|_{L^p(B_1)}).$$

By the expression for $\tilde{Q}$ and estimates on ψ and P_d, we obtain

$$\begin{aligned}C_* \le & C\bigg([f]_{C^{d-1+\alpha}_{L^p}}(0) + [f]_{C^{d-2+\alpha}_{L^p}}(0) \\ & \quad + \|Q_{d-2}\|_{L^p(B_{\frac{1}{2}})} + \|Q_{d-1}\|_{L^p(B_1)} + \|u\|_{L^p(B_1)}\bigg) \\ \le & C\bigg(\|u\|_{L^p(B_1)} + [f]_{C^{d-2}_{L^p}}(0) + [f]_{C^{d-1}_{L^p}}(0) + [f]_{C^{d-1+\alpha}_{L^p}}(0)\bigg).\end{aligned}$$

This finishes the proof for $l = 1$. $\square$

Theorem 4.2.10. *Let $\mathcal{L}$ be a second order elliptic operator in B_1 with the form (4.2.10) satisfying (4.2.11)-(4.2.13) and let u be a $W^{2,p}$ solution of $\mathcal{L}u = f$ in B_1 for some $f \in L^p(B_1)$ with $n < p \le \infty$. If, for some constant $\alpha \in (0,1)$ and some integer $d \ge 2$, $f \in C^{d-2,\alpha}_{L^p}(0)$ and $a_{ij}, b_i, c \in C^{d-2,\alpha}_{L^p}(0)$, then $u \in C^{d,\alpha}_{L^\infty}(0)$. Moreover,*

$$[u]_{C^{d,\alpha}_{L^\infty}}(0) \le C\bigg\{\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)} + [f]_{C^{d-2,\alpha}_{L^p}}(0)\bigg\},$$

where C depends only on $n, p, d, \lambda, \kappa, \alpha, \omega$ and the $C^{d-2,\alpha}_{L^p}(0)$-norms of a_{ij}, b_i, c.

The requirement on p in Theorem 4.2.10 is stronger than that in Theorem 4.2.9.

Proof. By $W^{2,p}$ estimates and the Sobolev embedding, we have

$$|u(x) - P_*(x)| \le C\left\{\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}\right\}|x|^{2-\frac{n}{p}} \quad \text{for any } |x| < \frac{1}{2},$$

where P_* is a polynomial of degree not greater than 1 and satisfies

$$\sum_{i=0}^{1} |D^i P_*(0)| \le C\left\{\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}\right\}.$$

Consider the equation satisfied by $u - P_*$

$$L(u - P_*) = f - b_i\partial_i P_* - cP_* \equiv F.$$

It is straightforward to check that $u - P_*$ and F satisfy the assumptions in Theorem 4.2.9 for u and f, with d and l in Theorem 4.2.9 replaced by 2 and $d-2$. Hence Theorem 4.2.10 follows from Theorem 4.2.9 directly. $\square$

Remark 4.2.11. Both Theorem 4.2.9 and Theorem 4.2.10 hold for integer d with $d = 1$. No extra assumptions are needed for coefficients a_{ij}, b_i, c and nonhomogeneous term f. Only $\|f\|_{L^p(B_1)}$ appears in the right side of the estimates.

4.3. Singular Sets of Non-Smooth Solutions

In this section, we study the structure of singular sets of non-smooth solutions. The uniform asymptotic expansion discussed in the previous section plays an important role. First, we generalize the notion of the vanishing orders and leading polynomials to nonsmooth solutions.

We consider the homogeneous elliptic equation of the form

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u = 0 \quad \text{in } B_1 \subset \mathbb{R}^n, \tag{4.3.1}$$

where the coefficients satisfy the following assumptions

$$\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^n,\ x \in B_1, \\ &\sum_{i,j=1}^{n} |a_{ij}(x)| + \sum_{i=1}^{n} |b_i(x)| + |c(x)| \leq \kappa, \quad \text{for any } x \in B_1, \end{aligned} \tag{4.3.2}$$

and

$$\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| \leq \Gamma|x-y|, \quad \text{for any } x, y \in B_1, \tag{4.3.3}$$

for some positive constants λ, κ and Γ. The Lipschitz condition (4.3.3) for the leading coefficients is essential. It implies the unique continuation for the operator $\mathcal{L}$. In other words, if a solution u of (4.3.1) vanishes to an infinite order at a point in B_1, then u is identically zero. For details, see Chapter 3.

Suppose u is a C^2 solution of (4.3.1) in B_1. By the unique continuation, for any $p \in B_1$, there exists an integer d satisfying

$$\limsup_{x \to p} \frac{|u(x)|}{|x-p|^d} < \infty,$$
$$\limsup_{x \to p} \frac{|u(x)|}{|x-p|^{d+1}} = \infty.$$

Then by Theorem 4.2.5, there exists a homogeneous polynomial P of degree d such that

$$u(x) = P(x-p) + O(|x-p|^{d+\alpha}).$$

In fact, we also have a priori estimates on P and the error term. We claim that P is not identical zero. Otherwise, by Theorem 4.2.5 $u(x) = O(|x-p|^{d+\alpha})$ implies $u(x) = O(|x-p|^{d+1})$, which is a contradiction. Naturally the integer d, the degree of the polynomial, is called *the vanishing order* of u at p, denoted by $\mathcal{O}(p)$ or $\mathcal{O}_u(p)$. For convenience, we call the nonzero homogeneous polynomial P *the leading polynomial* of u at p.

LEMMA 4.3.1. *Suppose $\{\mathcal{L}_k\}_{k=0}^{\infty}$ is a family of second order elliptic operators in B_1 of form (4.2.10) satisfying (4.2.11)-(4.2.14) and that u_k is a $W^{2,p}$ solution of $\mathcal{L}_k u_k = 0$ in B_1 for $k = 0, 1, 2, \cdots$, and for some $p > n/2$. Suppose that $\mathcal{L}_k \to \mathcal{L}_0$ in the sense that the corresponding coefficients converge uniformly and that $u_k \to u_0$ in $L^p(B_1)$. Then*

$$\limsup_{k \to \infty} \mathcal{O}_{u_k}(0) \leq \mathcal{O}_{u_0}(0). \tag{4.3.4}$$

If, in addition, $\mathcal{O}_{u_k}(0) = d$ and P_k is the leading polynomial of u_k at 0 for $k = 1, 2, \cdots$, then the following conclusions hold:

(i) if $\mathcal{O}_{u_0}(0) > d$, then

$$P_k \to 0 \quad \textit{uniformly in } B_1 \textit{ as } k \to \infty;$$

(ii) if $\mathcal{O}_{u_0}(0) = d$, then

$$P_k \to P_0 \quad \textit{uniformly in } B_1 \textit{ as } k \to \infty,$$

where P_0 is the leading polynomial of u_0 at 0.

PROOF. We prove (4.3.4) only for the case $\mathcal{O}_{u_k}(0) = d$ for $k = 1, 2, \cdots$. By Theorem 4.2.5, we have for $k = 1, 2, \cdots$,

$$|u_k(x)| \le C|x|^d \|u_k\|_{L^p(B_1)} \quad \text{in } B_R, \tag{4.3.5}$$

for some $R < 1$, where C and R are independent of k. By the Sobolev embedding theorem, $u_k \to u_0$ in $L^\infty(B_1)$ as $k \to \infty$. Hence by taking the limit, it is easy to see that u vanishes at 0 with the order at least d. This proves (4.3.4), and hence (4.3.5) holds also for $k = 0$. Moreover, by interior estimates, we get

$$\sum_{i=1}^{2} r^i \|D^i u_0\|_{L^P(B_r)} \le C r^{d+\frac{n}{p}} \|u_0\|_{L^p(B_1)} \quad \text{for any } r \le R.$$

This implies

$$\|(\mathcal{L}_k - \mathcal{L}_0)u_0\|_{L^P(B_r)} \le C r^{d-2+\frac{n}{p}} |\mathcal{L}_k - \mathcal{L}_0| \|u_0\|_{L^p(B_1)} \quad \text{for any } r \le R,$$

where $|\mathcal{L}_k - \mathcal{L}_0|$ denotes the maximal difference of corresponding coefficients of $\mathcal{L}_k$ and $\mathcal{L}_0$. Note $|\mathcal{L}_k - \mathcal{L}_0| \to 0$ as $k \to \infty$.

It is obvious that $u_k - u_0$ vanishes at 0 with an order at least d and satisfies

$$\mathcal{L}_k(u_k - u_0) = (\mathcal{L}_0 - \mathcal{L}_k)u_0 \quad \text{in } B_1.$$

By Theorem 4.2.5 again, we get

$$|u_k(x) - u_0(x)| \le C|x|^d \{|\mathcal{L}_k - \mathcal{L}_0| \|u_0\|_{L^p(B_1)} + \|u_k - u_0\|_{L^p(B_1)}\} \quad \text{in } B_R.$$

If u_0 vanishes at 0 with an order more than d, we let $P_0 \equiv 0$. If u_0 vanishes at 0 with order d, we let P_0 be the leading polynomial of u_0. In both cases $u_k - u_0$ begin with a polynomial $P_k - P_0$ of degree d. By Theorem 4.2.5, we obtain

$$|P_k(x) - P_0(x)| \le C|x|^d \{|\mathcal{L}_k - \mathcal{L}_0| \|u_0\|_{L^p(B_1)} + \|u_k - u_0\|_{L^p(B_1)}\} \quad \text{in } B_R.$$

This implies that $P_k \to P_0$ in $C^d(B_1)$ as $k \to \infty$. □

Now we state the main result concerning the structure of singular sets in this section, which was proved by Han [**43**].

THEOREM 4.3.2. *Let u be an H^2-solution of (4.3.1) in B_1 with (4.3.2)-(4.3.3). Then there exists the following decomposition*

$$\mathcal{S}(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^j(u),$$

where each $\mathcal{S}^j(u)$ is on a countable union of j-dimensional C^1 manifolds, $j = 0, 1, \cdots, n-3$, and $\mathcal{S}^{n-2}(u)$ is on a countable union of $(n-2)$-dimensional $C^{1,\beta}$ manifolds for some $\beta \in (0,1)$.

PROOF. The proof is similar to that of Theorem 4.1.3, with Lemma 4.1.2 replaced by Lemma 4.3.1. In the proof of Step 3, instead of (4.1.11) and (4.1.12), we have by (4.2.21) and the Sobolev embedding theorem, for a fixed $\alpha \in (0,1)$,

$$|x^1|^{d-1} \le C|x|^{d-1+\alpha},$$

or

$$|x^1| \le C|x|^{1+\frac{\alpha}{d-1}} \tag{4.3.6}$$

for some constant $C > 0$. □

Similar to Corollary 4.1.4, we also have the following result.

Corollary 4.3.3. *Let u be a solution as in Theorem 4.3.2. Then*

$$\mathcal{S}(u)=\mathcal{S}_g(u)\bigcup\mathcal{S}_b(u),$$

where the Hausdorff dimension of $\mathcal{S}_b(u)$ is at most $n-3$ and $\mathcal{S}_g(u)$ is on a countable union of $(n-2)$-dimensional $C^{1,\beta}$ manifolds for some $\beta\in(0,1)$. Moreover, for any $p\in\mathcal{S}_g(u)$ the leading polynomial of u at p is a polynomial of two variables after some rotation of coordinates.

CHAPTER 5

Measure Estimates of Nodal Sets of Solutions

In this chapter, we discuss measure estimates of nodal sets of solutions of linear elliptic equations of the second order. In Section 2.3, we proved the following result for harmonic functions. Suppose u is a harmonic function in $B_1 \subset \mathbb{R}^n$ with

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}.$$

Then

$$\mathcal{H}^{n-1}\left\{x \in B_{\frac{1}{2}};\ u(x) = 0\right\} \le cN,$$

where c is a positive constant depending only on n. The quantity N is the frequency and it controls the local growth of harmonic functions. In this chapter, we generalize this result to solutions of general linear elliptic differential equations of the second order.

Suppose u is a nonzero solution of the second-order linear elliptic differential equation of the form

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{x_i x_j} u + \sum_{i=1}^{n} b_i(x) D_{x_i} u + c(x)u = 0 \quad \text{in } B_1 \subset R^n.$$

We assume that the leading coefficients are Lipschitz and all other coefficients are bounded. As for harmonic functions, we introduce the frequency of u as before by

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}.$$

We proved in Section 3.2 that u satisfies the following doubling condition

$$\left(\fint_{B_{2r}(p)} u^2\right)^{\frac{1}{2}} \le 2^{c_1 N + c_2} \left(\fint_{B_r(p)} u^2\right)^{\frac{1}{2}},$$

for any $p \in B_{1/2}$ and $r < r_0$, where c_1, c_2 and r_0 are positive constants depending only on n, the ellipticity constant, the Lipschitz constant for leading coefficients and the L^∞-norms of all coefficients. This implies in particular that u cannot vanish to order more than $c_1 N + c_2$. Hence u is approximated by polynomials of degree not more than $c_1 N + c_2$, if u is smooth.

The following conjecture was proposed by Lin [**69**], motivated by the desire to understand to what extent the nodal sets of solutions can be described quantitatively by those of harmonic polynomials.

Conjecture 5.0.1. There holds

$$\mathcal{H}^{n-1}(\{x \in B_{\frac{1}{2}}; u(x) = 0\}) \le CN,$$

where C is a positive constant depending only on the coefficients.

In this chapter, we prove that Conjecture 5.0.1 holds for the analytic case. We also present a partial result for the smooth case.

5.1. Nodal Sets of Analytic Solutions

In this section, we will discuss nodal sets of solutions of elliptic differential equations with analytic coefficients.

Suppose that $\mathcal{L}$ is a homogeneous linear elliptic operator of the second order in $B_1 \subset \mathbb{R}^n$ given by

$$\mathcal{L} = a_{ij}(x)\partial_{x_ix_j} + b_i(x)\partial_{x_i} + c(x). \tag{5.1.1}$$

We assume

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \quad \text{for any } x \in B_1 \text{ and } \xi \in \mathbb{R}^n, \tag{5.1.2}$$

for some positive constant λ.

Throughout this section, we assume that all coefficients a_{ij}, b_i and c are analytic in B_1. A theorem due to Morrey and Nirenberg [**74**] asserts that any solution u of (5.1.1) is analytic. Moreover, there exists a positive constant $R \in (0,1)$, depending only on a_{ij}, b_i and c, such that u can be extended to a holomorphic function $\tilde{u}$ in $D_R = \{z \in \mathbb{R}^n; |z| < R\}$ and $\tilde{u}$ satisfies

$$\|\tilde{u}\|_{L^2(D_R)} \le C\|u\|_{L^2(B_1)},$$

where C is a positive constant depending only on a_{ij}, b_i and c.

In Section 2.3, we derived an optimal measure estimate of nodal sets of harmonic functions. A similar result due to Lin [**69**] holds for solutions of (5.1.1) with analytic coefficients.

Theorem 5.1.1. *Let a_{ij}, b_i and c be analytic functions in B_1 satisfying (5.1.2) and u be a solution of (5.1.1) satisfying*

$$\frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2} \le N,$$

for a positive constant N. Then

$$\mathcal{H}^{n-1}(\{x \in B_{\frac{1}{2}}; u(x) = 0\}) \le CN,$$

where C is a positive constant depending only on n, λ and the coefficients.

The proof of Theorem 5.1.1 follows closely that of Theorem 2.3.1 for harmonic functions. The doubling condition of harmonic functions plays an important role in the proof. Such a doubling condition was proved for arbitrary solutions of (5.1.1) in Chapter 3, specifically in Theorem 3.2.7. We omit details.

5.2. Nodal Sets of Non-Analytic Solutions: A Compactness Argument

In this section and the next, we will discuss nodal sets of solutions of elliptic differential equations with nonanalytic coefficients. Again, $\mathcal{L}$ is a homogeneous linear elliptic operator of the second order in $B_1 \subset \mathbb{R}^n$ of the form (5.1.1) with leading coefficients (a_{ij}) satisfying (5.1.2). Furthermore, we assume

$$\sum_{i=1}^{n} |b_i(x)| + |c(x)| \leq \kappa \quad \text{for any } x \in B_1; \tag{5.2.1}$$

$$\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| \leq \Gamma |x-y| \quad \text{for any } x, y \in B_1, \tag{5.2.2}$$

for some positive constants κ and Γ. We denote by $\mathcal{L}(\lambda, \kappa, \Gamma)$ the collection of all such linear operators. We also denote by $\mathcal{S}_N$ the collection of all H^1-functions u in B_1 satisfying

$$\frac{r\int_{B_r(p)} |Du|^2}{\int_{\partial B_r(p)} u^2} \leq N \quad \text{for any } B_r(p) \subset B_1.$$

For any $B_r(y) \subset B_1$, set

$$\omega(y; r) = \sup_{B_r(y)} (r|Da_{ij}| + r|b_i| + r^2|c|). \tag{5.2.3}$$

Obviously, we have

$$\omega(y; r) \leq r(\kappa + \Gamma) \quad \text{for any } B_r(y) \subset B_1.$$

The main result in this section is the following theorem due to Han and Lin [**49**]. The proof here follows an argument employed first by Hardt and Simon [**51**].

THEOREM 5.2.1. *Suppose $u \in \mathcal{S}_N$ is a nonzero solution of $\mathcal{L}u = 0$ in $B_1 \subset \mathbb{R}^n$ for some $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \Gamma)$. Then there exists a positive constant r_N, depending only on n, N and λ, such that*

$$\mathcal{H}^{n-1}\{x \in B_r(x_0);\, u(x) = 0\} \leq C_0 N r^{n-1} \quad \textit{for any } x_0 \in B_{\frac{1}{2}} \textit{ and } r \leq r_N, \tag{5.2.4}$$

where C_0 is a positive constant depending only on n, λ, κ and Γ.

We first describe ideas in proving Theorem 5.2.1. The nodal set $u^{-1}(0)$ is naturally decomposed into two parts, a good part, where the gradient is not zero, and a bad part, where the gradient is zero. A subset of the good part where the gradient is not small is well approximated by the nodal set of a harmonic function and hence enjoys a good measure estimate. At the same time, the rest of the good part and the bad part, where the gradient is small including zero, can be covered by finitely many balls. Such a decomposition with estimates and finite coverings can be carried out in a uniform way at any small scale. Theorem 5.2.1 is then proved by an iteration and a compactness argument.

We now prove several lemmas.

LEMMA 5.2.2. *Suppose that $\mathcal{L}$ is a linear operator in $B_1 \subset \mathbb{R}^n$ of the form (5.1.1) satisfying (5.1.2) with $a_{ij}(0) = \delta_{ij}$ and $\omega(0; 1) < \varepsilon_n$ for a small positive constant ε_n depending only on n, and that u is a solution of $\mathcal{L}u = 0$ in B_1 satisfying for some $N \geq 1$*

$$\frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2} \leq N.$$

Then there exists a harmonic function φ in $B_{3/4}$ such that

$$\frac{\frac{3}{4}\int_{B_{\frac{3}{4}}}|\nabla\varphi|^2}{\int_{\partial B_{\frac{3}{4}}}\varphi^2}\leq CN, \tag{5.2.5}$$

and for any $\alpha\in(0,1)$

$$\begin{aligned}|u|_{C^{1,\alpha}(B_{\frac{3}{4}})}+|\varphi|_{C^{1,\alpha}(B_{\frac{3}{4}})}\leq &C\left(\int_{\partial B_1}u^2\right)^{\frac{1}{2}},\\ |u-\varphi|_{C^{1,\alpha}(B_{\frac{3}{4}})}\leq &C\omega(0;1)\left(\int_{\partial B_1}u^2\right)^{\frac{1}{2}},\end{aligned} \tag{5.2.6}$$

where C is a positive constant depending only on n, α and λ.

Proof. Without loss of generality, we assume

$$\int_{\partial B_1}u^2=1.$$

Then we have by Corollary 3.2.8

$$\int_{B_1}u^2\leq C,$$

and

$$\int_{B_1}|\nabla u|^2\leq N.$$

Here, by Remark 3.2.4 we may take $r_0=1$ in Corollary 3.2.8. An interior $W^{2,p}$-estimate applied to $\mathcal{L}u=0$ in B_1 implies for any $p>1$

$$\|u\|_{W^{2,p}(B_{\frac{3}{4}})}\leq C\|u\|_{L^2(B_1)}\leq C, \tag{5.2.7}$$

where C is a positive constant depending only on n, p, λ and $\omega(0;1)$.

Now, we write (5.1.1) as

$$-\triangle u=(a_{ij}-\delta_{ij})\partial_{ij}u+b_i\partial_i u+cu,$$

and solve

$$\begin{aligned}-\triangle\varphi&=0\quad\text{in } B_{\frac{3}{4}},\\ \varphi&=u\quad\text{on } \partial B_{\frac{3}{4}}.\end{aligned} \tag{5.2.8}$$

Multiplying (5.2.8) by $\varphi-u$ and integrating in $B_{3/4}$, we obtain

$$\int_{B_{\frac{3}{4}}}\nabla(\varphi-u)\cdot\nabla\varphi=0,$$

or

$$\int_{B_{\frac{3}{4}}}|\nabla\varphi|^2=\int_{B_{\frac{3}{4}}}\nabla u\cdot\nabla\varphi.$$

By the Cauchy inequality, we have

$$\int_{B_{\frac{3}{4}}}|\nabla\varphi|^2\leq\int_{B_{\frac{3}{4}}}|\nabla u|^2.$$

This implies (5.2.5). Next, we note

$$\begin{aligned} -\triangle(u-\varphi) &= (a_{ij}-\delta_{ij})\partial_{ij}u + b_i\partial_i u + cu \equiv \tilde{f} \quad \text{in } B_{\frac{3}{4}}, \\ u-\varphi &= 0 \quad \text{on } \partial B_{\frac{3}{4}}. \end{aligned} \tag{5.2.9}$$

By (5.2.7) and the global $W^{2,p}$ estimate, we have

$$\|u-\varphi\|_{W^{2,p}(B_{\frac{3}{4}})} \le C\|\tilde{f}\|_{L^p(B_{\frac{3}{4}})} \le C\omega(0;1)\|u\|_{W^{2,p}(B_{\frac{3}{4}})} \le C\omega(0;1),$$

where C is a positive constant depending only on n and p. By taking p large and the Sobolev embedding theorem, we obtain

$$|u|_{C^{1,\alpha}(B_{\frac{3}{4}})} \le C,$$

and

$$|u-\varphi|_{C^{1,\alpha}(B_{\frac{3}{4}})} \le C\omega(0;1).$$

This implies (5.2.6) easily. ☐

Now we set

$$\mathcal{H}^1_N = \{u \in H^1(B_1); u \text{ is harmonic in } B_1, \frac{\int_{B_1}|Du|^2}{\int_{\partial B_1} u^2} \le N, \int_{\partial B_{\frac{1}{2}}} u^2 = 1\}.$$

Lemma 5.2.3. *The set $\mathcal{H}^1_N$ is compact in local L^2-norm or local C^k-norm for any $k \ge 0$.*

Proof. For any $u \in \mathcal{H}^1_N$, we have by Corollary 2.2.6

$$\int_{\partial B_1} u^2 \le c(N),$$

for a constant $c(N)$ depending only on N. This implies by Corollary 2.2.7

$$\int_{B_1} u^2 + \int_{B_1} |Du|^2 \le c(N).$$

Therefore, any sequence $\{u_m\}$ in $\mathcal{H}^1_N$ has a uniform H^1-bound in B_1. Then there exists a subsequence $\{u_{m'}\}$ and a $u \in H^1(B_1)$ such that

$$u_{m'} \to u \text{ strongly in } L^2(B_1) \quad \text{and} \quad u_{m'} \rightharpoonup u \text{ weakly in } H^1(B_1).$$

Obviously, u in a harmonic function in B_1. Moreover, by interior estimates for harmonic functions, we get for any integer $k \ge 0$ and any $r \in (0,1)$

$$u_{m'} \to u \text{ in } C^k(B_r).$$

This implies $\int_{\partial B_{1/2}} u^2 = 1$ and by Theorem 2.2.3 for each $u_{m'}$

$$\frac{r\int_{B_r}|Du|^2}{\int_{\partial B_r} u^2} = \lim_{m'\to\infty} \frac{r\int_{B_r}|Du_{m'}|^2}{\int_{\partial B_r} u_{m'}^2} \le N \quad \text{for any } r \in (0,1).$$

By Theorem 2.2.3 again, we have $u \in \mathcal{H}^1_N$ easily. ☐

Lemma 5.2.4. *For any harmonic function* $u \in \mathcal{H}_N^1$ *in* B_1*, there exist finitely many balls* $B_{r_i}(x_i)$ *with* $r_i \leq 1/2$ *such that*

$$\{x \in B_{\frac{1}{2}};\, |Du(x)| < \gamma(N)\} \subset \bigcup_i B_{r_i}(x_i), \tag{5.2.10}$$

and

$$\sum_i r_i^{n-1} < \frac{1}{2}, \tag{5.2.11}$$

where $\gamma(N)$ *is a positive constant depending only* n *and* N*.*

The proof is based on a "doubling compactness argument", the local compactness of $\{|Du| = 0\}$ for any C^1-function u and the local compactness of $\mathcal{H}_N^1$ as in Lemma 5.2.3.

Proof. Take an arbitrary $u_0 \in \mathcal{H}_N$. By Lemma 4.1.1, we have

$$\mathcal{H}^{n-1}\{|Du_0| = 0\} = 0.$$

Then there exist countably many balls $B_{r_i}(x_i)$ with $r_i \leq 1/2$ such that

$$\{|Du_0| = 0\} \subset \bigcup_i B_{r_i}(x_i), \tag{5.2.12}$$

and

$$\sum r_i^{n-1} \leq \frac{1}{2}. \tag{5.2.13}$$

By the compactness of $\{|Du_0| = 0\}$, we assume that $\{B_{r_i}(x_i)\}$, satisfying (5.2.12) and (5.2.13), consists of finitely many balls. Set

$$\gamma(u_0) = \frac{1}{3} \inf_{B_1 \setminus \cup B_{r_i}(x_i)} |Du_0|.$$

Obviously $\gamma(u_0) > 0$. Consider any $u \in \mathcal{H}_N$ such that

$$|u - u_0|_{C^1(B_{\frac{3}{4}})} < \gamma(u_0).$$

It is easy to see

$$\left\{x \in B_{\frac{1}{2}};\ |Du(x)| < \gamma(u_0)\right\} \subset \bigcup_i B_{r_i}(x_i).$$

Now the collection

$$\mathcal{B}_{\gamma(u_0)}(u_0) = \left\{u \in \mathcal{H}_N;\ |u - u_0|_{C^1(B_{\frac{3}{4}})} < \gamma(u_0)\right\}$$

is an open ball in $\mathcal{H}_N$. Note that $\mathcal{H}_N^1$ is compact under the local C^1-norm by Lemma 5.2.3. Hence, there exist $u_1, \ldots, u_m \in \mathcal{H}_N$ such that

$$\mathcal{H}_N \subset \bigcup_{i=1}^m \mathcal{B}_{\gamma(u_i)}(u_i),$$

where $m = m(N)$ is a positive constant depending only on n and N. Set

$$\gamma(N) = \min_{1 \leq i \leq m(N)} \gamma(u_i).$$

This finishes the proof. ☐

Next, we prove the following "nodal set comparison lemma".

LEMMA 5.2.5. *There exists an* $\eta_0 = \eta_0(n) \in (0, 1/2]$ *such that for any* $\eta \in (0, \eta_0]$ *and any* $w_1, w_2 \in C^{1,1/2}(B_2)$ *with*

$$|w_j|_{C^{1,1/2}} \le 1,\ j = 1, 2, \quad \text{and} \quad |w_1 - w_2|_{C^1} < \frac{1}{2}\eta^5, \tag{5.2.14}$$

there holds

$$\begin{aligned}&\mathcal{H}^{n-1}\big(B_{2-\eta} \cap \{w_1 = 0, |Dw_1| > \eta\}\big)\\ &\qquad \le (1 + c\sqrt{\eta})\mathcal{H}^{n-1}\big(B_2 \cap \{w_2 = 0, |Dw_2| > \eta/2\}\big).\end{aligned}$$

PROOF. Set $S_0 = \{w_1 = 0\}$, $S_1 = \{w_1 = 0, |Dw_1| > \eta\}$ and $S_2 = \{w_2 = 0, |Dw_2| > \eta/2\}$, and take an $x \in \bar{B}_{2-\eta} \cap S_1$. By (5.2.14), we have for η small

$$\begin{aligned}&|Dw_1| \ge \eta - \frac{\eta}{5} > \frac{3}{4}\eta \quad \text{on } B_{\frac{\eta^2}{25}}(x),\\ &|Dw_2| \ge \eta - \frac{\eta}{5} - \eta^5 > \frac{3}{4}\eta \quad \text{on } B_{\frac{\eta^2}{25}}(x).\end{aligned} \tag{5.2.15}$$

Setting $\nu_j = |Dw_j|^{-1}Dw_j$, we get for $j = 1, 2$

$$|\nu_j(y_1) - \nu_j(y_2)| \le c\eta^{-1}|y_1 - y_2|^{\frac{1}{2}} \le c\eta^{\frac{1}{2}} \quad \text{for any } y_1, y_2 \in B_{2\eta^3}(x), \tag{5.2.16}$$

and

$$|\nu_1(y) - \nu_2(y)| \le c\eta^{-1}|Dw_1(y) - Dw_2(y)| \le c\eta^4 \quad \text{for any } y \in B_{2\eta^3}(x). \tag{5.2.17}$$

In particular, by (5.2.16) and (5.2.17) we have

$$|\nu_2(y) - \nu_1(x)| \le c\eta^{\frac{1}{2}} \quad \text{for any } y \in B_{2\eta^3}(x). \tag{5.2.18}$$

We note

$$S_2 \cap B_{\eta^4}(x) \ne \emptyset. \tag{5.2.19}$$

In fact, using (5.2.14), (5.2.15) and the fact that $w_1(x) = 0$, we have

$$w_1\big(x + \eta^4\nu_1(x)\big) > \frac{1}{2}\eta^5, \quad w_1\big(x - \eta^4\nu_1(x)\big) < \frac{1}{2}\eta^5,$$

and hence

$$w_2\big(x + \eta^4\nu_1(x)\big) > 0, \quad w_2\big(x - \eta^4\nu_1(x)\big) < 0.$$

We then get $w_2\big(x + \theta\eta^4\nu_1(x)\big) = 0$ for some $\theta \in (0, 1)$ and thus establish (5.2.19).

Next, with $x \in S_1 \cap \bar{B}_{2-\eta}$ as above and with T_x denoting the hyperplane containing x and normal to $\nu_1(x)$, we claim

$$\begin{aligned}S_k \cap B_{\eta^3}(x) &= B_{\eta^3}(x) \cap \operatorname{graph}_{T_x}\psi_x^k\\ &\equiv B_{\eta^3}(x) \cap \{y + \psi_x^k(y)\nu_1(x);\ y \in T_x \cap B_{\eta^3}(x)\}, \quad k = 0, 2,\end{aligned} \tag{5.2.20}$$

where $\psi_x^k \in C^1\big(T_x \cap B_{\eta^3}(x)\big)$ with

$$|D\psi_x^k(y)| \le c\eta^{\frac{1}{2}}, \quad k = 0, 2. \tag{5.2.21}$$

Indeed, by (5.2.16) and (5.2.18) it is clear that $S_k \cap B_{2\eta^3}(x)$ is contained in a union of such graphs over the larger domain $T_x \cap B_{2\eta^3}(x)$. An elementary argument using (5.2.15) and the mean-value theorem for functions of 1 variable then justifies (5.2.20) and (5.2.21), which assert that $S_k \cap B_{\eta^3}(x)$ consists of a single piece of graph over $T_x \cap B_{\eta^3}(x)$ with a controlled gradient.

Note that (5.2.19) and (5.2.21) guarantee

$$|\psi_x^0(y)-\psi_x^2(y)| \le c\eta^4 + c\eta^3\eta^{\frac{1}{2}} \le c\eta^{\frac{7}{2}} \quad \text{for any } y \in T_x \cap B_{\eta^3}(x). \tag{5.2.22}$$

The required area comparison is now fairly evident from (5.2.20)-(5.2.22). Specifically, let $\{B_{\eta^3/4}(x_j)\}_{j=1,\cdots,N}$ be a maximal pairwise-disjoint collection of balls with $x_j \in S_1 \cap \bar{B}_{2-\eta}$. Then

$$\{B_{\eta^3/2}(x_j)\}_{j=1,\cdots,N} \text{ cover } S_1 \cap \bar{B}_{2-\eta}, \tag{5.2.23}$$

and there is a $c = c(n)$ such that if $\mathcal{F} \subset \{1,\cdots,N\}$, then

$$\cap_{j\in\mathcal{F}} B_{\eta^3}(x_j) \ne \emptyset \quad \Rightarrow \quad \mathcal{F} \text{ has } \le c \text{ elements.} \tag{5.2.24}$$

Let $\phi_1,\cdots,\phi_N$ be a partition of unity for $S_1 \cap \bar{B}_{2-\eta}$ with

$$\text{support } \phi_j \subset B_{\eta^3}(x_j),\ \phi_j \ge c^{-1} \text{ on } B_{\eta^3/2}(x_j) \text{ and } |D\phi_j| \le c/\eta^3. \tag{5.2.25}$$

Then

$$\begin{aligned}
&\mathcal{H}^{n-1}(S_1 \cap \bar{B}_{2-\eta})\\
&= \sum_{j=1}^N \int_{S_1\cap B_{\eta^3}(x_j)} \phi_j d\mathcal{H}^{n-1} = \sum_{j=1}^N \int_{T_{x_j}\cap B_{\eta^3}(x_j)} F_j^0(y)dy\\
&= \sum_{j=1}^N \int_{T_{x_j}\cap B_{\eta^3}(x_j)} F_j^2(y)dy + \sum_{j=1}^N \int_{T_{x_j}\cap B_{\eta^3}(x_j)} (F_j^0 - F_j^2)(y)dy\\
&\le \mathcal{H}^{n-1}(S_2 \cap B_2) + \sum_{j=1}^N \int_{T_{x_j}\cap B_{\eta^3}(x_j)} (F_j^0 - F_j^2)(y)dy,
\end{aligned}$$

where (in the notation of (5.2.20))

$$F_j^k(y) = \phi_j\big(y + \psi_{x_j}^k(y)\nu_1(x_j)\big)\sqrt{1+|D\psi_{x_j}^k(y)|^2}, \quad k = 0, 2.$$

In view of (5.2.20), (5.2.21), (5.2.22), (5.2.24) and (5.2.25), we conclude

$$\mathcal{H}^{n-1}(S_1 \cap B_{2-\eta}) \le (1 + c\sqrt{\eta})\mathcal{H}^{n-1}(S_2 \cap B_2),$$

as required. □

The following lemma plays an important role in the proof of Theorem 5.2.1.

LEMMA 5.2.6. *Suppose N is a positive integer. Then there exists a positive constant ω_N such that, for any $y \in B_{1/2}$ and $r \le 1/2$, if $\omega(y;2r) < \omega_N$, then there exist finitely many balls $\{B_i\}$ in $B_{2r}(y)$, with rad$(B_i) \le r/2$ for each i, such that*

$$B_r(y) \cap u^{-1}\{0\} \cap |Du|^{-1}\{0\} \subset \bigcup_i B_i,$$

$$\mathcal{H}^{n-1}\big(u^{-1}\{0\} \cap B_r(y) \setminus \bigcup_i B_i\big) \le CNr^{n-1}, \tag{5.2.26}$$

and

$$\sum_i r_i^{n-1} \le \frac{1}{2} r^{n-1}. \tag{5.2.27}$$

Lemma 5.2.6 asserts that the nodal set $u^{-1}(0)$ is decomposed into two parts, a good part and a bad part. The good part has an optimal measure estimates as in (5.2.26) and the bad part is covered, roughly speaking, by finitely many balls, as in (5.2.27).

PROOF. Consider the transformation

$$x \mapsto y + 2rx.$$

Then, by $\mathcal{L}u = 0$ in $B_{2r}(y)$, we have

$$\tilde{\mathcal{L}}\tilde{u} = 0 \quad \text{in } B_1,$$

where

$$\begin{aligned} \tilde{\mathcal{L}} &= \sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\partial_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}_i(x)\partial_{x_i} + \tilde{c}(x) \\ &= \sum_{i,j=1}^{n} a_{ij}(y+2rx)\partial_{x_i x_j} + \sum_{i=1}^{n} 2rb_i(y+2rx)\partial_{x_i} + (2r)^2 c(y+2rx), \end{aligned}$$

and

$$\tilde{u}(x) = \frac{u(y+2rx)}{\|u\|_{L^2(\partial B_{2r}(y))}}.$$

Obviously, we have

$$\begin{aligned} &\sup_{B_1} \big(|D\tilde{a}_{ij}| + |\tilde{b}_i| + |\tilde{c}|\big) \\ = &\sup_{B_{2r}(y)} \big(2r|Da_{ij}| + 2r|b_i| + (2r)^2|c|\big) = \omega(y; 2r). \end{aligned}$$

Note

$$\|\tilde{u}\|_{L^2(\partial B_1)} = 1 \quad \text{and} \quad \|D\tilde{u}\|_{L^2(B_1)} \le C\sqrt{N}.$$

By Lemma 5.2.2, we get a function φ in $B_{3/4}$ satisfying

$$\tilde{a}_{ij}(0)\partial_{ij}\varphi = 0 \quad \text{in } B_{\frac{3}{4}},$$

$$\frac{\frac{3}{4}\int_{B_{\frac{3}{4}}} |D\varphi|^2}{\int_{\partial B_{\frac{3}{4}}} \varphi^2} \le CN, \tag{5.2.28}$$

and

$$\begin{aligned} &|\tilde{u}|_{C^{1,\frac{1}{2}}(B_{\frac{3}{4}})} + |\varphi|_{C^{1,\frac{1}{2}}(B_{\frac{3}{4}})} \le C, \\ &|\tilde{u} - \varphi|_{C^1(B_{\frac{3}{4}})} < C\omega(y; 2r), \end{aligned} \tag{5.2.29}$$

where C is a positive constant depending on n, λ and $\omega(y; 2r)$. By Corollary 2.2.6, we have

$$\begin{aligned} \|\varphi\|_{L^2(\partial B_{\frac{1}{2}})} \ge &2^{-CN}\|\varphi\|_{L^2(\partial B_{\frac{3}{4}})} = 2^{-CN}\|\tilde{u}\|_{L^2(\partial B_{\frac{3}{4}})} \\ \ge &2^{-CN}\|\tilde{u}\|_{L^2(\partial B_1)} = 2^{-CN}, \end{aligned}$$

and hence

$$\Big\{x \in B_{\frac{3}{4}};\ |D\varphi(x)| < 2^{-CN}\gamma(N)\Big\} \subset \left\{x \in B_{\frac{3}{4}};\ \frac{|D\varphi(x)|}{\|\varphi\|_{L^2(\partial B_{\frac{1}{2}})}} < \gamma(N)\right\},$$

where $\gamma(N)$ is as in Lemma 5.2.4. By applying Lemma 5.2.4 to $\varphi/\|\varphi\|_{L^2(\partial B_{1/2})}$ in $B_{3/4}$, we obtain finitely many balls $\{B_{\tilde{r}_i}(\tilde{x}_i)\}$ such that

$$\Big\{x \in B_{\frac{1}{2}};\ |D\varphi(x)| < 2^{-CN}\gamma(N)\Big\} \subset \bigcup_i B_{\tilde{r}_i}(\tilde{x}_i), \tag{5.2.30}$$

and

$$\sum_i \tilde{r}_i^{n-1} < \frac{1}{2}. \tag{5.2.31}$$

If $C\omega(y;2r) \leq 2^{-CN}\gamma(N)$, then

$$\tilde{u}^{-1}\{0\} \cap B_{\frac{1}{2}} \cap |D\tilde{u}|^{-1}\{0\} \subset \tilde{u}^{-1}\{0\} \cap B_{\frac{1}{2}} \cap \{|D\varphi| < 2^{-CN}\gamma(N)\}. \tag{5.2.32}$$

If $C\omega(y;2r) \leq \big(2^{-CN}\gamma(N)\big)^5/2$, then we have by (5.2.29) and Lemma 5.2.5

$$\begin{aligned}&\mathcal{H}^{n-1}\big(u^{-1}\{0\} \cap B_{\frac{1}{2}} \cap \big\{|D\tilde{u}| > \frac{1}{2}\cdot 2^{-CN}\gamma(N)\big\}\big)\\ \leq &2\mathcal{H}^{n-1}\big(\varphi^{-1}\{0\} \cap B_{\frac{2}{3}} \cap \big\{|D\varphi| > \frac{1}{4}\cdot 2^{-CN}\gamma(N)\big\}\big).\end{aligned}$$

Note that φ is a solution of a constant coefficient elliptic equation in $B_{3/4}$ with the frequency at most CN. By Theorem 2.3.1, we have

$$\mathcal{H}^{n-1}\big(\varphi^{-1}\{0\} \cap B_{\frac{2}{3}}\big) \leq CN,$$

and hence

$$\mathcal{H}^{n-1}\big(\tilde{u}^{-1}\{0\} \cap B_{\frac{1}{2}} \cap \{|D\varphi| > 2^{-CN}\gamma(N)\}\big) \leq CN. \tag{5.2.33}$$

By (5.2.32) and (5.2.33), we obtain

$$\tilde{u}^{-1}\{0\} \cap B_{\frac{1}{2}} \cap |D\tilde{u}|^{-1}\{0\} \subset \bigcup_i B_{\tilde{r}_i}(\tilde{x}_i),$$

and

$$\mathcal{H}^{n-1}\{\tilde{u}^{-1}\{0\} \cap B_{\frac{1}{2}} \setminus \bigcup_i B_{\tilde{r}_i}(\tilde{x}_i)\} \leq CN.$$

We complete the proof by transforming $B_{1/2}$ back to $B_r(y)$ by $x \mapsto (x-y)/2r$. □

REMARK 5.2.7. We note that the constant ω_N in Lemma 5.2.6 has the form

$$\omega_N = 2^{-C_0N}\big(\gamma(N)\big)^5,$$

where C_0 is a positive constant depending only on n, λ, κ and Γ and $\gamma(N)$ is as in Lemma 5.2.4.

Now we are ready to prove Theorem 5.2.1. The proof is based on an iteration of Lemma 5.2.6.

PROOF OF THEOREM 5.2.1. We only consider the case $x_0 = 0$. Choose any $r < 1/2$ such that $\omega(0;2r) \leq \omega_N$, with ω_N as in Lemma 5.2.6. We use an iteration process to prove Theorem 5.2.1. To begin with, we define

$$\phi_0 = \{B_r\}.$$

We claim that we may find $\phi_1, \phi_2, \cdots$, each of which consists of a collection of balls, such that for any $\ell \geq 1$

$$\text{rad}(B) \leq \frac{r}{2^\ell} \quad \text{for any } B \in \phi_\ell,$$

$$\sum_{B\in\phi_\ell}\big(\mathrm{rad}(B)\big)^{n-1}\le\frac{1}{2^\ell}r^{n-1},$$

and

$$\mathcal{H}^{n-1}\big(u^{-1}(0)\cap\bigcup_{B\in\phi_{\ell-1}}B\setminus\bigcup_{B\in\phi_\ell}B\big)\le\frac{1}{2^{\ell-1}}CNr^{n-1},$$

where C is a positive constant depending only on n and λ. We should note that ϕ_ℓ is constructed in such a way that

$$B_r\cap u^{-1}\{0\}\cap|Du|^{-1}\{0\}\subset\bigcup_{B\in\phi_\ell}B.$$

Observe that

$$\begin{aligned}u^{-1}(0)\cap B_r\subset&\bigcup_{\ell=1}^{\infty}\big(u^{-1}(0)\cap\big(\bigcup_{B\in\phi_{\ell-1}}B\setminus\bigcup_{B\in\phi_\ell}B\big)\big)\\&\cup\bigcap_{\ell=0}^{\infty}\big(u^{-1}(0)\cap\bigcup_{j=\ell}^{\infty}\bigcup_{B\in\phi_j}B\big).\end{aligned}$$

Hence we have

$$\mathcal{H}^{n-1}\big(u^{-1}(0)\cap B_r\big)\le CN\big(\sum_{\ell\ge1}\frac{1}{2^{\ell-1}}+\inf_{\ell\ge0}\sum_{j=\ell}^{\infty}\frac{1}{2^j}\big)r^{n-1}\le 2CNr^{n-1}.$$

To prove the claim, we construct $\{\phi_\ell\}$ by an induction. Note $\phi_0=\{B_r\}$. Suppose $\phi_0,\phi_1,\dots,\phi_{\ell-1}$ are already defined for $\ell\ge1$. To construct ϕ_ℓ, we take an arbitrary $B=B_\rho(y)\in\phi_{\ell-1}$, with $\rho\le1/2$. Obviously, $\omega(y;2\rho)\le\omega(0;2r)\le\omega_N$. Then by Lemma 5.2.6, there exist finitely many balls $\{B_{r_i}(x_i)\}$, with $B_i=B_{r_i}(x_i)\subset B_{2\rho}(y)$ for each i, such that

$$B_\rho(y)\cap u^{-1}\{0\}\cap|Du|^{-1}\{0\}\subset\bigcup_i B_i,$$

$$\mathcal{H}^{n-1}\big(u^{-1}\{0\}\cap B_\rho(y)\setminus\bigcup_i B_i\big)\le CN\rho^{n-1},\tag{5.2.34}$$

and

$$\sum_i r_i^{n-1}\le\frac12\rho^{n-1}.\tag{5.2.35}$$

To conclude, we set

$$\phi_\ell^B=\bigcup_i\{B_i\},$$

and

$$\phi_\ell=\bigcup_{B\in\phi_{\ell-1}}\phi_\ell^B.$$

Then we have by (5.2.34)

$$\mathcal{H}^{n-1}\big(u^{-1}\{0\}\cap\bigcup_{B\in\phi_{\ell-1}}B\setminus\bigcup_{B\in\phi_\ell}B\big)\le CN\sum_{B_{r_i}(x_i)\in\phi_{\ell-1}}r_i^{n-1},$$

and by (5.2.35) and by an induction

$$\sum_{B_{r_i}(x_i)\in\phi_\ell} r_i^{n-1} \le \frac{1}{2^\ell} r^{n-1}, \tag{5.2.36}$$

for each $\ell \ge 1$. Therefore, we obtain

$$\mathcal{H}^{n-1}\big(u^{-1}\{0\} \cap \bigcup_{B\in\phi_{\ell-1}} B \setminus \bigcup_{B\in\phi_\ell} B\big) \le \frac{1}{2^{\ell-1}} CNr^{n-1}. \tag{5.2.37}$$

This finishes the proof. ☐

Now we can estimate the nodal set in any ball.

Corollary 5.2.8. *Suppose $u \in \mathcal{S}_N$ is a nonzero solution of $\mathcal{L}u = 0$ in $B_1 \subset \mathbb{R}^n$ for some $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \Gamma)$. Then*

$$\mathcal{H}^{n-1}\big(\{x \in B_{\frac{1}{2}};\, u(x) = 0\}\big) \le C(N), \tag{5.2.38}$$

where $C(N)$ is a positive constant depending only on N, as well as n, λ, κ and Γ.

Proof. We cover $B_{1/2}$ by balls $\{B_{r_N}(x_k)\}$ with the number of balls $\le c(n)r_N^{-n}$. Therefore, we get

$$\mathcal{H}^{n-1}\big(\{x \in B_{\frac{1}{2}};\ u(x) = 0\}\big) \le r_N^{-1} C_0 N.$$

This finishes the proof. ☐

Remark 5.2.9. We note that the constant r_N in Theorem 5.2.1 and the constant $C(N)$ in Corollary 5.2.8 are given by

$$\begin{aligned} r_N =& C_0^{-1} 2^{-C_0 N} \big(\gamma(N)\big)^5, \\ C(N) =& C_0 2^{C_0 N} \big(\gamma(N)\big)^{-5} \cdot N, \end{aligned}$$

where C_0 is a positive constant depending only on n, λ, κ and Γ and $\gamma(N)$ is as in Lemma 5.2.4.

5.3. Nodal Sets of Non-Analytic Solutions: An Explicit Estimate

In this section, we give an explicit estimate on nodal sets of solutions of elliptic differential equation. The entire section follows closely Hardt and Simon [**51**].

To obtain an explicit estimate on nodal sets, we need to find an explicit expression of $\gamma(N)$ in Lemma 5.2.4 in terms of N. We will do this only for harmonic polynomials.

If we consider a homogeneous harmonic polynomial of the form $u = r^d \cos(d\theta)$ of degree d in $\mathbb{R}^2$, then $|Du(x)| = d|x|^{d-1}$ and

$$\{x \in \mathbb{R}^2; |Du(x)| < dr^{d-1}\} = B_r.$$

Hence in this case we simply take $\gamma(d) = d/2^{d-1}$ and one ball $B_{1/2}$ to satisfy Lemma 5.2.4.

We first prove a result concerning the growth of harmonic polynomials.

LEMMA 5.3.1. *Let φ be a harmonic polynomial of degree d in $\mathbb{R}^n$. Then for any $r > 1$*

$$|\varphi|_{L^\infty(B_r)} \le c d^{\frac{n}{2}} r^d |\varphi|_{L^\infty(B_1)}, \tag{5.3.1}$$

where c is a positive constant depending only on n.

It is likely that (5.3.1) holds without the factor $d^{n/2}$.

PROOF. We set for any $r > 0$

$$\|\varphi\|_r^2 = \fint_{B_r} \varphi^2,$$

and claim for any $r > 0$ and $\theta \ge 1$

$$\|\varphi\|_{\theta r} \le \theta^d \|\varphi\|_r.$$

To see this, we simply note that

$$\|\varphi\|_r^2 = \sum_{i=0}^{d} a_i^2 r^{2i},$$

for some constants $a_0, a_1, \cdots, a_d$, and then

$$\|\varphi\|_{\theta r}^2 = \sum_{i=0}^{d} a_i^2 \theta^{2i} r^{2i} \le \theta^{2d} \sum_{i=0}^{d} a_i^2 r^{2i} = \theta^{2d} \|\varphi\|_r^2.$$

Fix an $r > 1$ and consider any $x \in \partial B_r$ and $s > r$. Then we have

$$\varphi(x) = \fint_{B_{s-r}(x)} \varphi,$$

and hence

$$\begin{aligned} |\varphi(x)| &\le \left(\fint_{B_{s-r}(x)} \varphi^2 \right)^{\frac{1}{2}} \le \left(\frac{s}{s-r} \right)^{\frac{n}{2}} \left(\fint_{B_s} \varphi^2 \right)^{\frac{1}{2}} \\ &\le \left(\frac{s^{n+2d}}{(s-r)^n} \right)^{\frac{1}{2}} \left(\fint_{B_1} \varphi^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now we choose s appropriately to minimize the function

$$f(s) = \frac{s^{n+2d}}{(s-r)^n}.$$

A simple calculation shows that $f(s)$ attains its minimum in (r, ∞) at

$$s_{\min} = \left(1 + \frac{n}{2d}\right) r,$$

and the corresponding minimum is given by

$$f_{\min} = \left(\frac{2}{n} + \frac{1}{d}\right)^n \left(1 + \frac{n}{2d}\right)^{2d} \cdot d^n r^{2d}.$$

Hence, we obtain for any $r > 1$

$$|\varphi|_{L^\infty(B_r)} \le c(n) d^{\frac{n}{2}} r^d \|\varphi\|_{L^2(B_1)}.$$

This ends the proof. □

Now we prove the following result.

Lemma 5.3.2. *There are positive constants* $\theta \in (0,1/2)$, c_0 *and* c, *depending only on* n, *such that, if* φ *is a harmonic polynomial of degree* d *in* $\mathbb{R}^n$ *with* $\sup_{B_1}|\varphi - \varphi(0)| = 1$ *and* $|D\varphi(0)| \le (\theta\varepsilon)^{d-1}$, *then for any* $\varepsilon \in (0,1/2]$

$$\mathcal{L}^n\big(\{x \in B_1; dist(x, \{|D\varphi| \le (\theta\varepsilon)^{d-1}\}) < \varepsilon\}\big) \le cd^{c_0}\varepsilon^2 \log \varepsilon^{-1}.$$

From the proof below, we have

$$c_0 = (2n-1)(1+\frac{n}{2}) + 3.$$

Moreover, we may write the conclusion in the form

$$\mathcal{H}^n_\varepsilon\big(\{x \in B_1; |D\varphi| \le (\theta\varepsilon)^{d-1}\}\big) \le cd^{c_0}\varepsilon^2 \log \varepsilon^{-1}.$$

It seems likely that Lemma 5.3.2 may be true without the factor $\log \varepsilon^{-1}$ in the right hand side, but such an inequality would not significantly improve the main results in this section.

Proof. The proof is a fairly straightforward application of Lemma 1.3.4 together with standard estimates for harmonic functions and the coarea formula.

First note that we have $d \ge 2$ and that for each $z \in B_1$ and each $\varepsilon \in (0,1/2]$

$$\sup_{B_\varepsilon(z)} |D^2\varphi| \ge \theta(2\theta\varepsilon)^{d-2}, \tag{5.3.2}$$

where $\theta = \theta(n) \in (0,1/2)$ is a constant depending only on n. Indeed, the given facts about φ imply for some positive constant $c = c(n)$

$$\sup_{B_1} |D^2\varphi| \ge c^{-1}. \tag{5.3.3}$$

Then we have $d \ge 2$. Also if (5.3.2) were false for a given $z \in B_1$, then standard estimates for derivatives of harmonic functions would imply for any multi-index α with $2 \le |\alpha| \le d$

$$\frac{1}{\alpha!}|D^\alpha\varphi(z)| \le c\theta(c\theta)^{d-|\alpha|},$$

which for small $\theta = \theta(n)$ contradicts (5.3.3) and the fact that each component of $D^2\varphi$ is a polynomial of degree $\le d-2$. In the following, we fix $\theta = \theta(n)$ so that (5.3.2) holds.

Take any $y \in B_1$ with $|D\varphi(y)| \le (\theta\varepsilon)^{d-1}$. Since each component of $D^2\varphi$ is a harmonic polynomial of degree $\le d-2$, by Lemma 5.3.1 we have

$$\sigma\varepsilon \sup_{B_{(1+\sigma)\varepsilon}} |D^3\varphi| + \sup_{B_{(1+\sigma)\varepsilon}} |D^2\varphi| \le cd^{\frac{n}{2}}(1+2\sigma)^d \sup_{B_\varepsilon} |D^2\varphi|, \tag{5.3.4}$$

for any $\sigma \in (0,1/4]$, where $c = c(n)$.

Let τ_1 be a unit vector in $\mathbb{R}^n$ and $\bar{y} \in \bar{B}_\varepsilon(y)$ such that

$$|D_{\tau_1\tau_1}\varphi(\bar{y})| = \sup_{B_\varepsilon(y)} |D^2\varphi|. \tag{5.3.5}$$

Now we claim for a $\gamma = \gamma(n)$ small enough and any unit vectors ξ and η in $\mathbb{R}^n$

$$|D_{\xi\eta}\varphi(x) - D_{\xi\eta}\varphi(\bar{y})| \le c\gamma|D_{\tau_1\tau_1}\varphi(\bar{y})| \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}), \tag{5.3.6}$$

and

$$|D_\xi\varphi(x)| \le c\varepsilon|D_{\tau_1\tau_1}\varphi(\bar{y})| \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}), \tag{5.3.7}$$

where

$$\beta = \gamma d^{-(1+\frac{n}{2})}, \tag{5.3.8}$$

for a small $\gamma = \gamma(n)$. To see this, we take any $\beta \in (0, \sigma)$. Then by (5.3.4), we get

$$|D_{\xi\eta}\varphi(x) - D_{\xi\eta}\varphi(\bar{y})| \leq c\beta\sigma^{-1}d^{\frac{n}{2}}(1+2\sigma)^d|D_{\tau_1\tau_1}\varphi(\bar{y})| \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}). \tag{5.3.9}$$

Thus we select $\sigma = c^{-1}d^{-1}$ and β as in (5.3.8) for suitable $c = c(n)$ and $\gamma = \gamma(n)$ such that

$$c\beta\sigma^{-1}d^{\frac{n}{2}}(1+2\sigma)^d \leq c\gamma \leq \frac{1}{2}.$$

Later on, we may choose γ smaller. This implies (5.3.6) easily. We also have

$$|D_{\xi\eta}\varphi(x)| \leq 2|D_{\tau_1\tau_1}\varphi(\bar{y})| \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}), \tag{5.3.10}$$

and

$$|D_{\tau_1\tau_1}\varphi(\bar{y})| \leq 2|D_{\tau_1\tau_1}\varphi(x)| \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}). \tag{5.3.11}$$

Also for any $x \in B_{\beta\varepsilon}(\bar{y})$ and any unit vector $\xi \in \mathbb{R}^n$, we have by (5.3.2) and (5.3.11)

$$\begin{aligned}
|D_\xi\varphi(x)| \leq & |D_\xi\varphi(y)| + |D_\xi\varphi(y) - D_\xi\varphi(\bar{y})| + |D_\xi\varphi(\bar{y}) - D_\xi\varphi(x)| \\
\leq & (\theta\varepsilon)^{d-1} + \varepsilon|D_{\tau_1\tau_1}\varphi(\bar{y})| + \frac{2\gamma\varepsilon}{d}|D_{\tau_1\tau_1}\varphi(\bar{y})| \\
\leq & \ c\varepsilon|D_{\tau_1\tau_1}\varphi(\bar{y})|.
\end{aligned}$$

This is (5.3.7).

Next, let $\mathcal{O}_\gamma$ be a collection of orthonormal basis of $\mathbb{R}^n$ such that for any orthonormal basis $\{\tau_1, \cdots, \tau_n\}$ of $\mathbb{R}^n$ there exists an orthonormal basis $\{\tilde{\tau}_1, \cdots, \tilde{\tau}_n\} \in \mathcal{O}_\gamma$ with

$$|\tau_i - \tilde{\tau}_i| < \beta, \quad i = 1, \cdots, n. \tag{5.3.12}$$

We can, and we will, choose $\mathcal{O}_\gamma$ in such a way that

$$\text{the number of elements in } \mathcal{O}_\gamma \leq c\beta^{1-n}, \tag{5.3.13}$$

for some constant $c = c(n)$.

With $\bar{y} \in \bar{B}_\varepsilon(y)$ as in (5.3.5), let $\tau_1, \cdots, \tau_n$ be an orthonormal basis of $\mathbb{R}^n$ such that

$$\begin{aligned}
|D_{\tau_1\tau_1}\varphi(\bar{y})| =& \max_{j\in\{1,\cdots,n\}} |D_{\tau_j\tau_j}\varphi(\bar{y})|, \\
|D_{\tau_2\tau_2}\varphi(\bar{y})| =& \max_{j\in\{2,\cdots,n\}} |D_{\tau_j\tau_j}\varphi(\bar{y})|, \\
D_{\tau_i\tau_j}\varphi(\bar{y}) =& 0, \quad i \neq j.
\end{aligned}$$

Furthermore, since $\sum_{j=1}^n D_{\tau_j\tau_j}\varphi = 0$, we have

$$|D_{\tau_2\tau_2}\varphi(\bar{y})| \geq (n-1)^{-1}|D_{\tau_1\tau_1}\varphi(\bar{y})|.$$

Then by (5.3.6), (5.3.10) and (5.3.12), we can select an orthonormal basis $\tilde{\tau}_1, \cdots, \tilde{\tau}_n \in \mathcal{O}_\gamma$ such that

$$\begin{aligned}
&|\varphi_{11}(x)| \geq c^{-1} \max_{j\in\{1,\cdots,n\}} |\varphi_{jj}(x)|, \\
&|\varphi_{22}(x)| \geq c^{-1} \max_{j\in\{2,\cdots,n\}} |\varphi_{jj}(x)|, \\
&|\varphi_{ij}(x)| \leq c\gamma|\varphi_{11}(x)|, \quad i \neq j,
\end{aligned} \tag{5.3.14}$$

for any $x \in B_{\beta\varepsilon}(\bar{y})$, where we use the notation $\varphi_{ij}(x) = D_{\tilde{\tau}_j\tilde{\tau}_j}\varphi(x)$. To see (5.3.14), we note

$$\varphi_{ij}(x) = a_i^k a_j^l D_{\tau_k\tau_l}\varphi(x),$$

for an orthogonal matrix (a_i^k). Then

$$\varphi_{ij}(x) - D_{\tau_i\tau_j}\varphi(x) = (a_i^k a_j^l - \delta_i^k\delta_j^l)D_{\tau_k\tau_l}\varphi(x).$$

By (5.3.12), we have $|a_i^k a_j^l - \delta_i^k\delta_j^l| < \gamma$ for each i, j, k and l. Then (5.3.10) implies

$$|\varphi_{ij}(x) - D_{\tau_i\tau_j}\varphi(x)| \le C\gamma|D_{\tau_1\tau_1}\varphi(\bar{y})| \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}).$$

With (5.3.6), we have

$$|\varphi_{ij}(x) - D_{\tau_i\tau_j}\varphi(\bar{y})| \le C\gamma|D_{\tau_1\tau_1}\varphi(\bar{y})| \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}).$$

Taking $i = j = 1$, $i = j = 2$ and other combinations of i and j respectively, we obtain

$$\begin{aligned}(1-c\gamma)|D_{\tau_1\tau_1}\varphi(\bar{y})| \le |\varphi_{11}(x)| \le (1+c\gamma)|D_{\tau_1\tau_1}\varphi(\bar{y})|,\\ (1-c\gamma)|D_{\tau_2\tau_2}\varphi(\bar{y})| \le |\varphi_{22}(x)| \le (1+c\gamma)|D_{\tau_2\tau_2}\varphi(\bar{y})|,\end{aligned} \tag{5.3.15}$$

and

$$|\varphi_{ij}(x)| \le (\delta_{ij} + C\gamma)|D_{\tau_1\tau_1}\varphi(\bar{y})|,$$

for any $x \in B_{\beta\varepsilon}(\bar{y})$. We then have (5.3.14) easily.

With $\varphi_j = D_{\tilde{\tau}_j}\varphi$, we consider the map $(\varphi_1, \varphi_2) : B_{\beta\varepsilon}(\bar{y}) \subset \mathbb{R}^n \to \mathbb{R}^2$ and denote by J its Jacobian, that is,

$$J = \sqrt{|D\varphi_1|^2|D\varphi_2|^2 - (D\varphi_1 \cdot D\varphi_2)^2}.$$

Then, we have by (5.3.14) and (5.3.15)

$$J^2 = \sum_{1\le i<j\le n}(\varphi_{i1}\varphi_{j2} - \varphi_{j1}\varphi_{i2})^2 \ge (\varphi_{11}\varphi_{22} - \varphi_{12}^2)^2 \ge c|D_{\tau_1\tau_1}\varphi(\bar{y})|^4 \quad \text{in } B_{\beta\varepsilon}(\bar{y}),$$

and by (5.3.7)

$$c^{-1} \le \varepsilon^2\big(\varphi_1^2(x) + \varphi_2^2(x)\big)^{-1}J(x) \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}). \tag{5.3.16}$$

Also by (5.3.14), (5.3.15) and (5.3.2), we have

$$|\varphi_{22}(x)|,\ |\varphi_{11}(x)| \ge (\theta\varepsilon)^{d-2} \quad \text{for any } x \in B_{\beta\varepsilon}(\bar{y}),$$

and hence

$$|\varphi_2|, |\varphi_1| \ge c^{-1}\beta(\theta\varepsilon)^{d-1}, \tag{5.3.17}$$

on a subset A of $B_{\beta\varepsilon}(\bar{y})$ with $\mathcal{L}^n(A) \ge c^{-1}(\beta\varepsilon)^n$, where $c = c(n)$.

We set $f = (\varphi_1, \varphi_2)$ and $g = 1/(\varphi_1^2 + \varphi_2^2)$ in Theorem 1.2.8, the coarea formula. By (5.3.16) and (5.3.17), we obtain

$$\begin{aligned}c^{-1}d^{-n(1+\frac{n}{2})}\varepsilon^n \le \varepsilon^2 \int_{\beta(c^{-1}\varepsilon)^{d-1}}^{c^{d-1}} \int_{\beta(c^{-1}\varepsilon)^{d-1}}^{c^{d-1}} (s^2+t^2)^{-1}\\ \cdot \sum_{\mathcal{O}_\gamma}\mathcal{H}^{n-2}\big(\{x \in B_\varepsilon(y); \varphi_1(x) = s, \varphi_2(x) = t\}\big)dsdt,\end{aligned} \tag{5.3.18}$$

for a suitable $c = c(n)$. Here we used the fact that $|D\varphi| \le c^{d-1}$ in B_2.

Now select a maximal pairwise-disjoint collection of balls $\{B_{\varepsilon/2}(y_j)\}_{j=1,\cdots,m}$ with $y_j \in \{x \in B_1; |D\varphi| \le (\theta\varepsilon)^{d-1}\}$, and sum over j after replacing y by y_j in

(5.3.18). Then (keeping in mind that $\mathcal{O}_\gamma$ has $\leq cd^{(n-1)(1+n/2)}$ elements by (5.3.13)), we get

$$\begin{aligned}&\mathcal{L}^n\big(\{x \in B_1; \operatorname{dist}(x, \{|D\varphi| \leq (\theta\varepsilon)^{d-1}\}) < \varepsilon\}\big)\\ &\quad\leq cd^{(2n-1)(1+\frac{n}{2})}\varepsilon^2 \int_{\beta(c^{-1}\varepsilon)^{d-1}}^{c^{d-1}} \int_{\beta(c^{-1}\varepsilon)^{d-1}}^{c^{d-1}} (s^2+t^2)^{-1}\\ &\quad\quad\cdot \max_{\mathcal{O}_\gamma} \mathcal{H}^{n-2}\big(\{x \in B_2; \varphi_1(x) = s, \varphi_2(x) = t\}\big)dsdt.\end{aligned}$$

The required inequality now follows from Lemma 1.3.4, because

$$\dim_{\mathcal{H}}\{x \in B_2; \varphi_1(x) = s, \varphi_2(x) = t\} \leq n-2 \quad \text{for a.e. } (s,t) \in (0,1) \times (0,1),$$

by the coarea formula, and because

$$\int_\alpha^{\alpha^{-1}} \int_\alpha^{\alpha^{-1}} (s^2+t^2)^{-1} dsdt \leq c \log \alpha^{-1}$$

for each $\alpha \in (0, 1/2)$. ☐

REMARK 5.3.3. We may cover $\{|D\varphi| \leq (\theta\varepsilon)^{d-1}\}$ by a collection of balls of radius ε such that the balls of the same centers and radius $\varepsilon/2$ are pairwise disjoint. Then by Lemma 5.3.2, the number of balls is $\leq cd^{c_0}\varepsilon^{2-n}\log\varepsilon^{-1}$.

Now let us recall that $\mathcal{L}(\lambda, \kappa, \Gamma)$ is the collection of all linear operators of the form (5.1.1) with coefficients satisfying (5.1.2), (5.2.1) and (5.2.2) and that $\mathcal{S}_N$ is the collection of all H^1-functions u in B_1 satisfying

$$\frac{r\int_{B_r(p)} |Du|^2}{\int_{\partial B_r(p)} u^2} \leq N \quad \text{for any } B_r(p) \subset B_1.$$

In the following, we always assume that $u \in H^1(B_1)$ is a solution of $\mathcal{L}u = 0$ for some $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \Gamma)$. We further assume that $u \in \mathcal{S}_N$ satisfies the following doubling condition

$$\fint_{\partial B_{2r}} u^2 \leq 2^{2N} \fint_{\partial B_r} u^2 \quad \text{for any } r \in (0, \frac{1}{4}).$$

For any $y \in B_1$, we introduce $\omega(y; r)$ as in (5.2.3), i.e.,

$$\omega(y; r) = \sup_{B_r(y)} (r|Da_{ij}| + r|b_i| + r^2|c|).$$

Obviously, $\omega(y; r) \to 0$ as $r \to 0$.

Now we prove that any solution can be well approximated by harmonic polynomials in small balls. Compare this with Lemma 5.2.2.

LEMMA 5.3.4. *Suppose* $u \in \mathcal{S}_N$ *is a nonzero solution of* $\mathcal{L}u = 0$ *in* $B_1 \subset \mathbb{R}^n$ *for some* $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \Gamma)$. *For any* $B = B_r(y)$ *with* $y \in B_{1/2}$ *and* $r \in (0, 1/4)$, *let*

$$u_B(x) = \frac{u(y + rx)}{(\fint_{\partial B_{3r}(y)} u^2)^{1/2}} \quad \textit{for } |x| \leq \frac{1}{2r}.$$

If for some $r < R < 1/(2r)$

$$\omega(y; Rr)(cR)^N < \frac{1}{2}, \tag{5.3.19}$$

then there exists a harmonic polynomial P_B *of degree* $\leq 2N$ *such that*

$$|u_B - P_B|_{C^1(B_2)} < \omega(y; Rr)(CR)^N + \big(\frac{C}{R}\big)^N,$$

and

$$|u_B|_{C^{1,\frac{1}{2}}(B_2)} + |P_B|_{C^{1,\frac{1}{2}}(B_2)} \leq C, \quad \fint_{\partial B_3} P_B^2 \geq \frac{1}{4},$$

where C *is a positive constant depending only on* n, λ, κ *and* Γ.

Proof. We first note that u_B is defined in $B_{1/r}$ with $\fint_{\partial B_3} u_B^2 = 1$. For any $R \in (3, 1/r)$, there exists an integer l such that $R/2^{l+1} < 3 \leq R/2^l$. Using the doubling condition l times, we have

$$\fint_{\partial B_R} u_B^2 \leq \big(\frac{R}{3}\big)^{2N} \fint_{\partial B_3} u_B^2 = \big(\frac{R}{3}\big)^{2N}.$$

Now u_B satisfies

$$\tilde{a}_{ij} u_{B,ij} + \tilde{b}_i u_{B,i} + \tilde{c} u_B = 0 \quad \text{in } B_{\frac{1}{r}},$$

where

$$\tilde{a}_{ij}(x) = a_{ij}(y + rx), \quad \tilde{b}_i(x) = r b_i(y + rx), \quad \tilde{c}(x) = r^2 c(y + rx).$$

By introducing

$$\|u\|_{p,r} = \big(\fint_{B_r} u^p\big)^{\frac{1}{p}}$$

and by the (scaled) interior $W^{2,p}$-estimates in $B_{2R} \subset B_{1/r}$, we have

$$\|u_B\|_{p,R} + R\|Du_B\|_{p,R} + R^2\|D^2 u_B\|_{p,R} \leq C\|u_B\|_{2,2R} \leq (CR)^N,$$

where C is a positive constant depending only on n, p, the module of $\tilde{a}_{ij}$ in B_{2R}, $|\tilde{a}_{ij}|_{L^\infty(B_{2R})}$, $R|\tilde{b}_i|_{L^\infty(B_{2R})}$ and $R^2|\tilde{c}|_{L^\infty(B_{2R})}$. We note

$$\begin{aligned}
&\sup_{B_{2R}}(|\tilde{a}_{ij}| + R|\tilde{b}_i| + R^2|\tilde{c}|) = \sup_{B_{2Rr}(y)}(|a_{ij}| + Rr|b_i| + (Rr)^2|c|) \\
\leq &\sup_{B_1}(|a_{ij}| + |b_i| + |c|) \leq \lambda^{-1} + \kappa.
\end{aligned}$$

For p large, we have by the Sobolev embedding

$$|u_B|_{L^\infty(B_r)} + R|Du_B|_{L^\infty(B_R)} + R^{1+\alpha}[Du_B]_{C^\alpha(B_R)} \leq (CR)^N. \tag{5.3.20}$$

We rewrite the equation as

$$-\Delta u_B \equiv f_B = \big(\tilde{a}_{ij} - \delta_{ij}\big) u_{B,ij} + \tilde{b}_i u_{B,i} + \tilde{c} u_B.$$

Note

$$\begin{aligned}
R^2\|f_B\|_{p,R} \leq &|\tilde{a}_{ij} - \delta_{ij}|_{L^\infty(B_r)} R^2\|D^2 u_B\|_{p,R} + R|\tilde{b}_i|_{L^\infty(B_R)} R\|Du_B\|_{p,R} \\
&+ R^2|\tilde{c}|_{L^\infty(B_R)}\|u_B\|_{p,R} \\
\leq &\sup_{B_{Rr}(y)} \big(|a_{ij} - \delta_{ij}| + Rr|b_i| + (Rr)^2|c|\big) \\
&\cdot \big(R^2\|D^2 u_B\|_{p,R} + R\|Du_B\|_{p,R} + |u_B\|_{p,R}\big) \\
\leq &\omega(y; Rr)(CR)^N.
\end{aligned}$$

Let v be the harmonic function in B_R with $v = u_B$ on ∂B_R. Note

$$\begin{aligned} -\Delta(u_B - v) &= f_B \quad \text{in } B_R, \\ u_B - v &= 0 \quad \text{on } \partial B_R. \end{aligned} \tag{5.3.21}$$

Then we obtain by the (scaled) global $W^{2,p}$-estimates

$$\begin{aligned} &\|u_B - v\|_{p,R} + R\|D(u_B - v)\|_{p,R} + R^2\|D^2(u_B - v)\|_{p,R} \\ \leq &CR^2\|f_B\|_{p,R} \leq \omega(y; Rr)(CR)^N \end{aligned}$$

and the Sobolev embedding

$$\begin{aligned} &|u_B - v|_{L^\infty(B_R)} + R|D(u_B - v)|_{L^\infty(B_R)} + R^{1+\alpha}[D(u_B - v)]_{C^\alpha(B_R)} \\ \leq &\omega(y; Rr)(CR)^N. \end{aligned} \tag{5.3.22}$$

This implies in particular

$$|u_B - v|_{C^{1,\frac{1}{2}}(B_2)} \leq \omega(y; Rr)(CR)^N,$$

and hence

$$\frac{1}{2} \leq \Big(\fint_{\partial B_3} v^2\Big)^{\frac{1}{2}} \leq \frac{3}{2},$$

if $\omega(y; Rr)(CR)^N < 1/2$.

Next, multiplying $\Delta v = 0$ by $u_B - v$ and integrating in B_R as in the proof of Lemma 5.2.2, we have

$$\int_{B_R} |\nabla v|^2 \leq \int_{B_R} |\nabla u_B|^2.$$

This implies in particular

$$\frac{R\int_{B_R} |\nabla v|^2}{\int_{\partial B_R} v^2} \leq N.$$

Then by Corollary 2.2.5, we have

$$\fint_{\partial B_R} v^2 \leq \Big(\frac{R}{3}\Big)^{2N} \fint_{\partial B_3} v^2 \leq 2\Big(\frac{R}{3}\Big)^{2N}.$$

Since v is a harmonic function, then for any $r > 0$

$$\fint_{\partial B_r} v^2 = \sum_{m=0}^{\infty} a_m^2 r^{2m},$$

and hence

$$\sum_{m=0}^{\infty} a_m^2 R^{2m} \leq 2\Big(\frac{R}{3}\Big)^{2N}.$$

Therefore, we get

$$\Big(\frac{R}{3}\Big)^{4N} \sum_{m=2N+1}^{\infty} a_m^2 3^{2m} \leq 2\Big(\frac{R}{3}\Big)^{2N},$$

and hence

$$\sum_{m=2N+1}^{\infty} a_m^2 3^{2m} \leq 2\Big(\frac{3}{R}\Big)^{2N}.$$

Let P_B be the polynomial of degree $\leq 2N$ in the Taylor expansion of v, we have

$$\fint_{\partial B_3} (v - P_B)^2 \leq 2\Big(\frac{3}{R}\Big)^{2N}.$$

We then get by interior estimates for harmonic functions

$$|v - P_B|_{C^{1,\frac{1}{2}}(B_2)} \le \big(\frac{C}{R}\big)^{2N}.$$

The $C^{1,1/2}$-bounds on u_B and P_B follow from the interior estimates in B_3 easily. $\square$

The following result is an improved version of Lemma 5.2.6.

LEMMA 5.3.5. *Suppose $u \in \mathcal{S}_N$ is a nonzero solution of $\mathcal{L}u = 0$ in $B_1 \subset \mathbb{R}^n$ for some $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \Gamma)$. For any $\varepsilon > 0$, there exists an $r(N, \varepsilon)$ such that, for any $B = B_r(y)$ with $u(y) = 0$, $y \in B_{1/2}$ and $r \le r(N, \varepsilon)$, there exist finitely many balls B_i with radius εr and centers in $B_r(y) \cap u^{-1}(0)$ such that*

$$\mathcal{H}^{n-1}\big(B_r(y) \cap u^{-1}(0) \setminus \cup_i B_i\big) \le c(n) N r^{n-1},$$

and

$$\#\{B_i\} \le cN^{c_0}\varepsilon^{2-n}\log\varepsilon^{-1}.$$

PROOF. For any $B = B_r(y)$ with $u(y) = 0$, $r \le 1/2$ and $|y| \le 1/2$, we write

$$u_B(x) = \frac{u(y+rx)}{(\fint_{\partial B_{3r}(y)} u^2)^{1/2}} \quad \text{for } |x| \le \frac{1}{2r}.$$

By Lemma 5.3.4, if for some $r < R < 1/(2r)$

$$\omega(y; Rr)(cR)^N < \frac{1}{2}, \tag{5.3.23}$$

then there exists a harmonic polynomial P_B of degree $\le 2N$ such that

$$|u_B - P_B|_{C^1(B_2)} < \omega(y; Rr)(CR)^N + \big(\frac{C}{R}\big)^N, \tag{5.3.24}$$

$$|u_B|_{C^{1,\frac{1}{2}}(B_2)} + |P_B|_{C^{1,\frac{1}{2}}(B_2)} \le C, \quad \fint_{\partial B_3} P_B^2 \ge \frac{1}{4}, \tag{5.3.25}$$

and

$$\frac{1}{2} \le \big(\fint_{\partial B_3} P^2\big)^{\frac{1}{2}} \le \frac{3}{2}, \tag{5.3.26}$$

where C is a positive constant depending only on n and λ.

Set

$$\eta^5 = \omega(y; Rr)(CR)^N + \big(\frac{C}{R}\big)^N.$$

We claim that, if $\eta < \eta_0$, with η_0 as in Lemma 5.2.5, then

$$\mathcal{H}^{n-1}\big(B_1 \cap u_B^{-1}(0) \cap \{|DP_B| > \eta\}\big) \le c(n)N.$$

In fact, we have by Lemma 5.2.5

$$\begin{aligned}
&\mathcal{H}^{n-1}\big(B_1 \cap u_B^{-1}(0) \cap \{|DP_B| > \eta\}\big)\\
\le&\mathcal{H}^{n-1}\big(B_1 \cap u_B^{-1}(0) \cap \{|Du_B| > \frac{1}{2}\eta\}\big)\\
\le&2\mathcal{H}^{n-1}\big(B_2 \cap P_B^{-1}(0) \cap \{|DP_B| > \frac{1}{4}\eta\}\big) \le c(n)N.
\end{aligned}$$

Now we discuss the set

$$B_1 \cap u_B^{-1}(0) \cap \{|DP_B| < \eta\}.$$

We assume this set is not empty. In other words, there exists an $x_0 \in B_1$ such that $u_B(x_0) = 0$ and $|DP_B(x_0)| < \eta$. By (5.3.24) and (5.3.26), we have $|P_B(x_0)| < \eta$ and $\int_{\partial B_3} P_B^2 \geq c$. Hence Lemma 5.3.2 is applicable. Now we require

$$\eta < (\theta\varepsilon)^{2N-1}. \tag{5.3.27}$$

By Remark 5.3.3, there are at most $cd^{c_0}\varepsilon^{2-n}\log\varepsilon^{-1}$ balls of radius $\varepsilon/2$ covering $\{|DP_B| < \eta\}$. We drop any ball B_i if $B_i \cap u_B^{-1}(0) = \emptyset$. Otherwise, we replace B_i by a ball with the center in $u_B^{-1}(0) \cap B_1$ and radius ε. To satisfy (5.3.27), we require

$$\omega(y; Rr)(CR)^N \leq \frac{1}{2}(\theta\varepsilon)^{5(2N-1)}, \quad \big(\frac{C}{R}\big)^N \leq \frac{1}{2}(\theta\varepsilon)^{5(2N-1)}.$$

Hence we take

$$R = \big(\frac{C}{\varepsilon}\big)^{10},$$

and require

$$\omega\big(y; \big(\frac{C}{\varepsilon}\big)^{10} r\big) \leq \big(\frac{C}{\varepsilon}\big)^{20N},$$

where C is a positive constant depending only on n, λ, κ and Γ. ☐

REMARK 5.3.6. We note that $\omega(y; Rr) < Rr(\kappa + \Gamma)$ for any $y \in B_{1/2}$ and any $Rr < 1/2$. Then we may take

$$r(N, \varepsilon) = \big(\frac{\varepsilon}{C}\big)^{20N+10}.$$

The main result in this section is the following theorem due to Hardt and Simon [**51**].

THEOREM 5.3.7. *Suppose $u \in \mathcal{S}_N$ is a nonzero solution of $\mathcal{L}u = 0$ in $B_1 \subset \mathbb{R}^n$ for some $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \Gamma)$. Then*

$$\begin{aligned} \mathcal{H}^{n-1}\big(\{x \in B_r(x_0);\, u(x) = 0\}\big) &\leq C_0 N r^{n-1} \\ \textit{for any } x_0 \in B_{\frac{1}{2}} \textit{ and } r &\leq (c_1 N)^{-c_2 N}, \end{aligned} \tag{5.3.28}$$

where C_0, c_1 and c_2 are positive constants depending only on n, λ, κ and Γ.

PROOF. We fix ε such that

$$cN^{c_0}\varepsilon\log\varepsilon^{-1} < \frac{1}{2},$$

or

$$\varepsilon = CN^{-(c_0+1)}.$$

Next, we take any $r \leq r(N, \varepsilon)$. Then we proceed as in the proof of Theorem 5.2.1 and get

$$\mathcal{H}^{n-1}\big(u^{-1}\{0\} \cap B_r\big) \leq cNr^{n-1}\Big(\sum_{\ell=1}^{\infty}\frac{1}{2^{\ell-1}} + \inf_{\ell\geq 0}\sum_{j=\ell}^{\infty}\frac{1}{2^j}\Big) \leq 2CNr^{n-1}.$$

We end the proof by substituting the expression of ε in $r(N, \varepsilon)$ in Remark 5.3.6. ☐

Similar to Corollary 5.2.8, we have the following result.

Corollary 5.3.8. *Suppose* $u \in \mathcal{S}_N$ *is a nonzero solution of* $\mathcal{L}u = 0$ *in* $B_1 \subset \mathbb{R}^n$ *for some* $\mathcal{L} \in \mathcal{L}(\lambda, \kappa, \Gamma)$. *Then*

$$\mathcal{H}^{n-1}\big(\{x \in B_{\frac{1}{2}};\, u(x) = 0\}\big) \le (c_1 N)^{c_2 N}, \tag{5.3.29}$$

where c_1 *and* c_2 *are positive constants depending only on* n, λ, κ *and* Γ.

CHAPTER 6

Measure Estimates of Nodal Sets of Eigenfunctions

In this chapter, we discuss measure estimates of nodal sets of eigenfunctions. The following conjecture was proposed by Yau [**89**]. Suppose (M^n, g) is a smooth n-dimensional connected and compact Riemannian manifold (without boundary). Consider an eigenfunction u corresponding to the eigenvalue λ, i.e.,

$$\Delta_g u + \lambda u = 0 \quad \text{on } M.$$

Conjecture 6.0.1. There holds

$$c_1\sqrt{\lambda} \le \mathcal{H}^{n-1}(\{x \in M; u(x) = 0\}) \le c_2\sqrt{\lambda},$$

where c_1 and c_2 are positive constants depending only on (M, g).

The optimality is shown by eigenfunctions $\sin(k_1x_1)\sin(k_2x_2)\cdots\sin(k_nx_n)$ on $\mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

In this chapter, we prove that Conjecture 6.0.1 holds for the analytic case. The main result is the following theorem due to Donnelly and Fefferman [**27**].

Theorem 6.0.2. *Suppose (M^n, g) is an analytic n-dimensional connected and compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. Then*

$$c_1\sqrt{\lambda} \le \mathcal{H}^{n-1}(\{x \in M; u(x) = 0\}) \le c_2\sqrt{\lambda},$$

where c_1 and c_2 are positive constants depending only on (M, g).

It is still open whether Conjecture 6.0.1 holds if (M, g) is only smooth. The result we have for the smooth case is far from optimal.

6.1. Upper Bounds

In this section, we will prove the upper bound in Theorem 6.0.2, following an argument by Lin [**69**].

We first prove that u satisfies a doubling condition.

Lemma 6.1.1. *Suppose (M^n, g) is a smooth n-dimensional connected and compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. Then*

$$\fint_{B_{2r}(x)} u^2 \le 2^{c\sqrt{\lambda}} \fint_{B_r(x)} u^2 \quad \text{for any } x \in M,\ r \in (0, r_0), \tag{6.1.1}$$

where r_0 and c are positive constants depending only on (M, g).

We note that the analyticity is not assumed for (M, g).

PROOF. We first note that $\lambda \geq \lambda_0$ for some positive constant λ_0 depending only on (M, g). For any $(x, x_{n+1}) \in M \times \mathbb{R}$, set

$$\bar{u}(x, x_{n+1}) = u(x) \cosh(\sqrt{\lambda} x_{n+1}).$$

Then $\bar{u}$ satisfies

$$\Delta_g \bar{u} + \bar{u}_{n+1\ n+1} = 0 \quad \text{in } M \times \mathbb{R}.$$

By a suitable scaling in the metric g, we assume that the injectivity radius of (M, g) is not less then two. We also normalize $\bar{u}$ so that

$$\int_{M\times(-1,1)} \bar{u}^2 = 1.$$

Then it is easy to see

$$\int_{M\times(-2,2)} \bar{u}^2 \leq e^{c\sqrt{\lambda}}.$$

Hence, for some $x_0 \in M$, we have

$$\int_{B_1(x_0)\times(-1,1)} \bar{u}^2 \geq c(g, M) > 0, \tag{6.1.2}$$

and

$$\int_{B_2(x_0)\times(-2,2)} \bar{u}^2 \leq e^{c\sqrt{\lambda}}. \tag{6.1.3}$$

Proceeding as in the proof of Lemma 2.2.8 with the help of Theorem 3.1.1, we obtain

$$\int_{B_r(x)\times(-r,r)} \bar{u}^2 \geq r^{c(\sqrt{\lambda}+1)} \quad \text{for any } x \in B_{5/3}(x_0), r \in (0, 1/4), \tag{6.1.4}$$

where c is a positive constant depending only on g in $B_2(x_0)$.

Recall that M is connected. For any $x \in M$, we join x_0 and x by an overlapping chain of balls with radius 1/4 whose centers are separated by a distance at most 1/8. Using (6.1.2), (6.1.3) and (6.1.4) inductively, we have for any $x \in M$

$$\int_{B_{\frac{1}{4}}(x)\times(-\frac{1}{4},\frac{1}{4})} \bar{u}^2 \geq 4^{-c(\sqrt{\lambda}+1)},$$

and in general for any $r \in (0, 1/4)$

$$\int_{B_r(x)\times(-r,r)} \bar{u}^2 \geq r^{c(\sqrt{\lambda}+1)},$$

where c is a positive constant depending only on the diameter of (M, g), the ellipticity bound of Δ_g and the Lipschitz continuity of g on any $B_1(x)$, $x \in M$. In particular, we get

$$\int_{B_1(x)\times(-1,1)} \bar{u}^2 \leq 4^{c(\sqrt{\lambda}+1)} \int_{B_{\frac{1}{4}}(x)\times(-\frac{1}{4},\frac{1}{4})} \bar{u}^2 \quad \text{for any } x \in M.$$

By arguments as in the proof of Lemma 2.2.8 again, we obtain

$$\int_{B_{2r}(x)\times(-2r,2r)} \bar{u}^2 \leq 2^{c(\sqrt{\lambda}+1)} \int_{B_r(x)\times(-r,r)} \bar{u}^2 \quad \text{for any } x \in M, r \in (0, \frac{1}{4}). \tag{6.1.5}$$

With the explicit expression of $\bar{u}$ in terms of u, we then have easily

$$\int_{B_{2r}(x)} u^2 \leq 2^{c(\sqrt{\lambda}+1)} \int_{B_r(x)} u^2 \quad \text{for any } x \in M, r \in (0, \frac{1}{4}).$$

This ends the proof. $\square$

Now we are ready to prove the upper bound in Theorem 6.0.2.

THEOREM 6.1.2. *Suppose* (M^n, g) *is an analytic* n*-dimensional connected and compact Riemannian manifold (without boundary) and* u *is a smooth function satisfying* $\Delta_g u + \lambda u = 0$ *on* M *for some positive constant* λ*. Then*

$$\mathcal{H}^{n-1}(\{x \in M; u(x) = 0\}) \leq c\sqrt{\lambda}, \tag{6.1.6}$$

where c *is positive constant depending only on* (M, g)*.*

PROOF. We use the same setting as in the proof of Lemma 6.1.1. In view of (6.1.5), we apply Theorem 5.1.1 to $\bar{u}$ in $B_{1/2}(x) \times (-1/2, 1/2)$ for any $x \in M$. Therefore, we get by a simple covering

$$\mathcal{H}^n\big(\{(x, x_{n+1}) \in M \times (-\frac{1}{2}, \frac{1}{2});\ \bar{u}(x, x_{n+1}) = 0\}\big) \leq c\sqrt{\lambda}.$$

Note $\bar{u}^{-1}\{0\} = u^{-1}\{0\} \times \mathbb{R}$. This implies (6.1.6) easily. $\square$

We have the following result when (M, g) is only smooth.

THEOREM 6.1.3. *Suppose* (M^n, g) *is a smooth* n*-dimensional connected and compact Riemannian manifold (without boundary) and* u *is a smooth function satisfying* $\Delta_g u + \lambda u = 0$ *on* M *for some positive constant* λ*. Then*

$$\mathcal{H}^{n-1}(\{x \in M; u(x) = 0\}) \leq c\lambda^{c\sqrt{\lambda}},$$

where c *is positive constant depending only on* (M, g)*.*

It is unknown whether the estimate in Theorem 6.1.3 can be improved.

PROOF. The proof is similar as that of Theorem 6.1.2 with Theorem 5.1.1 replaced by Corollary 5.3.8. $\square$

To end this section, we prove a growth estimate in sup-norms. Suppose (M^n, g) is an analytic n-dimensional connected and compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. As in Section 5.1, u can be extended to a holomorphic function in $\{x + iy; x \in M, |y| < R\}$, for some positive constant R depending only (M, g).

LEMMA 6.1.4. *Suppose* (M^n, g) *is an analytic* n*-dimensional connected and compact Riemannian manifold (without boundary) and* u *is a smooth function satisfying* $\Delta_g u + \lambda u = 0$ *on* M *for some positive constant* λ*. Then there exists a positive constant* R*, depending only on* (M, g)*, such that*

$$\max_{|z-p|<r} |u(z)| \leq c^{\sqrt{\lambda}} \max_{|x-p|<\frac{r}{2}} |u(x)| \quad \text{for any } p \in M,\ r \in (0, R), \tag{6.1.7}$$

where c *is a positive constant depending only on* (M, g)*.*

Proof. We use the same setting as in the proof of Lemma 6.1.1. As in Section 5.1, $\bar{u}$ can be extended to a holomorphic function $\bar{u}(w)$ for $\text{Re}w \in M \times (-R, R)$ and $|\text{Im}w| \in (-R, R)^{n+1} \in \mathbb{R}^{n+1}$ with the estimate

$$\max_{|w-(p,0)|<r} |\bar{u}(w)| \leq c \fint_{B_{2r}(p)\times(-2r,2r)} \bar{u}^2 \quad \text{for any } p \in M,\ r \in (0, R).$$

By (6.1.5), we have

$$\max_{|w-(p,0)|<r} |\bar{u}(w)| \leq c^{\sqrt{\lambda}} \max_{|y-(p,0)|<\frac{r}{2}} |\bar{u}(y)| \quad \text{for any } p \in M,\ r \in (0, R).$$

This implies (6.1.7) with help of the explicit expression of $\bar{u}$ in terms of u. □

6.2. Lower Bounds

In this section, we will prove the lower bound in Theorem 6.0.2. The entire section follows Donnelly and Fefferman [**27**].

We first show that there are sufficiently many nodal points on M.

Lemma 6.2.1. *Suppose (M^n, g) is a smooth n-dimensional compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. Then u vanishes at some point in each ball of radius at most $c\lambda^{-1/2}$, for some positive constant c depending only on (M, g).*

Proof. Let w be the first Dirichlet eigenfunction of Δ in a ball $B_{c\lambda^{-1/2}}(p)$. If u is positive in this ball, then $v = w/u$ assumes an interior maximum. At this interior maximum point,

$$0 = v_i = \frac{1}{u}(w_i u - w u_i),$$

and

$$0 \geq \Delta v = \frac{1}{u^2}(u\Delta w - w\Delta u) \geq \frac{1}{u^2}\big(-\frac{1}{2}\lambda uw + \lambda uw\big) > 0,$$

for c sufficiently large. This leads to a contradiction. □

Lemma 6.2.2. *Suppose (M^n, g) is a smooth n-dimensional compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. If $u(p) = 0$ for some $p \in M$, then*

$$\min\Big\{\int_{B_{a\lambda^{-1/2}}(p)} u^+, \int_{B_{a\lambda^{-1/2}}(p)} u^-\Big\} \geq c\int_{B_{a\lambda^{-1/2}}(p)} |u|, \tag{6.2.1}$$

where a and c are positive constants depending only on (M, g), a being sufficiently small.

Here u^+ and u^- denote the positive part and negative part of u respectively.

Proof. Let a be sufficiently small. If we rescale $B_{a\lambda^{-1/2}}(p)$ to the ball of radius 1, then the operator $\Delta + \lambda$ is a small perturbation of the Euclidean Laplacian on the unit ball. Thus, we have for any $r \in (0, a\lambda^{-1/2})$

$$0 = u(p) = \int_{\{|x-p|=r\}} \varphi(x)u(x)d\theta, \tag{6.2.2}$$

where φ is a positive function with $C^{-1} < \varphi(x) < C$ and $d\theta$ denotes the volume element on the standard unit sphere $\mathbb{S}^{n-1}$. Multiplying (6.2.2) by r^{n-1} and integrating in r, we get

$$\int_{B_{a\lambda^{-1/2}}(p)} \varphi u = 0.$$

We then obtain

$$\int_{B_{a\lambda^{-1/2}}(p)} \varphi u^+ = \int_{B_{a\lambda^{-1/2}}(p)} \varphi u^- = \frac{1}{2}\int_{B_{a\lambda^{-1/2}}(p)} \varphi |u|.$$

We obtain (6.2.1) from the bounds on φ. ☐

Next, we prove the following result due to Brüning [**15**], which yields an affirmative answer to the lower bound in Conjecture 6.0.1 for $n = 2$.

Theorem 6.2.3. *Suppose (M^2, g) is a smooth 2-dimensional compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. Then*

$$\mathcal{H}^1\big(\{x \in M; u(x) = 0\}\big) \geq c\sqrt{\lambda},$$

where c is a positive constant depending only on (M, g).

Proof. Take any $p \in M$ with $u(p) = 0$. Then close to p, $u^{-1}\{0\}$ has the same structure as $\mathrm{Re} z^k = 0$ for some k. Now consider a ball $B_{a\lambda^{-1/2}}(p)$ as in Lemma 6.2.2 for sufficiently small a. If we rescale $B_{a\lambda^{-1/2}}(p)$ to the ball of radius 1, then the operator $\Delta + \lambda$ is a small perturbation of the Euclidean Laplacian on the unit ball. The maximum principle implies that $u^{-1}\{0\}$ cannot form closed loops in $B_{a\lambda^{-1/2}}(p)$. Hence, we have

$$\mathcal{H}^1\big(u^{-1}\{0\} \cap B_{a\lambda^{-1/2}}(p)\big) \geq a\lambda^{-1/2}.$$

By Lemma 6.2.1, we take disjoint balls $\{B_{a\lambda^{-1/2}}(p)\}$ with $u(p) = 0$ and the number of the balls at the order of $O(\lambda)$. We finish the proof by a simple addition. ☐

The higher dimensions are more complicated. We describe a general method to get a lower bound of the measure of nodal sets. This is based on the isoperimetric inequality. Consider a continuous function u defined in a ball $B_r \subset \mathbb{R}^n$ such that $\{u = 0\}$ is countably $(n-1)$-rectifiable. We set

$$B_r^+ = \{x \in B_r; u(x) > 0\}, \quad B_r^- = \{x \in B_r; u(x) < 0\}.$$

The isoperimetric inequality implies

$$\mathcal{H}^{n-1}(u^{-1}\{0\} \cap B_r) \geq c(n)\left(\min\{\mathcal{L}^n(B_r^+), \mathcal{L}^n(B_r^-)\}\right)^{\frac{n-1}{n}}. \tag{6.2.3}$$

There are three cases for the relation between $\mathcal{L}^n(B_r^+)$ and $\mathcal{L}^n(B_r^-)$:

(i) $\mathcal{L}^n(B_r^+)$ and $\mathcal{L}^n(B_r^-)$ are comparable;
(ii) $\mathcal{L}^n(B_r^+) \ll \mathcal{L}^n(B_r^-)$;
(iii) $\mathcal{L}^n(B_r^+) \gg \mathcal{L}^n(B_r^-)$.

In order to get a lower bound, it suffices to prove that $\mathcal{L}^n(B_r^+)$ and $\mathcal{L}^n(B_r^-)$ are comparable. In other words, we need to prove that (ii) and (iii) cannot occur.

Next, we prove a general result.

LEMMA 6.2.4. *Let u be a continuous function defined in $B_r \subset \mathbb{R}^n$ such that $\{u = 0\}$ is countably $(n-1)$-rectifiable,*

$$\min\Big\{\int_{B_r} u^+, \int_{B_r} u^-\Big\} \geq c_0 \int_{B_r} |u|, \tag{6.2.4}$$

and

$$\Big(\fint_{B_r} u^2\Big)^{\frac{1}{2}} \leq \mu \fint_{B_r} |u|, \tag{6.2.5}$$

for some positive constants c_0 and μ. Then

$$\mathcal{H}^{n-1}(u^{-1}\{0\} \cap B_r) \geq c\mu^{-\frac{2(n-1)}{n}} r^{n-1}, \tag{6.2.6}$$

where c is a positive constant depending only on n and c_0.

PROOF. By the Cauchy inequality and (6.2.5), we have for any measurable set $G \subset B_r$

$$\Big(\int_G |u|\Big)^2 \leq \mathcal{L}^n(G)\int_G u^2 \leq \mathcal{L}^n(G)\int_{B_r} u^2 \leq \mu^2 \frac{\mathcal{L}^n(G)}{\mathcal{L}^n(B_r)}\Big(\int_{B_r} |u|\Big)^2,$$

or

$$\mathcal{L}^n(G) \geq \frac{1}{\mu^2}\Big(\frac{\int_G |u|}{\int_{B_r} |u|}\Big)^2 \mathcal{L}^n(B_r).$$

By (6.2.4) and taking $G = B_r^{\pm}$, we get

$$\mathcal{L}^n(B_r^+) \geq \frac{c}{\mu^2} r^n, \quad \mathcal{L}^n(B_r^-) \geq \frac{c}{\mu^2} r^n.$$

With (6.2.3), we finish the proof. □

With Lemmas 6.2.1-6.2.4, we will proceed to estimate the lower bound in Theorem 6.0.2 as follows. We take sufficiently many balls $B_{a\lambda^{-1/2}}(p)$ on M where (6.2.4) and (6.2.5) hold. Then we have

$$\mathcal{H}^{n-1}\big(u^{-1}\{0\} \cap B_{a\lambda^{-1/2}}(p)\big) \geq c\mu^{-\frac{2(n-1)}{n}} (\lambda^{-1/2})^{n-1}.$$

The number of balls is in the order of $\lambda^{n/2}$. Then by summing up in these balls, we obtain

$$\mathcal{H}^{n-1}(u^{-1}\{0\}) \geq c\mu^{-\frac{2(n-1)}{n}} \lambda^{1/2}.$$

In order to get the optimal lower bound, μ should be independent of λ.

We note that it is easy to get (6.2.5) for μ depending on λ. In fact, we have the following result.

THEOREM 6.2.5. *Suppose (M^n, g) is a smooth n-dimensional compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. Then*

$$\mathcal{H}^{n-1}\big(\{x \in M; u(x) = 0\}\big) \geq c^{-\sqrt{\lambda}},$$

where c is a positive constant depending only on (M, g).

PROOF. This is a simple consequence of the doubling condition. We take a ball $B_r(p)$, with $r = a\lambda^{-1/2}$, as in Lemma 6.2.2 for sufficiently small a. First, a standard a priori estimate implies

$$\sup_{B_r(p)} |u| \leq c\big(\fint_{B_{2r}(p)} |u|^2\big)^{\frac{1}{2}},$$

and hence with Lemma 6.1.1

$$\sup_{B_r(p)} |u| \leq c^{\sqrt{\lambda}}\big(\fint_{B_r(p)} |u|^2\big)^{\frac{1}{2}}.$$

Then we have

$$\fint_{B_r(p)} |u|^2 \leq |u|_{L^\infty(B_r(p))} \fint_{B_r(p)} |u| \leq c^{\sqrt{\lambda}}\big(\fint_{B_r(p)} |u|^2\big)^{\frac{1}{2}} \fint_{B_r(p)} |u|,$$

or

$$\big(\fint_{B_r(p)} |u|^2\big)^{\frac{1}{2}} \leq c^{\sqrt{\lambda}} \fint_{B_r(p)} |u|.$$

Therefore, we can take $\mu = c^{\sqrt{\lambda}}$ in (6.2.5). Lemma 6.2.4 implies

$$\mathcal{H}^{n-1}\big(u^{-1}\{0\} \cap B_r(p)\big) \geq c^{-\sqrt{\lambda}} r^{n-1}.$$

This completes the proof. □

Note that the previous proof does not yield the optimal lower bound. In the following, we will prove that (6.2.5) holds in sufficiently many balls $B_{a\lambda^{-1/2}}(p)$ on M with $u(p) = 0$ for a constant μ depending only on (M, g), independent of λ. It suffices to prove

$$\big(\fint_{B_{2a\lambda^{-1/2}}(p)} u^2\big)^{\frac{1}{2}} \leq c\big(\fint_{B_{a\lambda^{-1/2}}(p)} u^2\big)^{\frac{1}{2}},$$

for sufficiently many balls $B_{a\lambda^{-1/2}}(p)$ on M with $u(p) = 0$ for a constant c depending only on (M, g). This is a much improved doubling condition with the doubling coefficient independent of λ.

We need some preparations.

PROPOSITION 6.2.6. *Suppose F is holomorphic on $D_3 \subset \mathbb{C}$ and $|F|_{L^\infty(D_2)} \leq |F(0)|e^{c_1 d}$. Assume $F(x)$ is real and nonnegative for $|x| \leq 1$. For d sufficiently large, cover $|x| \leq 1$ by disjoint subintervals Q_i of length c_2/d. Then for any given $\varepsilon > 0$, there exists a subset E of measure less than ε such that*

$$|\log F(x) - \log \fint_{Q_i} F| \leq c_3 \quad \textit{for any } x \in Q_i \setminus E,$$

where c_3 is a positive constant depending on ε but not on d.

PROOF. Without loss of generality, we assume $F(0) = 1$. By Lemma 2.3.2,

$$F \text{ has at most } O(d) \text{ zeroes in } |z| < 3/2. \tag{6.2.7}$$

Now fix an r close to $3/2$ such that $F(z) \neq 0$ for $|z| = r$ and let $\{z_k\}$ be the collection of zeroes of F in $\{|z| < r\}$. We write

$$F(z) = e^{G(z)} \prod \frac{(z - z_k)/r}{1 - \bar{z}_k z/r^2} \quad \text{for } |z| \leq r.$$

Then it is easy to see

$$\max_{|z|=r} \mathrm{Re}G \le cd. \tag{6.2.8}$$

With $F(0) = 1$, we get $e^{\mathrm{Re}G(0)} \ge 1$ or $\mathrm{Re}G(0) \ge 0$. The mean value property of harmonic functions implies

$$\fint_{\{|z|=r\}} \mathrm{Re}G \ge 0. \tag{6.2.9}$$

By (6.2.8), (6.2.9) and the Poisson kernel representation of harmonic functions, we obtain

$$\max_{|z|=1} |\nabla \mathrm{Re}G| \le cd. \tag{6.2.10}$$

Define $f(z) = \sum_k \log|z - z_k|$. Using (6.2.10) and elementary arguments, we have

$$|\log F(x) - \log \fint_{Q_i} F| < |\log f(x) - \log \fint_{Q_i} e^f| + c \quad \text{for any } x \in Q_i.$$

Here we used the fact that the length of each Q_i is less than c_2/d. Now we claim

$$|f'| < cd \quad \text{outside a set } E_1 \text{ of measure less than } a_1\varepsilon. \tag{6.2.11}$$

To see this, we assume $\mathrm{Im}z_k \le 0$ since $|x - z_k| = |x - \bar{z}_k|$. If all $\mathrm{Im}z_k < 0$, then the definition of the Hilbert transform H ([**85**], P130) yields

$$f' = H(\sum_k q_k), \quad q_k = -\mathrm{Im}(\frac{1}{z - z_k}).$$

Clearly, $\|q_k\|_{L^1} < c$, with c independent of x_k. The weak type (1,1) property of H ([**85**], P187) completes the proof for $\mathrm{Im}z_k < 0$. Since these estimates are uniform in z_k, the result also holds for $\mathrm{Im}z_k \le 0$.

Suppose $x, x_i \in Q_i$ and let A_i be the set of roots z_k with $\mathrm{Re}z_k$ of distance less than $m(Q_i) = O(1/d)$ from Q_i. We decompose

$$f(x) = \sum_{z_k \in A_i} \log|x - z_k| + \sum_{z_k \notin A_i} \log|x - z_k| = b_i(x) + g_i(x)$$

and estimate each of the two terms.

Define E_2 to be the union of those Q_i with A_i containing more than c roots. By (6.2.7), we require the measure $m(E_2) < a_2\varepsilon$. If Q_i is not contained in E_2, then let $Q_{i\varepsilon} \subset Q_i$ be the subset of $x \in Q_i$ with $|x - z_k| < c/d$ for some z_k. We assume

$$m(Q_{i\varepsilon}) < a_3\frac{\varepsilon}{d} < \frac{1}{2}m(Q_i).$$

Now we claim that, if Q_i is not contained in E_2, then for any $x \in Q_i \setminus Q_{i\varepsilon}$,

$$|b_i(x) - \max_{Q_i} b_i(x)| < c, \tag{6.2.12}$$

and

$$|b_i'(x)| < cd. \tag{6.2.13}$$

In fact, for any $x \in Q_i \setminus Q_{i\varepsilon}$,

$$\log|x - z_k| \ge \max_{x \in Q_i} \log|x - z_k| - c \quad \text{for any } z_k \in A_i.$$

Summing over z_k yields

$$b_i(x) \geq \max_{x \in Q_i} b_i(x) - c,$$

which implies (6.2.12). Now (6.2.13) is immediate from the definitions of E_2 and $Q_{i\varepsilon}$.

We now turn to g_i. Note $g_i' = f' - b_i'$. If Q_i is not contained in $E_1 \cup E_2 \cup Q_{i\varepsilon}$, then

$$\text{there exists an } x_i \in Q_i \setminus Q_{i\varepsilon} \text{ with } |g_i'(x_i)| < cd. \tag{6.2.14}$$

This follows easily from (6.2.11) and (6.2.13).

It is also necessary to estimate the second derivative. Clearly

$$|g_i''(x)| \leq c \sum_{z_k \notin A_i} |x - z_k|^{-2}$$

and the right hand side has a constant order of magnitude for $x \in Q_i$. Thus

$$\sum_i \max_{Q_i} |g_i''(x)| \; |Q_i| \leq c \sum_k \int_{|x|<1, |x - \mathrm{Re} z_k| > cd^{-1}} |x - z_k|^{-2} \leq cd^2.$$

Hence there exists a union E_3 of intervals Q_i with $m(E_3) < a_4 \varepsilon$ so that if Q_i is not contained in E_3 then

$$\max_{x \in Q_i} |g_i''(x)| \leq cd^2. \tag{6.2.15}$$

Now suppose Q_i is not contained in $E_1 \cup E_2 \cup E_3 \cup Q_{i\varepsilon}$. Let $x_i \in Q_i$ be as in (6.2.14). By (6.2.14), (6.2.15) and Taylor's formula with remainder, we have

$$|g_i(x) - g_i(x_i)| < c \quad \text{for any } x \in Q_i. \tag{6.2.16}$$

Then using (6.2.12), (6.2.16) and elementary arguments, we obtain

$$|\log f(x) - \log \fint_{Q_i} e^f| < c \quad \text{for any } x \in Q_i \setminus Q_{i\varepsilon}.$$

This completes the proof. ☐

We now turn to several complex variables.

Proposition 6.2.7. *Suppose F is holomorphic on $D_3 \subset \mathbb{C}^n$ and $|F|_{L^\infty(D_2)} \leq |F(0)| e^{c_1 d}$. Assume $F(x)$ is real and nonnegative on the cube Q given by $|x_i| \leq 1$, $i = 1, \cdots, n$, in $\mathbb{R}^n$. Additionally, suppose that $F(x) = 1$ on any hyperplane $x_i = 0$, $i = 1, \cdots, n$. For d sufficiently large, subdivide Q into cubes Q_i of sides c_2/d. Then for any given $\varepsilon > 0$, there exists a subset E of measure less than ε such that*

$$|\log F(x) - \log \fint_{Q_i} F| \leq c_3 \quad \textit{for any } x \in Q_i \setminus E,$$

where c_3 is a positive constant depending on ε but not on d.

Proof. This follows from Proposition 6.2.6 by induction. We successively average over each coordinate direction. The extra technical hypothesis that $F = 1$ on each coordinate hyperplane is crucial for this argument. ☐

We proceed to remove the technical hypotheses in Proposition 6.2.7. Define maps T_j by

$$\begin{aligned} T_j(x_1, \cdots, x_n) =& (x_1, x_2, \cdots, x_j, x_j x_{j+1}, \cdots, x_n) \quad j = 1, \cdots, n-1, \\ T_n(x_1, \cdots, x_n) =& (x_n x_1, x_2, \cdots, x_j, \cdots, x_n). \end{aligned}$$

Set $T = T_n T_{n-1} \cdots T_1$ and $W = T^2$.

LEMMA 6.2.8. *W maps every coordinate hyperplane $\{x_i = 0\}$ to the origin. The Jacobian determinant of W vanishes along the coordinate hyperplanes only. There exists an open set $U \subset Q$ so that $W : U \to W(U)$ is a diffeomorphism.*

PROOF. The first two statements are verified by direct computations. The last assertion then follows from the inverse function theorem. □

Now we prove the main ingredient in the proof of the lower bound.

PROPOSITION 6.2.9. *Suppose k is a sufficiently large integer depending only on n and F is holomorphic on $D_{3^k} \subset \mathbb{C}^n$ and $|F|_{L^\infty(D_{2^k})} \leq |F(0)|e^{c_1 d}$. Assume $F(x)$ is real and nonnegative on the cube Q given by $|x_i| \leq 1$, $i = 1, \cdots, n$, in $\mathbb{R}^n$. Suppose R is a suitable cube contained in Q. For d sufficiently large, subdivide R into cubes R_i of sides c_2/d. Then for any given $\varepsilon > 0$, there exists a subset E of measure less than ε such that*

$$|\log F(x) - \log \fint_{R_i} F| \leq c_3 \quad \text{for any } x \in R_i \setminus E,$$

where c_3 is a positive constant depending on ε but not on d.

PROOF. We assume $F(0) = 1$. Choose $R \subset W(U)$ where $W(U)$ is obtained from Lemma 6.2.8. The function $F \circ W$ satisfies the hypotheses of Proposition 6.2.7. The conclusion of Proposition 6.2.7 then implies that if $Q_i \subset U$, we have outside a set of measure of $a_5 \varepsilon$

$$|\log F(x) - \log \fint_{W(Q_i)} F| \leq c,$$

since the Jacobian of W is bounded on U. For d sufficiently large, we assume that each R_i is contained in some $W(Q_i)$ except for a union of R_i whose total measure is less than $a_6 \varepsilon$. Also, we require that

$$c^{-1} \leq \frac{m(R_i)}{m(Q_i)} \leq c.$$

We then finish the proof by elementary arguments. □

Now we are ready to prove the lower bound in Theorem 6.0.2.

THEOREM 6.2.10. *Suppose (M^n, g) is an analytic n-dimensional compact Riemannian manifold (without boundary) and u is a smooth function satisfying $\Delta_g u + \lambda u = 0$ on M for some positive constant λ. Then*

$$\mathcal{H}^{n-1}(\{x \in M; u(x) = 0\}) \geq c\sqrt{\lambda}, \tag{6.2.17}$$

where c is positive constant depending only on (M, g).

Proof. It suffices to consider the nodal sets contained in a single coordinate patch U. By Lemma 6.2.1, there is at least one nodal point inside every ball of radius $a_1\lambda^{-1/2}$. Cover U by cubes Q_i of sides $a_2\lambda^{-1/2}$, $a_2 > a_1$, so that there exists a nodal point x_i in the middle tenth of Q_i. Choose a_3 so that $B_i = B_{a_3\lambda^{-1/2}}(x_i)$ is completely contained in the middle one half of Q_i.

We now apply Proposition 6.2.9 for the nonnegative function u^2. The required hypotheses are guaranteed by (6.1.7). Hence there exists a fixed cube $R \subset U$ so that for any sufficiently small $\varepsilon > 0$

$$|\log u^2(x) - \log \fint_{Q_i} u^2| \le c \quad \text{for any } x \in R \cap Q_i \setminus E,$$

where E is a subset of measure less than ε and c is a positive constant depending on ε but not on d. Let $R_i = R \cap Q_i \setminus E$. Then we conclude easily that half of the Q_i satisfy

$$m(R_i) \ge (1 - a_4\varepsilon)m(Q_i). \tag{6.2.18}$$

Here m denotes the measure. Let $\mathcal{S}$ denote the set of those Q_i satisfying (6.2.18). Fix $\varepsilon > 0$ sufficiently small and consider only those $Q_i \in \mathcal{S}$. Then clearly we have

$$\fint_{Q_i} u^2 \le c \fint_{B_i} u^2, \tag{6.2.19}$$

where c is a positive constant depending only on (M, g), independent of λ. By a similar argument as in the proof of Theorem 6.2.5, we get

$$\Big(\fint_{B_i} u^2\Big)^{\frac{1}{2}} \le c \fint_{B_i} |u|.$$

By Lemma 6.2.4, we obtain

$$\mathcal{H}^{n-1}(u^{-1}\{0\} \cap B_i) \ge c(\lambda^{-1/2})^{n-1}.$$

Summing over those $Q_i \in \mathcal{S}$ satisfying (6.2.18), we obtain

$$\mathcal{H}^{n-1}(u^{-1}\{0\}) \ge \sum_{Q_i \in \mathcal{S}} \mathcal{H}^{n-1}(u^{-1}\{0\} \cap B_i) \ge c\sqrt{\lambda}.$$

This completes the proof. □

CHAPTER 7

Measure Estimates of Singular Sets

In this chapter, we discuss the geometric measure of singular sets of solutions of linear elliptic differential equations of second order.

We begin with a simple example. Consider a homogenous harmonic polynomial of degree d in $\mathbb{R}^2$. By using polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ in $\mathbb{R}^2 = \{(x_1, x_2)\}$, we assume $P(x) = r^d \cos d\theta$. A direct calculation shows

$$\partial_1 P = dr^{d-1}\cos(d-1)\theta,\ \partial_2 P = -dr^{d-1}\sin(d-1)\theta.$$

Therefore, both $\partial_1 P$ and $\partial_2 P$ are products of $d-1$ distinct homogeneous linear functions. Now assume u is a smooth perturbation of P in B_1. Then, it is not hard to imagine that the critical set of u has at most $(d-1)^2$ points in B_1. As we see, this is quite difficult to prove if u is only smooth.

In Section 2.4, we discussed singular sets of planar harmonic functions and derived an optimal estimate on the number of points in singular sets. In this chapter, we discuss singular sets of solutions of general elliptic differential equations. We consider the homogeneous elliptic differential equation of the form

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u = 0 \quad \text{in } B_1 \subset \mathbb{R}^n,$$

where the coefficients a_{ij} satisfy

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n,\ x \in B_1,$$

for some positive constant λ. We assume a_{ij} are Lipschitz and b_i and c are at least bounded. The Lipschitz condition for the leading coefficients is essential. It implies the unique continuation for the operator $\mathcal{L}$. In other words, if a solution u vanishes to an infinite order at a point in B_1, then u is identically zero. For details, see Chapter 3.

Set

$$N = \frac{\int_{B_1}|\nabla u|^2}{\int_{\partial B_1} u^2}.$$

It is proved in Chapter 3 that u satisfies

$$\fint_{B_{2r}(x_0)} u^2 \leq 4^{c_1 N + c_2} \fint_{B_r(x_0)} u^2(x) \quad \text{for any } x_0 \in B_{\frac{1}{2}},\ r < r_0,$$

where c_1, c_2 and $r_0 < 1/3$ are positive constants depending only on n, the ellipticity constant λ, the Lipschitz constant of leading coefficients and the L^∞-norms of all coefficients. We then conclude that the vanishing order of u at any point $p \in B_{1/2}$

does not exceed $c_1N + c_2$. The quantity N is called the frequency of u in B_1. It controls the vanishing order of u.

The following conjecture concerns the size of singular sets in terms of the frequency. It was proposed by Lin [**69**].

CONJECTURE 7.0.1. Let u be a solution of $\mathcal{L}u = 0$ in $B_1 \subset \mathbb{R}^n$. Then

$$\mathcal{H}^{n-2}\{x \in B_{\frac{1}{2}}; u(x) = |Du|(x) = 0\} \le CN^2,$$

where C is a positive constant depending only on the elliptic operator $\mathcal{L}$.

At this time, it is known to be true only for planar harmonic functions. It is open even for harmonic functions in the general dimension.

7.1. Singular Sets of Harmonic Functions

In this section, we discuss singular sets of harmonic functions in arbitrary dimension. We first examine an example.

EXAMPLE 7.1.1. Consider the harmonic polynomial in $\mathbb{R}^3$

$$u(x_1, x_2, x_3) = x_1^2x_3 + x_2^2x_3 - \frac{2}{3}x_3^3 - \varepsilon x_3.$$

A simple calculation shows

$$Du(x_1, x_2, x_3) = (2x_1x_3, x_2x_3, x_1^2 + x_2^2 - 2x_3^2 - \varepsilon).$$

Then we have

$$\mathcal{S}(u) = \begin{cases} \emptyset & \text{for } \varepsilon < 0, \\ \{(0,0,0)\} & \text{for } \varepsilon = 0, \\ \{(x_1, x_2, 0); x_1^1 + x_2^2 = \varepsilon\} & \text{for } \varepsilon > 0. \end{cases}$$

The dimension of the singular set $\mathcal{S}(u)$ changes according to the sign of ε.

Example 7.1.1 illustrates that a serious problem arises if we study singular sets only in real spaces. This suggests that we need to study the singular set for holomorphic extensions of harmonic functions in the complex space. Results on the structure of singular sets in Chapter 4 play an important role.

The main result in this section is the following theorem, which was proved by Han, Hardt and Lin [**47**] for solutions of general elliptic differential equations with smooth coefficients.

THEOREM 7.1.2. *Suppose u is a harmonic function in $B_1 \subset \mathbb{R}^n$ with*

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}.$$

Then

$$\mathcal{H}^{n-2}(\mathcal{S}(u) \cap B_{\frac{1}{2}}) \le C(N),$$

where $C(N)$ is a positive constant depending only on n and N.

The key result is the following lemma. We denote by B_r and D_r balls of radius r in $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively.

LEMMA 7.1.3. *Let P be a homogeneous harmonic polynomial of degree $d \geq 2$ and of two variables in $\mathbb{R}^n$. Then there exist positive constants ε and r, depending on P, such that for any harmonic function u in B_1 with*

$$|u - P|_{L^\infty(B_1)} < \varepsilon,$$

there holds

$$\mathcal{H}^{n-2}(|Du|^{-1}\{0\} \cap B_r) \leq c(n)(d-1)^2 r^{n-2}.$$

PROOF. First, there exists a universal constant $R \in (0,1)$ such that $u(x)$ can be extended to a holomorphic function $u(z)$ in D_R with

$$|u - P|_{C^1(D_R)} \leq c(n)|u - P|_{L^\infty(B_1)} \leq c\varepsilon. \tag{7.1.1}$$

In the following, we assume $\int_{\partial B_1} P^2 = 1$.

We first prove for $n = 2$. By (2.4.11), we have

$$|DP(z)| \geq c_0|z|^{d-1},$$

where c_0 is a positive universal constant. By taking ε small in (7.1.1), we get

$$|Du(z) - DP(z)| < |PD(z)| \quad \text{for any } |z| = R.$$

Note that Lemma 1.3.1, the Bezout's formula, implies

$$\#\{|DP|^{-1}(0)\} = (d-1)^2 \quad \text{(including the multiplicity)}.$$

By Lemma 1.3.2, the Rouché Theorem in $\mathbb{C}^2$, we have

$$\#\{|Du|^{-1}(0) \cap D_R\} \leq (d-1)^2.$$

This implies in particular

$$\#\{|Du|^{-1}(0) \cap B_R\} \leq (d-1)^2.$$

Next, we discuss the general dimension. We temporarily denote by $\tilde{x}$ the coordinates in $\mathbb{R}^n$ and by $\tilde{z}$ the corresponding complex coordinates. In the following, we set $Du = (f, \tilde{f})$ and $DP = (g, 0)$. Here we treat f and g as maps from $\mathbb{C}^n$ to $\mathbb{C}^2$. Then we have

$$|g(\tilde{z})|^2 \geq c_0(|\tilde{z}_1|^2 + |\tilde{z}_2|^2)^{\frac{d-1}{2}},$$

and

$$|f - g|_{L^\infty(D_R)} \leq c\varepsilon.$$

Now we introduce a change of coordinate $\tilde{x} = Ox$ in $\mathbb{R}^n$ with an orthogonal matrix $O = (o_{ij})$ to be chosen. Then in $\mathbb{C}^n$, we have $\tilde{z} = Oz$. In the following, we evaluate f and g in z. For simplicity, we still write f and g, instead of $f \circ O$ and $g \circ O$. Then we have

$$|g(z)|^2 \geq c_0 \left(|\sum_{i=1}^n o_{1i}z_i|^2 + |\sum_{i=1}^n o_{2i}z_i|^2 \right)^{\frac{d-1}{2}}. \tag{7.1.2}$$

Note that only the first two rows of the matrix O appear in (7.1.2).

For any $p \in \mathbb{R}^n$ and any $1 \leq i < j \leq n$, denote by $\mathbb{P}_{ij}(p)$ the 2-dimensional hyperplane

$$\{(p_1, \cdots, p_{i-1}, z_i, p_{i+1}, \cdots, p_{j-1}, z_j, p_{j+1}, \cdots, p_n)\},$$

and simply write $\mathbb{P}_{ij}(p) = \{(z_i, z_j)\}$ when there is no confusion. We also set $\mathbb{P}_{ij} = \mathbb{P}_{ij}(0)$.

Fix any $1 \leq i < j \leq n$. We consider f and g restricted on $\mathbb{P}_{ij}$. A straightforward calculation shows that

$$\begin{aligned} \left|g|_{\mathbb{P}_{ij}}\right|^2 \geq & c_0 \left(|o_{1i}z_i + o_{1j}z_j|^2 + |o_{2i}z_i + o_{2j}z_j|^2\right)^{\frac{d-1}{2}} \\ \geq & c_0 \left(c_*(|z_i|^2 + |z_j|^2)\right)^{\frac{d-1}{2}}, \end{aligned} \tag{7.1.3}$$

where

$$c_* = \min\left\{\frac{1}{2}(o_{1i}^2 + o_{1j}^2 + o_{2i}^2 + o_{2j}^2), \frac{(o_{1i}o_{2j} - o_{1j}o_{2i})^2}{o_{1i}^2 + o_{1j}^2 + o_{2i}^2 + o_{2j}^2}\right\}.$$

Therefore, we require that, in the orthogonal matrix O, any 2×2 submatrices in the first two rows have nonzero determinants. If we write $g = (g_1, g_2)$, then each g_i is a product of $d-1$ homogeneous linear functions with real-valued coefficients. We again apply Lemma 1.3.2, the Rouché Theorem in $\mathbb{C}^2$, to get the following conclusion. There exists a constant η_{ij} such that for any holomorphic function $v : D_R \subset \mathbb{C}^2 = \{(z_i, z_j)\} \to \mathbb{C}^2$ with

$$\left|v - g|_{\mathbb{P}_{ij}}\right|_{L^\infty(D_R^2)} < \eta_{ij}, \tag{7.1.4}$$

there holds

$$\#(v^{-1}\{0\} \cap D_R^2) \leq (d-1)^2. \tag{7.1.5}$$

Here we denote by D_R^2 the ball (centered at origin) with radius R in $\mathbb{C}^2$.

Take

$$\eta = \frac{1}{2}\min_{1\leq i<j\leq n} \eta_{ij}.$$

For any $p \in \mathbb{R}^n$ and any $1 \leq i < j \leq n$, set $v_{ij,p} = f\big|\mathbb{P}_{ij}(p)$. By taking ε small in (7.1.1), we have

$$\left|v_{ij,p} - g|_{\mathbb{P}_{ij}}\right|_{L^\infty(D_R^2)} < 2\eta \leq \eta_{ij}.$$

Then we have

$$\#(v_{ij,p}^{-1}\{0\} \cap D_R^2) \leq (d-1)^2.$$

Obviously $|Du|^{-1}\{0\} \cap \mathbb{P}_{ij}(p) \subset v_{ij,p}^{-1}\{0\}$. If we set π_{ij} as the projection

$$\pi_{ij}(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \in \mathbb{R}^{n-2},$$

then we have shown, in particular, that for any $q \in B_R^{n-2} \subset \mathbb{R}^{n-2}$ and any $1 \leq i < j \leq n$

$$\#(|Du|^{-1}\{0\} \cap \pi_{ij}^{-1}(q) \cap B_R) \leq (d-1)^2.$$

Hence Theorem 1.2.10, the integral geometric formula, implies

$$\begin{aligned} &\mathcal{H}^{n-2}(|Du|^{-1}\{0\} \cap B_R) \\ &\leq \sum_{1\leq i<j\leq n} \int_{B_r^{n-2}} \#(|Du|^{-1}\{0\} \cap \pi_{ij}^{-1}(q) \cap B_R) d\mathcal{H}^{n-2}q \leq c(n)(d-1)^2 R^{n-2}. \end{aligned}$$

This finishes the proof. □

As the first step in proving Theorem 7.1.2, we show the following result.

LEMMA 7.1.4. *Suppose u is a nonconstant harmonic function in $B_1 \subset \mathbb{R}^n$ with $\|u\|_{L^2(B_1)} = 1$. Then there exist positive constants $C(u)$ and $\varepsilon(u)$, depending on n and u, and a finite collection of balls $\{B_{r_i}(x_i)\}$ with $r_i \le 1/8$ and $x_i \in \mathcal{S}(u)$ such that for any harmonic function v in B_1 with*

$$|u - v|_{L^\infty(B_1)} < \varepsilon(u),$$

there hold

$$\mathcal{H}^{n-2}\left(\mathcal{S}(v) \cap B_{1/2} \backslash \cup B_{r_i}(x_i)\right) \le C(u),$$

and

$$\sum r_i^{n-2} \le \frac{1}{2^{n-1}}.$$

Lemma 7.1.4 illustrates that the singular set $\mathcal{S}(v)$ of v is decomposed into two parts, a good part and a bad part. The good part has a measure estimate and the bad part is covered by small balls. The key point here is that the estimate for the good part and the covering for the bad part can be made uniform for harmonic functions v close to some u.

PROOF. Let u be given in Lemma 7.1.4. By Lemma 4.1.4, we have

$$\mathcal{S}(u) = \mathcal{S}_b(u) \cup \mathcal{S}_g(u),$$

where $\mathcal{S}_b(u)$ has the Hausdorff dimension not exceeding $n-3$, $\mathcal{S}_g(u)$ is on a countable union of $(n-2)$-dimensional C^1 manifolds and for any $p \in \mathcal{S}_g(u)$ the leading polynomial of u at p is a homogeneous harmonic polynomial of 2 variables after an appropriate rotation. In particular, we have

$$\mathcal{H}^{n-2}\big(\mathcal{S}_b(u)\big) = 0.$$

Then, there exist at most countably many balls $B_{r_i}(x_i)$ with $r_i \le 1/8$ and $x_i \in \mathcal{S}_b(u)$ such that

$$\mathcal{S}_b(u) \subset \bigcup_i B_{r_i}(x_i), \tag{7.1.6}$$

and

$$\sum r_i^{n-2} \le \frac{1}{2^{n-1}}. \tag{7.1.7}$$

We claim for any $y \in \mathcal{S}_g(u) \cap B_{3/4}$, there exist positive constants $R < 1/8$, $r < R$, η and c, depending only on u and y, such that, for any harmonic function v satisfying

$$|u - v|_{L^\infty\left(B_R(y)\right)} < \eta, \tag{7.1.8}$$

then

$$\mathcal{H}^{n-2}\left\{\mathcal{S}(v) \cap B_r(y)\right\} \le c r^{n-2}. \tag{7.1.9}$$

We postpone the proof of (7.1.9).

It is obvious that the collection of $\{B_{r_i}(x_i)\}$ and $\{B_{r(y)}(y)\}$, $y \in \mathcal{S}_g(u)$, covers $\mathcal{S}(u)$. By the compactness of $\mathcal{S}(u)$, there exist $x_i \in \mathcal{S}_b(u)$, $i = 1, \cdots, k = k(u)$, and $y_j \in \mathcal{S}_g(u)$, $j = 1, \cdots, l = l(u)$, such that

$$\mathcal{S}(u) \cap B_{3/4} \subset \left(\bigcup_{i=1}^{k} B_{r_i}(x_i)\right) \bigcup \left(\bigcup_{j=1}^{l} B_{s_j}(y_j)\right), \tag{7.1.10}$$

with $r_i \le 1/8$, $i = 1, \cdots, k$, and $s_j \le 1/8$, $j = 1, \cdots, l$. Since $\mathcal{S}(u)$ is closed, there exists a positive constant $\rho = \rho(u)$ such that

$$\big\{x \in B_{3/4};\ \mathrm{dist}\big(x, \mathcal{S}(u)\big) < \rho\big\} \subset \left(\bigcup_{i=1}^{k} B_{r_i}(x_i)\right) \bigcup \left(\bigcup_{j=1}^{l} B_{s_j}(y_j)\right). \tag{7.1.11}$$

It is easy to see that for such a ρ there exists a positive constant $\delta = \delta(u)$ such that $|u - v|_{C^1(B_{3/4})} < \delta$ implies

$$\mathcal{S}(v) \cap B_{1/2} \subset \big\{x \in B_{3/4};\ \mathrm{dist}\big(x, \mathcal{S}(u)\big) < \rho\big\}. \tag{7.1.12}$$

Denote

$$\mathcal{B}_u = \bigcup_{i=1}^{k} B_{r_i}(x_i), \quad \mathcal{G}_u = \bigcup_{j=1}^{l} B_{s_j}(y_j).$$

Now we take $\varepsilon(u) < \delta(u)$ small enough such that, for any harmonic function v in B_1, the condition

$$|u - v|_{L^\infty(B_1)} < \varepsilon(u)$$

implies for each $j = 1, \cdots, l = l(u)$,

$$|u - v|_{L^\infty(B_R(y_j))} < \eta(y_j, u).$$

Therefore, there hold by (7.1.6), (7.1.7), (7.1.10)-(7.1.12),

$$\mathcal{S}(v) \cap B_{1/2} \subset \big(\mathcal{S}(v) \cap \mathcal{B}_u\big) \cup \big(\mathcal{S}(v) \cap \mathcal{G}_u\big),$$

$$\mathcal{H}^{n-2}\big(\mathcal{S}(v) \cap \mathcal{G}_u\big) \le c \sum_{j=1}^{l} s_j^{n-2} \equiv C(u),$$

and

$$\mathcal{B}_u = \bigcup_{i=1}^{k} B_{r_i}(x_i),\ r_i \le \frac{1}{8} \ \text{ and } \ \sum_{i=1}^{k} r_i^{n-2} \le \frac{1}{2^{n-1}}.$$

Now we prove (7.1.9) under the assumption (7.1.8). For any $y \in \mathcal{S}_g(u) \cap B_{3/4}$, we have

$$u(x + y) = P(x) + \psi(x) \quad \text{for any } x \in B_{\frac{1}{4}},$$

where P is a nonzero homogeneous harmonic polynomial of degree d with $2 \le d \le N$ and $\psi(x)$ satisfies, by interior estimates,

$$|\psi(x)| \le C|x|^{d+1} \quad \text{for any } |x| < 1/8, \tag{7.1.13}$$

for a positive constant C depending only on N and n. Here N is the largest vanishing order of u in $B_{1/2}$. By an appropriate rotation, P is a function of two variables. Hence we may assume P is defined in $\mathbb{R}^2 \times \{0\}$ with $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$. We abuse the notation by saying that P is defined in $\mathbb{R}^2$. Let ε_* and r_* be the constants given in Lemma 7.1.3 for P. By (7.1.13), we take a positive constant $R = R(y, u) < 1/8$ such that

$$\Big|\frac{1}{R^d}\psi\Big|_{L^\infty(B_R)} < \frac{1}{2}\varepsilon_*.$$

Choose η small, depending on R and ε_*, such that (7.1.8) implies

$$\Big|\frac{1}{R^d}(u - v)\Big|_{L^\infty(B_R(y))} < \frac{1}{2}\varepsilon_*.$$

Then

$$|\frac{1}{R^d}\big(v - P(\cdot - y)\big)|_{L^\infty(B_R(y))} < \varepsilon_*.$$

By considering the transformation $x \mapsto y + Rx$, we have

$$|\frac{1}{R^d}v(y + R\,\cdot) - P|_{L^\infty(B_1)} < \varepsilon_*.$$

Hence we apply Lemma 7.1.3 to P. After transforming back to $B_R(y)$, we get for some $r \le Rr_*$

$$\mathcal{H}^{n-2}(|Dv|^{-1}\{0\} \cap B_r) \le c(n)(d-1)^2 r^{n-2}.$$

Therefore, we obtain (7.1.9). □

The proof of Theorem 7.1.2 is based on an iteration of Lemma 7.1.4. In order to do this, we need to introduce a class of compact harmonic functions. Consider a positive integer N and denote by $\mathcal{H}_N$ the collection of all harmonic functions u in $B_1 \subset \mathbb{R}^n$ satisfying

$$\text{(7.1.14)} \qquad \fint_{B_{2r}(x_0)} u^2(x)\,dx \le 4^N \fint_{B_r(x_0)} u^2(x)\,dx,$$

for any $x_0 \in B_{2/3}$ and $0 < 2r < \text{dist}(x_0, \partial B_1)$. Obviously, $\mathcal{H}_N$ is invariant under dilations and translations. Specifically, if $u \in \mathcal{H}_N$, then $u_{x_0,r} = u(x_0 + r\cdot) \in \mathcal{H}_N$ for any $x_0 \in B_{2/3}$ and $0 < 2r < \text{dist}(x_0, \partial B_1)$. The class $\mathcal{H}_N$ has the following important compactness property.

Lemma 7.1.5. *For any fixed positive integer N, the collection*

$$\mathcal{H}_N^1 = \{u \in \mathcal{H}_N; \int_{B_{\frac{1}{2}}} u^2(x)dx = 1\}$$

is compact under the local L^∞-metric.

Proof. The proof is straightforward. Suppose $u_k \in \mathcal{H}_N^1$. By (7.1.14) and a covering argument, we have for any $R \in (0,1)$

$$\|u_k\|_{L^2(B_R)} \le c(N,R), \quad k = 1, 2, \cdots.$$

Then there is a subsequence $u_{k'}$ such that $u_{k'}$ converges to a harmonic function u locally in $C^2(B_1)$. In (7.1.14) with u replaced with u_k, we take the limit $k \to \infty$. Hence (7.1.14) holds for u and then $u \in \mathcal{H}_N$. It is obvious that $\int_{B_{1/2}} u^2(x)dx = 1$. □

Now we prove the following result.

Theorem 7.1.6. *Let N be a positive integer. Then for any $u \in \mathcal{H}_N$,*

$$\mathcal{H}^{n-2}\left\{\mathcal{S}(u) \cap B_{1/2}\right\} \le C,$$

where C is a positive constant depending on N and n.

Theorem 7.1.2 follows readily from Theorem 7.1.6. To prove Theorem 7.1.6, we need an improved version of Lemma 7.1.4.

LEMMA 7.1.7. *Suppose N is a positive integer. Then there exists a positive constant C, depending only on n and N, such that for any $u \in \mathcal{H}_N$ there exists a finite collection of balls $\{B_{r_i}(x_i)\}$, with $r_i \le 1/4$ and $x_i \in \mathcal{S}(u)$, such that*

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2} \backslash \cup B_{r_i}(x_i)\right) \le C,$$

and

$$\sum r_i^{n-2} \le \frac{1}{2}.$$

By comparing Lemma 7.1.4 and Lemma 7.1.7, we note that the constant C in Lemma 7.1.4 depends on the function u and that the constant C in Lemma 7.1.7 depends only on the class $\mathcal{H}_N$, independent of specific functions in this class.

PROOF. Take an arbitrary function $u_0 \in \mathcal{H}_N^1$. Consider any $u \in \mathcal{H}_N^1$ with $|u_0 - u|_{L^\infty(B_{7/8})} < \eta_0$. We take $\eta_0 = \eta_0(u_0)$ small such that

$$\eta_0 \le \varepsilon(u_0),$$

where $\varepsilon(u_0)$ is the constant given in Lemma 7.1.4. Then by Lemma 7.1.4, there exist a positive constant $C(u_0)$ and finitely many balls $\{B_{r_i}(x_i)\}$, with $x_i \in \mathcal{S}(u_0)$ and $r_i \le 1/8$, such that, for any $u \in \mathcal{H}_N^1$ with $|u_0 - u|_{L^\infty(B_{7/8})} < \eta_0$,

$$\mathcal{H}^{n-2}\Big(\mathcal{S}(u) \cap B_{\frac{1}{2}} \backslash \cup B_{r_i}(x_i)\Big) \le C(u_0),$$

and

$$\sum r_i^{n-2} \le \frac{1}{2^{n-1}}.$$

If $\mathcal{S}(u) \cap B_{r_i}(x_i) \neq \emptyset$, we take $\tilde{x}_i \in \mathcal{S}(u) \cap B_{r_i}(x_i)$. Obviously $B_{r_i}(x_i) \subset B_{2r_i}(\tilde{x}_i)$. Therefore, for such a u by renaming radii and centers, we find a finite collection of balls $\{B_{r_i}(x_i)\}$, with $x_i \in \mathcal{S}(u)$ and $r_i \le 1/4$, such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2} \backslash \cup B_{r_i}(x_i)\right) \le C(u_0),$$

and

$$\sum r_i^{n-2} \le \frac{1}{2}.$$

By Lemma 7.1.5, $\mathcal{H}_N^1$ is compact under local L^∞-metric. Hence there exist $u_1, \cdots, u_m \in \mathcal{H}_N^1$ and $\eta_1 = \eta(u_1), \cdots, \eta_m = \eta(u_m)$ such that for any $u \in \mathcal{H}_N^1$ there exists a k, $1 \le k \le m$, with the property

$$|u - u_k|_{L^\infty(B_{7/8})} \le \eta_k.$$

Denote

$$C = \max\{C(u_1), \cdots, C(u_p)\}.$$

Such a constant C is finite and depends only on the class $\mathcal{H}_N$. □

Now we are ready to prove Theorem 7.1.6. The iteration scheme in the proof was first introduced by Hardt and Simon [**51**] and already used in Chapter 5.

PROOF OF THEOREM 7.1.6. We use an iteration process to prove Theorem 7.1.6. To begin with, define

$$\phi_0 = \{B_{1/2}\}.$$

We claim that we may find $\phi_1, \phi_2, \cdots$, each of which consists of a collection of balls, such that for any $\ell \geq 1$

$$\mathrm{rad}(B) \leq \frac{1}{2} \cdot \frac{1}{2^\ell} \quad \text{for any } B \in \phi_\ell,$$

$$\sum_{B \in \phi_\ell} \big(\mathrm{rad}(B)\big)^{n-2} \leq \frac{1}{2^\ell},$$

and

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap \bigcup_{B \in \phi_{\ell-1}} B \setminus \bigcup_{B \in \phi_\ell} B\right) \leq \frac{C}{2^\ell},$$

where C is the positive constant given in Lemma 7.1.7. Observe that

$$\mathcal{S}(u) \cap B_{\frac{1}{2}} \subset \bigcup_{\ell=1}^{\infty}\left(\mathcal{S}(u) \cap \big(\bigcup_{B \in \phi_{\ell-1}} B \setminus \bigcup_{B \in \phi_\ell} B\big)\right)$$
$$\cup \bigcap_{\ell=0}^{\infty}\left(\mathcal{S}(u) \cap \bigcup_{j=\ell}^{\infty} \bigcup_{B \in \phi_j} B\right).$$

Hence we have

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2}\right) \leq C\left\{\sum_{\ell \geq 1} \frac{1}{2^{\ell-1}} + \inf_{\ell \geq 0} \sum_{j=\ell}^{\infty} \frac{1}{2^j}\right\} \leq 2C.$$

To prove the claim, we construct $\{\phi_\ell\}$ by an induction. Note $\phi_0 = \{B_{1/2}\}$. Suppose $\phi_0, \phi_1, \ldots, \phi_{\ell-1}$ are already defined for some $\ell \geq 1$. To construct ϕ_ℓ, we take $B = B_r(y) \in \phi_{\ell-1}$, with $r \leq 1/2$, and consider the transformation

$$x \mapsto y + 2rx.$$

Then, $\tilde{u}(x) = u(y + 2rx)$ is a harmonic function in B_1. Obviously, $\tilde{u} \in \mathcal{H}_N$. Hence we apply Lemma 7.1.7 to $\tilde{u}$ to obtain a collection of balls $\{B_{s_i}(z_i)\}$, with $s_i \leq 1/4$ and $z_i \in \mathcal{S}(\tilde{u})$, such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(\tilde{u}) \cap B_{\frac{1}{2}} \setminus \cup B_{s_i}(z_i)\right) \leq C,$$

and

$$\sum s_i^{n-2} \leq \frac{1}{2}.$$

Now transform $B_{1/2}$ back to $B_r(y)$ by

$$x \mapsto \frac{x-y}{2r}.$$

We obtain that, for $B = B_r(y) \in \phi_{\ell-1}$, there exist finitely many balls $\{B_{r_i}(x_i)\}$ in $B_{2r}(y)$, with $r_i \leq r/2$, such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_r(y) \setminus \bigcup_i B_{r_i}(x_i)\right) \leq Cr^{n-2},$$

and

$$\sum_i r_i^{n-2} \leq \frac{1}{2} r^{n-2}.$$

Then we set

$$\phi_\ell^B = \bigcup_i \{B_i(x_i)\},$$

and

$$\phi_\ell = \bigcup_{B \in \phi_{\ell-1}} \phi_\ell^B .$$

Hence we obtain

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap \bigcup_{B\in\phi_{\ell-1}} B \setminus \bigcup_{B\in\phi_\ell} B\right) \le C \left(\sum_{B_{r_i}(x_i)\in\phi_{\ell-1}} r_i^{n-2}\right),$$

and by induction

$$r_i \le \frac{1}{2}\cdot\frac{1}{2^\ell}, \qquad \sum_{B_{r_i}(x_i)\in\phi_\ell} r_i^{n-2} \le \frac{1}{2^\ell},$$

for each $\ell \ge 1$. This concludes the proof. □

7.2. Singular Sets of Smooth Solutions

In this section, we discuss the geometric measure of singular sets for smooth solutions. The main result is the following theorem proved by Han, Hardt and Lin [**47**].

THEOREM 7.2.1. *Suppose that $\mathcal{L}$ is an elliptic operator of the form (4.3.1) satisfying (4.3.2) and (4.3.3) and that u is a C^∞-solution of $\mathcal{L}u = 0$ in B_1 with*

$$\frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2} \le N_0,$$

for some positive constant N_0. Then

$$\mathcal{H}^{n-2}(\mathcal{S}(u)\cap B_{\frac{1}{2}}) \le C,$$

where C is a positive constant depending on N_0, λ and the C^M-norms of the coefficients a_{ij}, b_i and c, for some positive constant M depending on N_0, λ, κ and Γ.

The key result is the following lemma for functions in $\mathbb{R}^2$.

LEMMA 7.2.2. *Let P be a homogeneous harmonic polynomial of degree $d \ge 2$ in $\mathbb{R}^2$. Then there exists a positive constant δ, depending only on P, such that for any $u \in C^{2d^2}(B_1)$ with*

$$|u - P|_{C^{2d^2}(B_1)} < \delta,$$

there holds

$$\#(|Du|^{-1}\{0\}\cap B_{\frac{1}{2}}) \le c(d-1)^2,$$

where c is a universal constant.

Lemma 7.2.2 replaces Lemma 1.3.2, the Rouché Theorem in $\mathbb{C}^2$, in the proof of a result similar to Lemma 7.1.3. Lemma 7.2.2 follows easily from Lemma 1.3.3 with $(P_1, P_2) = \nabla P$.

Now we sketch the proof of Theorem 7.2.1.

Proof of Theorem 7.2.1. The proof consists of several steps.
Step 1. Set

$$\begin{aligned}\mathcal{S}_g(u) = \{p \in & \mathcal{S}(u);\ \text{the leading polynomial of } u \text{ at } p \text{ is} \\ & \text{a polynomial of two variables by an appropriate rotation}\}.\end{aligned}$$

By Corollary 4.1.4, we have

$$\mathcal{H}^{n-2}\big(\mathcal{S}(u) \setminus \mathcal{S}_g(u)\big) = 0.$$

Then for any $\varepsilon > 0$, there exist at most countably many balls $B_{r_i}(x_i)$ with $r_i \leq \varepsilon$ and $x_i \in \mathcal{S}(u) \setminus \mathcal{S}_g(u)$ such that

$$\mathcal{S}(u) \setminus \mathcal{S}_g(u) \subset \bigcup_i B_{r_i}(x_i), \tag{7.2.1}$$

and

$$\sum r_i^{n-2} \leq \gamma(\varepsilon, u), \tag{7.2.2}$$

where $\gamma(\varepsilon, u) \to 0$ as $\varepsilon \to 0$.

We claim for any $y \in \mathcal{S}_g(u) \cap B_{3/4}$, there exist $R = R(y, u)$, $r = r(y, u)$ and $c = c(y, u)$, with $r < R$, such that

$$\mathcal{H}^{n-2}\left\{B_r(y) \cap \mathcal{S}(u)\right\} \leq c r^{n-2}. \tag{7.2.3}$$

The proof of (7.2.3) is based on Lemma 7.2.2 and the fact that the degree of the leading polynomial at any $p \in \mathcal{S}_g(u)$ is at most $c_0 N$. We omit details.

It is obvious that the collection of $\{B_{r_i}(x_i)\}$ and $\{B_{r(y)}(y)\}$, $y \in \mathcal{S}_g(u)$, covers $\mathcal{S}(u)$. By the compactness of $\mathcal{S}(u)$, there exist $x_i \in \mathcal{S}(u) \backslash \mathcal{S}_g(u)$, $i = 1, \cdots, k = k(\varepsilon, u)$, and $y_j \in \mathcal{S}_*(u)$, $j = 1, \cdots, l = l(\varepsilon, u)$, such that

$$\mathcal{S}(u) \cap B_{3/4} \subset \left(\cup_{i=1}^k B_{r_i}(x_i)\right) \bigcup \left(\cup_{j=1}^l B_{s_j}(y_j)\right), \tag{7.2.4}$$

with $r_i \leq \varepsilon$, $i = 1, \cdots, k$, and $s_j \leq \varepsilon$, $j = 1, \cdots, l$.

Step 2. In Step 1, The constant γ in (7.2.2) and c in (7.2.3) depend on u. To improve the results established in Step 1, we should work in a compact class of elliptic operators satisfying (4.3.1)-(4.3.3) and in a compact class of solutions with controlled frequency. Then by a compactness argument, we conclude the following result. Let u be as given in Theorem 7.2.1. For any $\varepsilon > 0$, there exist positive constants $C(\varepsilon)$ and $\gamma(\varepsilon)$, depending only on ε and N_0, as well as λ, κ, Γ and n, with $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that there exists a collection of balls $\{B_{r_i}(x_i)\}$ with $r_i \leq \varepsilon$ and $x_i \in \mathcal{S}(u)$ such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2} \setminus \cup B_{r_i}(x_i)\right) \leq C(\varepsilon),$$

and

$$\sum r_i^{n-2} \leq \gamma(\varepsilon).$$

We emphasize that $C(\varepsilon)$ and $\gamma(\varepsilon)$ are independent of u.

Step 3. We use the standard iteration process to prove Theorem 7.2.1. To begin with, define

$$\phi_0 = \{B_{1/2}\}\ .$$

Fix an $\varepsilon > 0$. We claim that we may find $\phi_1, \phi_2, \cdots$, each of which consists of a collection of balls, such that for any $\ell \geq 1$

$$\operatorname{rad}(B) \leq \frac{(2\varepsilon)^\ell}{2} \quad \text{for any } B \in \phi_\ell,$$

$$\sum_{B\in\phi_\ell} \big(\operatorname{rad}(B)\big)^{n-2} \leq \gamma(\varepsilon)^\ell,$$

and

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap \bigcup_{B\in\phi_{\ell-1}} B \setminus \bigcup_{B\in\phi_\ell} B\right) \leq C(\varepsilon)[\gamma(\varepsilon)]^{\ell-1},$$

where $C(\varepsilon)$ and $\gamma(\varepsilon)$ are given in Step 2. Observe that

$$\begin{aligned}\mathcal{S}(u) \cap B_{1/2} \subset & \bigcup_{\ell=1}^{\infty}\left(\mathcal{S}(u) \cap (\bigcup_{B\in\phi_{\ell-1}} B \setminus \bigcup_{B\in\phi_\ell} B)\right)\\ & \cup \bigcap_{\ell=0}^{\infty}\left(\mathcal{S}(u) \cap \bigcup_{j=\ell}^{\infty}\bigcup_{B\in\phi_j} B\right).\end{aligned}$$

Hence we have

$$\begin{aligned}\mathcal{H}^{n-2}\big(\mathcal{S}(u) \cap B_{1/2}(0)\big) &\leq C(\varepsilon)\left\{\sum_{\ell\geq1}[\gamma(\varepsilon)]^{\ell-1} + \inf_{\ell\geq0}\sum_{j=\ell}^{\infty}[\gamma(\varepsilon)]^j\right\}\\ &\leq 2C(\varepsilon),\end{aligned}$$

provided we take ε small so that $\gamma(\varepsilon) \leq 1/2$.

To prove the claim, we construct $\{\phi_\ell\}$ by an induction. Note $\phi_0 = \{B_{1/2}\}$, independent of ε. Suppose $\phi_0, \phi_1, \ldots, \phi_{\ell-1}$ are already defined for some $\ell \geq 1$. To construct ϕ_ℓ, we take $B = B_r(y) \in \phi_{\ell-1}$, with $r \leq 1/2$. Consider the transformation $x \mapsto y + 2rx$. Then, via $\mathcal{L}u = 0$ in $B_{2r}(y)$, we have

$$\tilde{\mathcal{L}}\tilde{u} = 0 \quad \text{in} \quad B_1,$$

where

$$\tilde{\mathcal{L}} = \sum_{i,j=1}^{n} a_{ij}(y + 2rx)\partial_{x_i x_j} + \sum_{i=1}^{n} 2rb_i(y + 2rx)\partial_{x_i} + (2r)^2 c(y + 2rx),$$

and

$$\tilde{u}(x) = u(y + 2rx).$$

Note that Step 2 can be applied to $\tilde{\mathcal{L}}$ and $\tilde{u}$. Hence we obtain a collection of balls $\{B_{s_i}(z_i)\}$ with $s_i \leq \varepsilon$ and $z_i \in \mathcal{S}(\tilde{u})$ such that

$$\mathcal{H}^{n-2}\big(\mathcal{S}(\tilde{u}) \cap B_{1/2} \setminus \cup B_{s_i}(z_i)\big) \leq C(\varepsilon),$$

and

$$\sum s_i^{n-2} \leq \gamma(\varepsilon).$$

Now transform $B_{1/2}$ back to $B_r(y)$ by $x \mapsto (x-y)/2r$. Then for $B = B_r(y) \in \phi_{\ell-1}$, there exist finitely many balls $\{B_{r_i}(x_i)\}$ in $B_{2r}(y)$, with $r_i \leq 2\varepsilon r$, such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_r(y) \setminus \bigcup_i B_{r_i}(x_i)\right) \leq C(\varepsilon) r^{n-2},$$

and

$$\sum_i r_i^{n-2} \leq r^{n-2}\gamma(\varepsilon).$$

Now we set

$$\phi_\ell^B = \bigcup_i \{B_i(x_i)\},$$

and

$$\phi_\ell = \bigcup_{B \in \phi_{\ell-1}} \phi_\ell^B.$$

Hence we obtain

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap \bigcup_{B \in \phi_{\ell-1}} B \setminus \bigcup_{B \in \phi_\ell} B\right) \leq C(\varepsilon)\left(\sum_{B_{r_i}(x_i) \in \phi_{\ell-1}} r_i^{n-2}\right),$$

and by an induction

$$r_i \leq \frac{(2\varepsilon)^\ell}{2}, \qquad \sum_{B_{r_i}(x_i) \in \phi_\ell} r_i^{n-2} \leq [\gamma(\varepsilon)]^\ell,$$

for each $\ell \geq 1$. This concludes the proof. $\square$

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