LINEARIZATIONS OF THE DARBOUX EQUATION AND SMOOTH ISOMETRIC EMBEDDINGS

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ABSTRACT. In this paper, we introduce a new linearization process for the Darboux equation, a fully nonlinear differential equation for the isometric embedding of 2-dim Riemannian manifolds in $\mathbb{R}^3$. We prove that the local solvability of the Darboux equation is determined *solely* by the local solvability of this linearized equation.

1. INTRODUCTION

In 1873, Schlaefli conjectured that *every* 2*-dimensional smooth Riemannian manifold admits a smooth local isometric embedding in* $\mathbb{R}^3$. It was more than 50 years later that an affirmative answer was given for the analytic case by Janet; he proved in 1926 that any 2-dimensional analytic Riemannian manifold admits a local analytic isometric embedding in $\mathbb{R}^3$. Schlaefli's conjecture for the smooth case was given a renewed attention by Yau [13] in the 1980s and 1990s.

The existence of a local isometric embedding of a 2-dim smooth Riemannian manifold in $\mathbb{R}^3$ can be shown easily to be equivalent to the existence of a local solution of a fully nonlinear equation of the Monge-Ampère type. Suppose g is a smooth metric in an open set $\Omega \subset \mathbb{R}^2$ containing the origin. Isometrically immersing g in $\mathbb{R}^3$ is equivalent to finding a function $\mathbf{r}= (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$ such that

$$dX_1^2 + dX_2^2 + dX_3^2 = g. \tag{1.1}$$

With $X_3 = u(x_1, x_2)$, the metric $dX_1^2 + dX_2^2 = g - dX_3^2 = g - du^2$ is flat and hence has a zero Gauss curvature. By a straightforward calculation, we conclude that the solvability of (1.1) is equivalent to the solvability of

$$\det(\nabla_g^2 u) = K \det(g_{ij})(1 - |\nabla_g u|^2), \tag{1.2}$$

with a subsidiary condition $|\nabla_g u| < 1$. Equation (1.2) is referred to as the Darboux equation. Here and thereafter, K is the Gauss curvature of g.

The difficulty in solving (1.2) locally arises from the vanishing of the Gauss curvature. In 1985 and 1986, Lin made important breakthroughs by proving the existence of a sufficiently smooth local isometric embedding in $\mathbb{R}^3$ of sufficiently smooth surfaces if the Gauss curvature is nonnegative [10] or if the Gauss curvature changes sign cleanly, i.e., the gradient of the Gauss curvature is not zero [11]. In 2003, Han, Hong and Lin [5] proved the existence of a smooth local isometric embedding of smooth surfaces if the Gauss curvature is nonpositive and satisfies some stability condition. In 2005, Han [2] gave a simple proof of Lin's result established in [11]. Later, Han [3] proved the existence

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The author is supported in part by an NSF grant.

of a sufficiently smooth local isometric embedding in $\mathbb{R}^3$ if the Gauss curvature changes sign at any (finite) odd order across a curve. A similar result was also obtained by Khuri [8]. (See [4] for details.)

The first step in proving these results all involves linearizing (1.2) at an appropriately chosen initial approximate solution. In [10], [11], [5], [2], [3] and [8], different initial approximate solutions and hence different linearization process were introduced for different cases, according to whether the Gauss curvature is nonnegative, nonpositive or of a mixed sign. A major contribution in this paper is to introduce a new process, independent of the Gauss curvature, to linearize the Darboux equation (1.2) so that the type of the resulting linearized equation is determined *solely* by the Gauss curvature. The local solvability of the *fully nonlinear* Darboux equation (1.2) is then determined completely by whether we are able to solve this particular *linear* equation. It is believed that this linearization process will lead to more results on the existence of local isometric embedding of surfaces in $\mathbb{R}^3$.

Now we describe briefly this linearization process. We intend to find a solution u of (1.2) of the form

$$u(x_1, x_2) = \frac{1}{2}x_1^2 + w(x_1, x_2), \tag{1.3}$$

where w has a small C^2-norm. The solution u found in this way satisfies $\nabla_{11}u \neq 0$ and corresponds to an isometric embedding with a nonvanishing second fundamental form. The linearized equation of (1.2) at w, although easy to calculate, is difficult to analyze, especially when K is allowed to vanish. We have a crucial observation in regard to this linearized equation. By choosing an appropriate coordinate system and adding some correcting terms, we are able to write the principal part of this linearized equation in the form

$$\partial_{22}u + a(x)K(x)\partial_{11}\rho = f, \tag{1.4}$$

where a is a positive function in a neighborhood of $0 \in \mathbb{R}^2$. Clearly, (1.4) is elliptic if $K > 0$ and hyperbolic if $K < 0$. In deriving (1.4), we take an advantage of the fact that coefficients of the second order terms in the linearized equation consist of the cofactor matrix of a Hessian (with respect to the metric g). The method here is limited to the 2-dimensional case. See Lemma 2.1 for a complete statement of this linearization.

As an application of this general linearization process, we will revisit results in [11], [2], [3] and [8], where only sufficiently smooth local isometric embeddings are established even if metrics are smooth. The isometric embeddings in these papers fail to be smooth, because the domain of the existence shrinks to a point as the regularity of the isometric embedding is raised to infinity. In this paper, we will prove that there indeed exist smooth local isometric embeddings in these cases.

To author's opinion, the general linearization process described above is the major contribution in this paper. Theorems to be presented are only corollaries of this general process. Next, we state a simple case and postpone a general case till Section 4.

Theorem 1.1. *Suppose g is a smooth metric in a neighborhood of $0 \in \mathbb{R}^2$ such that its Gauss curvature K satisfies $K(0) = 0$ and $\nabla K(0) \neq 0$. Then g, restricted to some neighborhood of 0, admits a smooth isometric embedding in $\mathbb{R}^3$.*

In the rest of the introduction, we describe the proof. The aim is to prove the existence of a smooth local solution of (1.2) under the assumptions in Theorem 1.1 or in Theorem 4.1. With a slight modification of the linearization process described before, (1.4) has the following form instead

$$\partial_{22}\rho + a(x)\eta(x_2)\partial_{11}\rho = f, \tag{1.5}$$

where $\eta(t) = t$ for Theorem 1.1 or η is a smooth increasing function in a neighborhood of $0 \in \mathbb{R}$ with $\eta(0) = 0$ for Theorem 4.1 and a is a positive function in a neighborhood of $0 \in \mathbb{R}^2$. Clearly, (1.5) is elliptic if $x_2 > 0$ and hyperbolic if $x_2 < 0$. We note that (1.5) is a Tricomi-type equation if $\eta(t) = t$. The Tricomi equation has been studied extensively by many people. A recent survey paper [12] by Morawetz gave a detailed account of historical backgrounds and known results. Since we are only concerned with the existence of local solutions, we have freedom to choose domains and boundary/initial conditions. In [11] and [2], (1.5) was changed to a positive symmetric differential system introduced by Friedrichs [1]. By this method, we can only obtain sufficiently smooth local solutions.

In this paper, we provide an efficient method to construct a smooth solution of (1.5) with standard energy estimates. We discuss (1.5) separately in the elliptic region $\{x; x_2 > 0\}$ and in the hyperbolic region $\{x; x_2 < 0\}$. The discussion in the elliptic region is similar to that in [9] and [7], while the discussion in the hyperbolic region is adapted from [6]. As $\{x; x_2 = 0\}$ consists of a single piece of curve, solutions in two separate regions can be combined together to form a smooth solution of (1.5) in a neighborhood of the origin.

The paper consists of 4 sections, including this introduction. In Section 2, we linearize the Darboux equation (1.2). By changing coordinates and adding some correcting terms, we change the linearized equation into (1.4). In Section 3, we derive a priori estimates for solutions of (1.5), in $\{x; x_2 \geq 0\}$ and $\{x; x_2 \leq 0\}$ separately. In Section 4, we use the Nash-Moser iteration to prove the existence of a smooth solution of (1.2).

2. A Linearization Process

In this section, we discuss linearizations of the Darboux equation.

We denote by $\tilde{g}$ a metric in a neighborhood of $0 \in \mathbb{R}^2 = \{(\tilde{x}_1, \tilde{x}_2)\}$ and consider

$$\tilde{\mathcal{F}}(\tilde{u}) = \det(\tilde{\nabla}_{ij}\tilde{u}) - \tilde{K}|\tilde{g}|(1 - |\tilde{\nabla}\tilde{u}|^2), \tag{2.1}$$

where $\tilde{\nabla}$ is the covariant derivative with respect to $\tilde{g}$ and $\tilde{K}$ is the Gauss curvature of $\tilde{g}$. We always assume $\tilde{K}(0) = 0$. For $\varepsilon > 0$, we set

$$\tilde{x} = \varepsilon x,$$

and

$$\tilde{u}(\tilde{x}) = \frac{1}{2}\tilde{x}_1^2 + \varepsilon^{\frac{5}{2}} w\big(\frac{\tilde{x}}{\varepsilon}\big) = u(x). \tag{2.2}$$

Now, we introduce a new metric

$$g = g_{ij}(x)dx_i dx_j = \tilde{g}_{ij}(\varepsilon x)dx_i dx_j.$$

Then we have

$$\tilde{\nabla}\tilde{u} = \frac{1}{\varepsilon}\nabla u, \quad \tilde{\nabla}^2\tilde{u} = \frac{1}{\varepsilon^2}\nabla^2 u,$$

where we denote by ∇ the covariant derivative with respect to g. We also write $\partial_i = \partial_{x_i}$ and $\tilde{\partial}_i = \partial_{\tilde{x}_i}$.

To proceed, we set

$$\begin{aligned}\mathcal{F}(w) =& \frac{1}{\sqrt{\varepsilon}}\tilde{\mathcal{F}}(\tilde{u}) = \frac{1}{\sqrt{\varepsilon}}\big(\det(\tilde{\nabla}_{ij}\tilde{u}) - \tilde{K}|\tilde{g}|(1-|\tilde{\nabla}\tilde{u}|^2)\big)\\ =& \frac{1}{\sqrt{\varepsilon}}\left(\det\big(\frac{\nabla_{ij}u}{\varepsilon^2}\big) - \tilde{K}|\tilde{g}|(1-\frac{1}{\varepsilon^2}|\nabla u|^2)\right),\end{aligned} \tag{2.3}$$

where $\tilde{K}$ and $|\tilde{g}|$ are evaluated at εx. Next, we set

$$\Omega = \left\{(x_1,x_2)\in\mathbb{R}^2; |x_1|\le 2, |x_2|\le 2\right\}.$$

For ε small, $\mathcal{F}(w)$ is well defined in Ω. For $w=0$, we have

$$\mathcal{F}(0) = \frac{1}{\sqrt{\varepsilon}}\det(\delta_{1i}\delta_{1j} - \varepsilon\tilde{\Gamma}^1_{ij}x_1) - \frac{1}{\sqrt{\varepsilon}}\tilde{K}|\tilde{g}|(1-\varepsilon^2\tilde{g}^{11}x_1^2),$$

where $\tilde{K}, |\tilde{g}|, \tilde{\Gamma}^k_{ij}, \tilde{g}^{ij}$ are evaluated at εx. As $\tilde{K} = O(\varepsilon)$ since $\tilde{K}(0)=0$, we obtain

$$\mathcal{F}(0) = \sqrt{\varepsilon}F_0(\varepsilon, x), \tag{2.4}$$

where F_0 is a smooth function in $\sqrt{\varepsilon}$ and x.

Now we consider the linearized operator of $\mathcal{F}$ at a smooth function w in Ω. We recall from (2.2)

$$u(x) = \frac{1}{2}\varepsilon^2 x_1^2 + \varepsilon^{\frac{5}{2}}w(x). \tag{2.5}$$

By (2.3), the linearized operator of $\mathcal{F}$ at w is given by

$$\mathcal{F}'(w)\rho = \Phi^{ij}\nabla_{ij}\rho + 2\tilde{K}|\tilde{g}|\nabla u\nabla\rho, \tag{2.6}$$

where (Φ^{ij}) is the cofactor matrix of $(\nabla_{ij}u/\varepsilon^2)$. In (2.6), it is not clear how $\mathcal{F}'(w)$ depends on $\tilde{K}$. The following result is the general linearization described in the introduction.

Lemma 2.1. *For any $\varepsilon\in(0,\varepsilon_0]$ and any smooth function w in Ω with $|w|_{C^2}\le 1$, there exist a transform $T:\Omega\to T(\Omega)$, smooth in x, D^2w and D^3w, of the form*

$$x \mapsto y, \tag{2.7}$$

such that, in the new coordinate y, the operator $\mathcal{F}'(w)$ is given by

$$\begin{aligned}\mathcal{F}'(w)\rho =& a_{22}\partial_{y_2y_2}\rho + a_{11}K\partial_{y_1y_1}\rho + (b_{10}K + b_{11}\partial_{y_1}K)\partial_{y_1}\rho + b_2\partial_{y_2}\rho\\ &+ \tilde{a}_{11}\mathcal{F}(w)\partial_{y_1y_1}\rho + \big(\tilde{b}_{10}\mathcal{F}(w) + \tilde{b}_{11}\partial_{y_1}\mathcal{F}(w)\big)\partial_{y_1}\rho,\end{aligned} \tag{2.8}$$

where K is given by

$$K(y) = \tilde{K}(\varepsilon x), \tag{2.9}$$

and a_{11}, a_{22}, b_{10}, b_{11}, b_2, $\tilde{a}_{11}$, $\tilde{b}_{10}$ and $\tilde{b}_{11}$ are smooth functions in $\sqrt{\varepsilon}$, y, Dw, D^2w, D^3w and D^4w, with

$$a_{ii} = 1 + O(\varepsilon) \quad \text{in } T(\Omega), \quad i=1,2. \tag{2.10}$$

Moreover, $y_i = y_i(x)$ *in (2.7) satisfies*

$$y_i = x_i + O(\sqrt{\varepsilon}) \quad \text{in } \Omega, \quad i = 1, 2, \tag{2.11}$$

and for any $s \geq 0$

$$\|y_i\|_s \leq c_s(1 + \|w\|_{s+3}), \tag{2.12}$$

for some positive constant c_s.

Before the proof, we first discuss the significance of Lemma 2.1. We define

$$\mathcal{L}(w)\rho = a_{22}\partial_{y_2y_2}\rho + a_{11}K\partial_{y_1y_1}\rho + (b_{10}K + b_{11}\partial_{y_1}K)\partial_{y_1}\rho + b_2\partial_{y_2}\rho. \tag{2.13}$$

Obviously, this is a part of $\mathcal{F}'(w)$ in (2.8), since we may write

$$\mathcal{F}'(w)\rho = \mathcal{L}(w)\rho + \tilde{a}_{11}\mathcal{F}(w)\partial_{y_1y_1}\rho + \big(\tilde{b}_{10}\mathcal{F}(w) + \tilde{b}_{11}\partial_{y_1}(\mathcal{F}(w))\big)\partial_{y_1}\rho. \tag{2.14}$$

The operator $\mathcal{L}(w)$ in (2.13) has a simple form in the new coordinate system (y_1, y_2). It is elliptic if $K > 0$ and hyperbolic if $K < 0$. (We note that the principal part of $\mathcal{L}(w)$ has the form (1.4).) We emphasize that the type of $\mathcal{L}(w)$ is determined *solely* by K and is independent of w, where the linearized operator is obtained. This is crucial in the iteration. We note that other terms in (2.14) are *quadratic* in $\mathcal{F}(w)$ and ρ, and their derivatives. Hence, we will regard them as quadratic errors and solve $\mathcal{L}(w)\rho = f$ instead of $\mathcal{F}'(w)\rho = f$ in the iteration process.

Proof. Recall that (Φ^{ij}) is the cofactor matrix of $(\nabla_{ij}u/\varepsilon^2)$, i.e.,

$$\Phi^{22} = \frac{\nabla_{11}u}{\varepsilon^2}, \quad \Phi^{12} = -\frac{\nabla_{12}u}{\varepsilon^2}, \quad \Phi^{11} = \frac{\nabla_{22}u}{\varepsilon^2}. \tag{2.15}$$

Then we have by (2.5)

$$\Phi^{ij} = \delta_2^i\delta_2^j + O(\sqrt{\varepsilon}). \tag{2.16}$$

Let $\hat{\varphi}$ be a smooth function in ε and x satisfying

$$\hat{\varphi}(\varepsilon, x) = x_2 + O(\varepsilon). \tag{2.17}$$

First, we set

$$y_2 = \hat{\varphi}(\varepsilon, x). \tag{2.18}$$

Obviously, y_2 is a smooth function in ε and x, with

$$y_2 = x_2 + O(\varepsilon).$$

Next, we set

$$e^i = \Phi^{ij}\partial_j y_2. \tag{2.19}$$

It is obvious that e^i is smooth in $\sqrt{\varepsilon}, x$ and Dw and D^2w, and satisfies

$$e^i = \delta_2^i + O(\sqrt{\varepsilon}).$$

We consider the following equation for y_1

$$\begin{aligned} e^i\partial_i y_1 &= 0, \\ y_1(x_1, 0) &= x_1. \end{aligned} \tag{2.20}$$

The coefficient of $\partial_2 y_1$ is given by e^2, which is not zero in Ω if ε is small. Hence (2.20) always has a unique solution y_1 in Ω, smooth in $\sqrt{\varepsilon}, x$, Dw and D^2w. Moreover, there holds

$$y_1(x) = x_1 + O(\sqrt{\varepsilon}).$$

Obviously, $y = \big(y_1(x), y_2(x)\big)$ forms new coordinates for small ε. This defines the transform T in (2.7). Then we can check (2.11) and (2.12) easily. By (2.18)-(2.20), we have

$$\partial_2 y_1 = -\frac{e^1}{e^2}\partial_1 y_1, \tag{2.21}$$

and

$$\begin{aligned} \Phi^{i1}\partial_i y_1 &= \det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\partial_2 y_2, \\ \Phi^{i2}\partial_i y_1 &= -\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\partial_1 y_2. \end{aligned} \tag{2.22}$$

In the following, we write $\mathcal{F}'(w)$ in the coordinate $y \in T(\Omega)$. First, we write $\mathcal{F}'(w)$ in the coordinate x as

$$\mathcal{F}'(w)\rho = \Phi^{ij}(\partial_{ij}\rho - \Gamma^k_{ij}\partial_k\rho) + 2K|g|\nabla u\nabla\rho,$$

where K is given by (2.9). We note that K here is not the Gauss curvature of g. By a simple calculation, we obtain

$$\mathcal{F}'(w)\rho = c_{kl}\partial_{y_k y_l}\rho + c_k\partial_{y_k}\rho, \tag{2.23}$$

where

$$c_{kl} = \Phi^{ij}\partial_i y_k\partial_j y_l, \tag{2.24}$$

and

$$c_k = \Phi^{ij}\nabla_{ij}y_k + 2K|g|\nabla u\nabla y_k. \tag{2.25}$$

By (2.21)-(2.22), it is easy to see

$$\begin{aligned} c_{11} &= \Phi^{1i}\partial_i y_1\partial_1 y_1 + \Phi^{2i}\partial_i y_1\partial_2 y_1 \\ &= \det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}(\partial_1 y_1\partial_2 y_2 - \partial_2 y_1\partial_1 y_2) = \det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)J\frac{\partial_1 y_1}{e^2}, \\ c_{12} &= \Phi^{ij}\partial_i y_1\partial_j y_2 = e^i\partial_i y_1 = 0, \\ c_{22} &= \Phi^{ij}\partial_i y_2\partial_j y_2, \end{aligned} \tag{2.26}$$

where J is the determinant of the Jacobian matrix for the transform $x \mapsto y$. Note

$$\begin{aligned} J =& \partial_1 y_1\partial_2 y_2 - \partial_2 y_1\partial_1 y_2 = (e^2\partial_2 y_2 + e^1\partial_1 y_2)\frac{\partial_1 y_1}{e^2} \\ =& e^i\partial_i y_2\frac{\partial_1 y_1}{e^2} = \Phi^{ij}\partial_i y_2\partial_j y_2\frac{\partial_1 y_1}{e^2}. \end{aligned}$$

It is easy to see by (2.11) and (2.15)

$$J = 1 + O(\sqrt{\varepsilon}).$$

For c_2, we have by (2.25)

$$c_2 = \Phi^{ij}\nabla_{ij}y_2 + 2K|g|\nabla u\nabla y_2. \tag{2.27}$$

For c_1, we need only discuss the first term in (2.25)

$$\begin{aligned}\tilde{c}_1 =&\Phi^{ij}\nabla_{ij}y_1 = \Phi^{ij}\partial_{ij}y_1 - \Phi^{ij}\Gamma^l_{ij}\partial_l y_1\\ =&\partial_j(\Phi^{ij}\partial_i y_1) - (\partial_j\Phi^{lj} + \Phi^{ij}\Gamma^l_{ij})\partial_l y_1.\end{aligned} \tag{2.28}$$

By (2.20) and (2.22), we have

$$\begin{aligned}\partial_j(\Phi^{ij}\partial_i y_1) =&\partial_j y_l\partial_{y_l}(\Phi^{ij}\partial_i y_1) = \partial_1 y_l\partial_{y_l}(\Phi^{i1}\partial_i y_1) + \partial_2 y_l\partial_{y_l}(\Phi^{i2}\partial_i y_1)\\ =&\partial_1 y_l\partial_{y_l}\big(\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\partial_2 y_2\big) - \partial_2 y_l\partial_{y_l}\big(\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\partial_1 y_2\big)\\ =&\partial_{y_l}\big(\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\big)(\partial_1 y_l\partial_2 y_2 - \partial_2 y_l\partial_1 y_2)\\ &+\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\big(\partial_1 y_l\partial_{y_l}(\partial_2 y_2) - \partial_2 y_l\partial_{y_l}(\partial_1 y_2)\big)\\ =&J\partial_{y_1}\big(\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\big).\end{aligned}$$

We emphasize that there is no ∂_{y_2}-derivative in the above expression! Now, we claim

$$\partial_j\Phi^{lj} + \Phi^{ij}\Gamma^l_{ij} = -K|g|g^{li}\partial_i u + \Phi^{il}\partial_i\ln\sqrt{|g|}. \tag{2.29}$$

With (2.25), we then have

$$c_1 = J\partial_{y_1}\left(\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\right) - \Phi^{ij}\partial_j y_1\partial_i\ln\sqrt{|g|} + 3K|g|g^{ij}\partial_i u\partial_j y_1. \tag{2.30}$$

Note

$$\begin{aligned}\Phi^{ij}\partial_i y_1\partial_j\ln\sqrt{|g|} =&\Phi^{1i}\partial_i y_1\partial_1\ln\sqrt{|g|} + \Phi^{2i}\partial_i y_1\partial_2\ln\sqrt{|g|}\\ =&\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\big(\partial_2 y_2\partial_1\ln\sqrt{|g|} - \partial_1 y_2\partial_2\ln\sqrt{|g|}\big).\end{aligned}$$

With (2.30), we obtain

$$\begin{aligned}c_1 =&J\partial_{y_1}\left(\det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\right) - \det\big(\frac{\nabla^2 u}{\varepsilon^2}\big)\frac{\partial_1 y_1}{e^2}\big(\partial_2 y_2\partial_1\ln\sqrt{|g|} - \partial_1 y_2\partial_2\ln\sqrt{|g|}\big)\\ &+3K|g|g^{ij}\partial_i u\partial_j y_1.\end{aligned} \tag{2.31}$$

Now, we start to prove (2.29) and we only discuss $l = 1$. By (2.15), we have

$$\varepsilon^2\Phi^{11} = \partial_{22}u - \Gamma^k_{22}\partial_k u,\quad \varepsilon^2\Phi^{12} = -\partial_{12}u + \Gamma^k_{12}\partial_k u,\quad \varepsilon^2\Phi^{22} = \partial_{11}u - \Gamma^k_{12}\partial_k u.$$

By a simple substitution, we collect terms according to the order of derivatives of u. We note that the third derivatives of u cancel among themselves. Hence, we have

$$\begin{aligned}\varepsilon^2(\partial_j\Phi^{1j} + \Phi^{ij}\Gamma^1_{ij}) =&(\Gamma^k_{12}\partial_{2k}u - \Gamma^k_{22}\partial_{1k}u + \Gamma^1_{11}\partial_{22}u - 2\Gamma^1_{12}\partial_{12}u + \Gamma^1_{22}\partial_{11}u)\\ &+(\partial_2\Gamma^k_{12} - \partial_1\Gamma^k_{22} - \Gamma^1_{11}\Gamma^k_{22} + 2\Gamma^1_{12}\Gamma^k_{12} - \Gamma^1_{22}\Gamma^k_{11})\partial_k u.\end{aligned}$$

After a simple calculation for terms involving the second derivatives, we obtain

$$\begin{aligned}\varepsilon^2(\partial_j\Phi^{1j} + \Phi^{ij}\Gamma^1_{ij}) =&(\Gamma^2_{12} + \Gamma^1_{11})\partial_{22}u - (\Gamma^1_{12} + \Gamma^2_{22})\partial_{12}u \\ &+ (\partial_2\Gamma^k_{12} - \partial_1\Gamma^k_{22} - \Gamma^1_{11}\Gamma^k_{22} + 2\Gamma^1_{12}\Gamma^k_{12} - \Gamma^1_{22}\Gamma^k_{11})\partial_k u.\end{aligned}$$

For the first term in the right hand side, we write

$$\begin{aligned}(\Gamma^2_{12} + \Gamma^1_{11})\partial_{22}u =&(\Gamma^2_{12} + \Gamma^1_{11})(\partial_{22}u - \Gamma^k_{22}\partial_k u) + (\Gamma^2_{12} + \Gamma^1_{11})\Gamma^k_{22}\partial_k u \\ =&\Gamma^i_{1i}\nabla_{22}u + (\Gamma^2_{12} + \Gamma^1_{11})\Gamma^k_{22}\partial_k u \\ =&\varepsilon^2\partial_1 \ln\sqrt{|g|}\Phi^{11} + (\Gamma^2_{12} + \Gamma^1_{11})\Gamma^k_{22}\partial_k u.\end{aligned}$$

Similarly, we get

$$(\Gamma^1_{12} + \Gamma^2_{22})\partial_{12}u = -\varepsilon^2\partial_2 \ln\sqrt{|g|}\Phi^{12} + (\Gamma^1_{12} + \Gamma^2_{22})\Gamma^k_{12}\partial_k u.$$

Then we obtain

$$\varepsilon^2(\partial_j\Phi^{1j} + \Phi^{ij}\Gamma^1_{ij}) = I_k \cdot \partial_k u + \varepsilon^2\partial_1 \ln\sqrt{|g|}\Phi^{11} + \varepsilon^2\partial_2 \ln\sqrt{|g|}\Phi^{12},$$

where I_k is given by

$$\begin{aligned}I_k =&\partial_2\Gamma^k_{12} - \partial_1\Gamma^k_{22} - \Gamma^1_{11}\Gamma^k_{22} + 2\Gamma^1_{12}\Gamma^k_{12} - \Gamma^1_{22}\Gamma^k_{11} \\ &+ (\Gamma^2_{12} + \Gamma^1_{11})\Gamma^k_{22} - (\Gamma^1_{12} + \Gamma^2_{22})\Gamma^k_{12}.\end{aligned}$$

A simple calculation shows

$$\begin{aligned}I_k =&\partial_2\Gamma^k_{12} - \partial_1\Gamma^k_{22} + \Gamma^m_{12}\Gamma^k_{m2} - \Gamma^m_{22}\Gamma^k_{1m} = g^{kl}R_{2l12} \\ =&g^{1k}R_{2112} = -g^{1k}R_{1212} = -g^{1k}K_g|g|,\end{aligned}$$

where K_g is the Gauss curvature of g. Note that $K_g = \varepsilon^2\tilde{K} = \varepsilon^2 K$. Hence we have by a simple substitution

$$\varepsilon^2(\partial_j\Phi^{1j} + \Phi^{ij}\Gamma^1_{ij}) = -\varepsilon^2 K|g|g^{1k}\partial_k u + \varepsilon^2\partial_i \ln\sqrt{|g|}\Phi^{1i}.$$

This is (2.29) for $l = 1$. The case $l = 2$ can be proved similarly.

By (2.3), we have

$$\det(\frac{\nabla^2 u}{\varepsilon^2}) = K|g|(1 - \frac{1}{\varepsilon^2}|\nabla u|^2) + \sqrt{\varepsilon}\mathcal{F}(w).$$

We may write c_{ij} and c_i in (2.26), (2.27) and (2.31) as follows

$$\begin{aligned}c_{11} &= K|g|(1 - \frac{1}{\varepsilon^2}|\nabla u|^2)J\frac{\partial_1 y_1}{e^2} + \sqrt{\varepsilon}\mathcal{F}(w)J\frac{\partial_1 y_1}{e^2}, \\ c_{12} &= 0, \\ c_{22} &= J\frac{e^2}{\partial_1 y_1},\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}c_1 =& J\partial_{y_1}\left(K|g|(1-\frac{1}{\varepsilon^2}|\nabla u|^2)\frac{\partial_1 y_1}{e^2}\right)+3K|g|\nabla u\cdot\nabla y_1\\&-K|g|(1-\frac{1}{\varepsilon^2}|\nabla u|^2)\frac{\partial_1 y_1}{e^2}\big(\partial_2 y_2\partial_1\ln\sqrt{|g|}-\partial_1 y_2\partial_2\ln\sqrt{|g|}\big)\\&+\sqrt{\varepsilon}J\partial_1\left(\mathcal{F}(u)\frac{\partial_1 y_1}{e^2}\right)-\sqrt{\varepsilon}\mathcal{F}(u)\frac{\partial_1 y_1}{e^2}\big(\partial_2 y_2\partial_1\ln\sqrt{|g|}-\partial_1 y_2\partial_2\ln\sqrt{|g|}\big),\\c_2 =& \Phi^{ij}\nabla_{ij}y_2+2K|g|\nabla u\cdot\nabla y_2.\end{aligned}\tag{2.33}$$

This finishes the proof. ☐

In the proof of Lemma 2.1, $\hat\varphi$ is an arbitrary smooth function in ε and x satisfying (2.17). Obviously, we may take $\hat\varphi(\varepsilon,x)=x_2$. However, such a choice may not lead to solvable linearized equations. We examine a special case.

Corollary 2.2. *Let $\tilde g$ be a smooth metric in Ω with its Gauss curvature $\tilde K$ given by $\tilde K=\eta\circ\varphi$, where*

$$\eta=\eta(t)\ \textit{is a smooth increasing function with}\ \eta(0)=0,\tag{2.34}$$

and φ is a smooth function in a neighborhood of $0\in\mathbb{R}^2$ satisfying

$$\varphi(0)=0,\quad \tilde\partial_1\varphi(0)=0,\quad \tilde\partial_2\varphi(0)=1.\tag{2.35}$$

Then Lemma 2.1 holds with (2.9) replaced by

$$K(y)=\eta(\varepsilon y_2).\tag{2.36}$$

To prove Corollary 2.2, we simply take

$$\hat\varphi(\varepsilon,x)=\frac{1}{\varepsilon}\varphi(\varepsilon x).\tag{2.37}$$

Obviously, (2.17) is satisfied because of (2.35).

Remark 2.3. The only essential difference lies between (2.36) and (2.9) for the expression of K. Under extra assumptions (2.34) and (2.35), K has a simple form in new coordinates. By (2.36), we have $\partial_{y_1}K=0$. Therefore, we may take $b_{11}=0$ in (2.8). This simple fact plays an important role in proving the existence of smooth solutions later on.

To proceed, we examine the new transform (2.7) introduced by y_1 and y_2 in (2.18) and (2.20). In fact, (2.7) can be decomposed into two parts

$$x\mapsto\hat x=\big(x_1,\hat\varphi(\varepsilon,x)\big),\tag{2.38}$$

and

$$\hat x\mapsto y=\big(y_1(\hat x),\hat x_2\big),\tag{2.39}$$

where $\hat\varphi$ is given in (2.37) and y_1 in (2.20). By (2.17), we note that $\hat x_2=\hat\varphi(\varepsilon,x)$, and hence the transform $x\mapsto\hat x$, are independent of w. In the $\hat x$ coordinate, $\mathcal{F}(w)$ in (2.3) can be written as

$$\mathcal{F}(w)=a(\varepsilon,\hat x,\partial w,\partial_{11}w,\partial_{12}w)\partial_{22}w+F_1(\varepsilon,\hat x,\partial w,\partial_{11}w,\partial_{12}w),\tag{2.40}$$

where a and F_1 are smooth in $\sqrt{\varepsilon}, \hat{x}$, ∂w, $\partial_{11}w$ and $\partial_{12}w$, with

$$a(\varepsilon, \hat{x}, \partial w, \partial_{11}w, \partial_{12}w) = 1 + O(\sqrt{\varepsilon}), \tag{2.41}$$

for any small ε and any smooth function w in Ω with $|\partial w| + |\partial_{11}w| + |\partial_{12}w| \leq 1$. We note in the coordinate $\hat{x}$

$$K = \eta(\varepsilon \hat{x}_2).$$

In the following sections, we will solve $\mathcal{F}(w) = 0$ in (2.40) in $\{\hat{x}_2 > 0\}$ and $\{\hat{x}_2 < 0\}$ separately. We note that a smooth solution w to $\mathcal{F}(w) = 0$ in $\{\hat{x}_2 > 0\}$ and $\{\hat{x}_2 < 0\}$ forms a smooth solution as long as it is C^1 across $\{\hat{x}_2 = 0\}$.

3. Smooth Solutions of Linearized Equations

In this section, we construct smooth solutions of the linearized equations we derived in the previous section. Instead of discussing the elliptic region and the hyperbolic region separately, we consider an elliptic equation and a hyperbolic equation in the same region. For degenerate elliptic equations, solutions with a high regularity exist when coefficients satisfy a *smallness* assumption. See (3.7) in Theorem 3.1. No such a smallness assumption is needed for degenerate hyperbolic equations. This difference results in different iteration schemes, specifically different modified linearized equations, in solving the Darboux equation in elliptic regions and hyperbolic regions. This explains different forms in (3.2) and (3.22). This difference is also the main reason that we need to carry out the Nash-Moser iteration separately in elliptic regions and hyperbolic regions.

In $\mathbb{R}^2$, we set for any $l > 0$

$$D_l = \{(x, t); |x| < l, \ 0 < t < 1\}. \tag{3.1}$$

We denote by $\| \cdot \|_{s,D_l}$ the H^s-norm in D_l. For $l = 1$, we simply write $D = D_1$ and $\| \cdot \|_s = \| \cdot \|_{s,D_1}$.

In the first part of this section, we discuss a class of elliptic equations in D. We consider

$$a\partial_{tt}u + (K+\theta)\partial_{xx}u + b_0\partial_t u + b(K+\theta)\partial_x u + B\partial_x u = f \quad \text{in } D, \tag{3.2}$$

where θ is a positive constant, and a, b_0, b, B and K are smooth functions in D satisfying for some constant $\lambda > 0$

$$a \geq \lambda \quad \text{in } D, \tag{3.3}$$

and

$$K = K(t) \geq 0 \quad \text{in } D. \tag{3.4}$$

Equation (3.2) is elliptic under assumptions (3.3)-(3.4). We note that K does not depend on x. We set

$$\Lambda_s^+ = \|a\|_{s+1} + \|b_0\|_{s+1} + \|b\|_{s+1} + \|B\|_s + \|K+\theta\|_s. \tag{3.5}$$

Next, we prove the existence of smooth solutions to (3.2). Our method is adapted from [10] and [7], where sufficiently smooth solutions are constructed. It is not clear whether solutions constructed in [10] and [7] are actually smooth. We are able to construct smooth solutions in our case because of the special structure of (3.2).

Theorem 3.1. *Suppose a, b_0, b, B and K are smooth functions in D satisfying (3.3)-(3.4) for some constant $\lambda > 0$. Then for any positive constant θ and any smooth function f in D, there exists a smooth solution u of (3.2) satisfying*

$$u(\cdot, 0) = u(\cdot, 1) = 0. \tag{3.6}$$

Moreover, for any integer $s \geq 0$ satisfying

$$|\partial_{tt}a|_{L^\infty} + |\partial_{tt}b|_{L^\infty} + |\partial_t b_0|_{L^\infty} + |\partial b|_{L^\infty} + |K + \theta|_{L^\infty} + s|B|_{C^1} \leq \varepsilon_0\lambda, \tag{3.7}$$

there holds

$$\|u\|_s \leq C_s\big(\|f\|_s + (s-2)^+\Lambda_s^+\|f\|_2\big), \tag{3.8}$$

where ε_0 is a (universal) constant, and C_s is a positive constant depending only s, λ and H^5-norms of a, b_0, b, B and K.

We note that (3.8) holds for any integer $s \geq 0$ and any small θ if $B \equiv 0$. However, (3.7) fails to hold for sufficiently large s if $|B|_{C^1} \neq 0$. It is not clear whether (3.8) always holds without (3.7).

Proof. Consider a smooth function φ in D satisfying

$$2\partial_x\varphi + b\varphi = 0. \tag{3.9}$$

We may simply take

$$\varphi(x, t) = \exp\{-\frac{1}{2}\int_0^x b(\cdot, t)\}. \tag{3.10}$$

Then we set

$$u = \varphi v. \tag{3.11}$$

A straightforward calculation yields

$$a\partial_{tt}v + (K + \theta)\partial_{xx}v + \tilde{b}_0\partial_t v + B\partial_x v + cv = \tilde{f} \quad \text{in } D, \tag{3.12}$$

where

$$\begin{aligned}\tilde{b}_0 =& 2a\frac{\partial_t\varphi}{\varphi} + b_0,\\ c =& a\frac{\partial_{tt}\varphi}{\varphi} + b_0\frac{\partial_t\varphi}{\varphi} - \frac{1}{4}(K + \theta)(2\partial_x b + b^2) - \frac{1}{2}bB,\\ \tilde{f} =& \frac{f}{\varphi}.\end{aligned} \tag{3.13}$$

To proceed, we extend $a, \tilde{b}_0, c$ and $\tilde{f}$ from $D = D_1$ to D_2 such that the extended functions are 4-periodic in x, with $a \geq \lambda/2$ in D_2, and that there hold for any $s \geq 0$

$$\|a\|_{s,D_2} \leq C_s\|a\|_{s,D_1}, \quad |a|_{C^s(D_2)} \leq C_s|a|_{C^s(D_1)}, \tag{3.14}$$

and similar estimates for $\tilde{b}_0, c$ and $\tilde{f}$. Now, we consider (3.12) in D_2. Since (3.12) is (strictly) elliptic in D_2, it admits a (unique) smooth solution v in D_2 which is 4-periodic in x and satisfies

$$v(\cdot, 0) = v(\cdot, 1) = 0.$$

Now, we derive estimates for v. Set for any integer $s \geq 0$

$$\tilde{\Lambda}_s = \|a\|_{s+1,D_2} + \|\tilde{b}_0\|_{s,D_2} + \|c\|_{s,D_2} + \|B\|_{s,D_2} + \|K+\theta\|_{s,D_2}. \tag{3.15}$$

We claim there exists a universal $\tilde{\varepsilon}_0 > 0$ such that, for any integer $s \geq 0$ satisfying

$$\left|\frac{1}{2}\partial_{tt}a - \frac{1}{2}\partial_t\tilde{b}_0 + c\right| + \left|(s-\frac{1}{2})\partial_x B\right| \leq \tilde{\varepsilon}_0\lambda \quad \text{in } D_2, \tag{3.16}$$

there holds

$$\|v\|_{s,D_2} + \|\partial_t v\|_{s,D_2} \leq C_s\left(\|\tilde{f}\|_{s,D_2} + (s-2)^+\tilde{\Lambda}_s|v|_{L^\infty(D_2)}\right), \tag{3.17}$$

where C_s is a positive constant depending only on s, the C^3-norm of a and C^2-norms of $\tilde{b}_0$, c and K.

We first prove $(3.17)_0$. We integrate the product of v and (3.12) in D_2. There are no boundary integrals since all functions involved are 4-periodic in x and v vanishes on $t = 0$ and $t = 1$. A simple calculation shows

$$\int_{D_2} a(\partial_t v)^2 + K(\partial_x v)^2 = \int_{D_2}\big(\frac{1}{2}\partial_{tt}a - \frac{1}{2}\partial_t\tilde{b}_0 + c - \frac{1}{2}\partial_x B\big)v^2 - \int_{D_2} v\tilde{f}.$$

By $(3.16)_0$, we get

$$\int_{D_2} a(\partial_t v)^2 + K(\partial_x v)^2 \leq \tilde{\varepsilon}_0\lambda\int_{D_2} v^2 - \int_{D_2} v\tilde{f}.$$

Since $v = 0$ as $t = 0$, we have by Poincaré inequality

$$\int_{D_2}\big(v^2 + (\partial_t v)^2 + K(\partial_x v)^2\big) \leq C_0\int_{D_2}\tilde{f}^2, \tag{3.18}$$

where C_0 is a positive constant depending only on λ. Hence we have for

$$\|v\|_{0,D_2} + \|\partial_t v\|_{0,D_2} \leq C_0\|\tilde{f}\|_{0,D_2}. \tag{3.19}$$

Next, we assume $(3.17)_{s-1}$ holds and prove $(3.17)_s$ under the assumption $(3.16)_s$. We note that $\partial_x^s v$ satisfies an equation similar to (3.12). Hence a similar argument leading to (3.19) and the induction hypothesis $(3.17)_{s-1}$ yield

$$\|\partial_x^s v\|_{0,D_2} + \|\partial_t\partial_x^s v\|_{0,D_2} \leq C(s,\lambda)\big(\|\tilde{f}\|_{s,D_2} + (s-2)^+\tilde{\Lambda}_s|v|_{L^\infty(D_2)}\big). \tag{3.20}$$

It remains to estimate other s-derivatives. By (3.3) and (3.12), we have

$$\partial_{tt}v = -\frac{K}{a}\partial_{xx}v - \frac{\tilde{b}_0}{a}\partial_t v - \frac{B}{a}\partial_x v - \frac{c}{a}v + \frac{\tilde{f}}{a} \quad \text{in } D_2.$$

Applying $\partial_t^k\partial_x^{s-k}$, with $2 \leq k \leq s$, and then using the estimate on $\partial_x^s v$ and $\partial_t\partial_x^{s-1}v$ in (3.20), we obtain

$$\sum_{k=2}^{s}\|\partial_t^k\partial_x^{s-k}v\|_{0,D_2} \leq C(s,\lambda)\big(\|\tilde{f}\|_{s,D_2} + (s-2)^+\tilde{\Lambda}_s|v|_{L^\infty(D_2)}\big). \tag{3.21}$$

Therefore, we get $(3.17)_s$ by (3.20)-(3.21).

With $(3.17)_s$ and the Sobolev embedding, we obtain

$$\|v\|_{s,D_2} \leq C_s\big(\|\tilde{f}\|_{s,D_2} + (s-2)^+\tilde{\Lambda}_s\|\tilde{f}\|_{2,D_2}\big).$$

By (3.5), (3.13) and (3.14) for $a, \tilde{b}_0$ and c, we have easily

$$\tilde{\Lambda}_s \leq C_s \Lambda_s^+.$$

Moreover, (3.7) implies (3.16) for sufficiently small ε. With (3.14) for $\tilde{f}$, we obtain

$$\|v\|_s \leq C_s\big(\|\tilde{f}\|_s + (s-2)^+ \Lambda_s^+ \|\tilde{f}\|_2\big),$$

where integrations are over $D = D_1$. By (3.11) and interpolation inequalities, we have (3.8) easily. □

In (3.8), there is a loss of derivatives in the elliptic sense. Note that the H^s-norm of f only controls the H^s-norm of u.

In the rest of this section, we discuss a class of hyperbolic equations in D. In the following, we set $\Sigma = \{t = 0\} \cap \partial D$ and consider

$$\begin{aligned} \partial_{tt} u - a(K+\theta)\partial_{xx} u - b_0 \partial_t u - b\partial_x u = f \quad &\text{in } D, \\ u = \psi_0,\ \partial_t u = \tilde{\psi}_0 \quad &\text{on } \Sigma, \end{aligned} \tag{3.22}$$

where θ is a positive constant, and a, b_0, b and K are smooth functions satisfying

$$a \geq \lambda \quad \text{in } D, \tag{3.23}$$

$$0 \leq K = K(t) \leq 1 \quad \text{in } D, \tag{3.24}$$

and

$$|b| \leq C\sqrt{K+\theta} \quad \text{in } D, \tag{3.25}$$

for some positive constants λ and C. The crucial assumption concerning K is

$$\partial_t K \geq 0 \quad \text{in } D. \tag{3.26}$$

We note that (3.25) is referred to as the *Levi condition.* We set

$$\Lambda_s^- = \|\partial^2\big(a(K+\theta)\big)\|_s + \|\partial b_0\|_s + \|\partial b\|_s.$$

The next theorem is an improved version compared with results in [6] and the proof is much simpler.

Theorem 3.2. *Suppose a, b_0, b and K are smooth functions in D satisfying (3.23)-(3.26). Then for any positive constant θ, any smooth function f in D and any smooth functions ψ_0 and $\tilde{\psi}_0$ on Σ, there exists a smooth solution u to (3.22). Moreover, there holds for any $s \geq 1$*

$$\begin{aligned} ||u||_s \leq C_s \Big(&\|\psi_0\|_{H^s(\Sigma)} + \|\tilde{\psi}_0\|_{H^s(\Sigma)} + ||f||_s \\ &+ (s-2)^+ \Lambda_s^- (\|\psi_0\|_{H^2(\Sigma)} + \|\tilde{\psi}_0\|_{H^2(\Sigma)} + \|f\|_2)\Big), \end{aligned} \tag{3.27}$$

where C_s is a positive constant depending only on s, λ, the Lipschitz norm of $\sqrt{K}$ and C^2-norms of a, b_0, b and K in D.

For any function u, we set for $s \geq 1$

$$|u|_s^2 = \sum_{|\gamma| \leq s} |\partial^\gamma u|^2, \quad |u|_{(0,s)}^2 = \sum_{i \leq s} |\partial_x^i u|^2.$$

Proof. We first extend a, b_0, b and f from D to D_∞ such that there hold $a \geq \lambda/2$ in D_∞ and for any integer $s \geq 0$

$$\|f\|_{s,D_\infty} \leq C_s \|f\|_{s,D}, \quad |f|_{C^s(D_\infty)} \leq C_s |f|_{C^s(D)}, \tag{3.28}$$

and similar estimates for ∂a, $\tilde{b}_0$ and b. Moreover, we may assume b_0, b and f have compact supports. Similarly, we extend ψ_0 and $\tilde{\psi}_0$ from Σ to $\{t = 0\}$, with a similar estimate as (3.28) for ψ_0 and $\tilde{\psi}_0$.

In the following, we set $k = \sqrt{K + \theta}$, and consider

$$\begin{aligned} \partial_{tt} u - ak^2 \partial_{xx} u - b_0 \partial_t u - b \partial_x u = f \quad &\text{in } D_\infty, \\ u = \psi_0, \ \partial_t u = \tilde{\psi}_0 \quad &\text{on } \{t = 0\}. \end{aligned} \tag{3.29}$$

Obviously, (3.29) admits a unique smooth solution u with a compact support since it is (strictly) hyperbolic. Set

$$\tilde{\Lambda}_s^- = \|\partial^2 (ak^2)\|_{s,D_\infty} + \|\partial b_0\|_{s,D_\infty} + \|\partial b\|_{s,D_\infty}.$$

We claim there holds for any integer $s \geq 1$

$$\begin{aligned} ||u||_{s,D_\infty} \leq C_s \Big(&\|\psi_0\|_{H^s(\{t=0\})} + \|\tilde{\psi}_0\|_{H^s(\{t=0\})} + ||f||_{s.D_\infty} \\ &+ (s-2)^+ \tilde{\Lambda}_s^- (\|\psi_0\|_{H^2(\{t=0\})} + \|\tilde{\psi}_0\|_{H^2(\{t=0\})} + \|f\|_{2,D_\infty}) \Big), \end{aligned} \tag{3.30}$$

where C_s is a positive constant depending only on s, λ, the Lipschitz norm of $\sqrt{K}$ and C^2-norms of a, b_0, b and K in D_∞.

We introduce a function z satisfying

$$\begin{aligned} \partial_{tt} z - b_0 \partial_t z = f \quad &\text{in } D_\infty, \\ z = u, \quad \partial_t z = u_t \quad &\text{on } \{t = 0\}. \end{aligned} \tag{3.31}$$

By setting $\hat{u} = u - z$, we obtain

$$\begin{aligned} \partial_{tt} \hat{u} - ak^2 \partial_{xx} \hat{u} - b_0 \partial_t \hat{u} - b \partial_x \hat{u} = \hat{f} \quad &\text{in } D_\infty, \\ \hat{u} = 0, \quad \partial_t \hat{u} = 0 \quad &\text{on } \{t = 0\}, \end{aligned} \tag{3.32}$$

where

$$\hat{f} = ak^2 \partial_{xx} z + b \partial_x z. \tag{3.33}$$

By a straightforward calculation, we have

$$\begin{aligned} &\partial_t \big\{ e^{-\mu t} \big(\frac{\hat{u}_t^2}{k^2} + \frac{\hat{u}^2}{k^2} + a \hat{u}_x^2 \big) \big\} - \partial_x \big(2e^{-\mu t} a \partial_t \hat{u} \partial_x \hat{u} \big) + \mu e^{-\mu t} \big(\frac{\hat{u}_t^2}{k^2} + \frac{\hat{u}^2}{k^2} + a \hat{u}_x^2 \big) \\ &+ e^{-\mu t} \partial_t \ln k^2 \big(\frac{\hat{u}_t^2}{k^2} + \frac{\hat{u}^2}{k^2} \big) = -2e^{-\mu t} a_x k \hat{u}_x \frac{\hat{u}_t}{k} + e^{-\mu t} a_t \hat{u}_x^2 \\ &+ 2e^{-\mu t} b_0 \frac{\hat{u}_t^2}{k^2} + 2e^{-\mu t} \hat{u}_x \frac{b \hat{u}_t}{k^2} + 2e^{-\mu t} \frac{\hat{u} \hat{u}_t}{k^2} + 2e^{-\mu t} \frac{\hat{f} \hat{u}_t}{k^2}. \end{aligned} \tag{3.34}$$

Note $\partial_{xx}z$ appears in the expression of $\hat{f}$. This would create an extra loss of derivative if we simply integrate (3.34) in D_∞. To avoid this loss, we write

$$\begin{aligned} e^{-\mu t}\frac{\hat{f}\hat{u}_t}{k^2} =& e^{-\mu t}az_{tx}\hat{u}_x - \mu e^{-\mu t}az_x\hat{u}_x + e^{-\mu t}z_x\frac{b\hat{u}_t}{k^2} \\ &- e^{-\mu t}a_xz_x\hat{u}_t + e^{-\mu t}a_tz_x\hat{u}_x \\ &+ \partial_x\big(e^{-\mu t}az_x\hat{u}_t\big) - \partial_t\big(e^{-\mu t}az_x\hat{u}_x\big). \end{aligned} \tag{3.35}$$

We note z_{tx} appears in the first term in the right hand side of (3.35) instead of z_{xx} in (3.33). Now we may integrate (3.34) in D_∞, with the last term in (3.34) replaced by (3.35). In view of the hypothesis $\partial_t k^2 \geq 0$, we obtain by Cauchy inequality

$$\hat{B}_+ + (\mu-\mu_1)\int_{D_\infty} e^{-\mu t}\big(\frac{\hat{u}^2}{k^2}+\frac{\hat{u}_t^2}{k^2}+a\hat{u}_x^2\big) \leq \int_{D_\infty} e^{-\mu t}\big(\mu z_x^2 + z_{xt}^2\big), \tag{3.36}$$

where $\hat{B}_+$ is the boundary integral over $\{t=1\}$

$$\hat{B}_+ = \int_{\{t=1\}} e^{-\mu t}\big(\frac{\hat{u}^2}{k^2}+\frac{\hat{u}_t^2}{k^2}+a\hat{u}_x^2 + az_x\hat{u}_x\big). \tag{3.37}$$

There is no boundary integral on $\{t=0\}$ since $\hat{u}=\hat{u}_t=0$ on $\{t=0\}$ by (3.32). We then have by Cauchy inequality

$$\hat{B}_+ \geq \int_{\{t=1\}} e^{-\mu t}\big(\frac{\hat{u}^2}{k^2}+\frac{\hat{u}_t^2}{K}+\frac{1}{2}a\hat{u}_x^2\big) - \frac{1}{2}\int_{\{t=1\}} e^{-\mu t}z_x^2. \tag{3.38}$$

By (3.24), we have

$$\frac{\hat{u}_t^2}{k^2}+\frac{\hat{u}^2}{k^2}+a\hat{u}_x^2 \geq (u-z)^2+(u_t-z_t)^2+a(u_x-z_x)^2.$$

This implies

$$\begin{aligned} &c_1\int_{\{t=1\}} e^{-\mu t}(u^2+u_t^2+u_x^2) + (\mu-\mu_1)\int_{D_\infty} e^{-\mu t}(u^2+u_t^2+u_x^2) \\ &\leq c_1'\int_{\{t=1\}} e^{-\mu t}(z^2+z_t^2+z_x^2) + c_1''\int_{D_\infty} e^{-\mu t}(\mu z^2+\mu z_t^2+\mu z_x^2+z_{xt}^2). \end{aligned} \tag{3.39}$$

We need to get estimates on z and its derivatives. First we have for any μ

$$\begin{aligned} &\int_{\{t=1\}} e^{-\mu t}(z^2+z_t^2) + (\mu-\mu_0)\int_{D_\infty} e^{-\mu t}(z^2+z_t^2) \\ \leq& c_0\left\{\int_{\{t=0\}}(u^2+u_t^2) + \int_{D_\infty} e^{-\mu t}f^2\right\}, \end{aligned} \tag{3.40}$$

for some constants μ_0 and c_0 depending only on the L^∞-norm of b_0. We obtain (3.40) by multiplying $2e^{-\mu t}z_t$ to (3.31) and integrating over D_∞. Now we differentiate (3.31) with respect to x. Similarly, we get for any μ

$$\begin{aligned}&\int_{\{t=1\}} e^{-\mu t}(z_x^2+z_{xt}^2)+(\mu-\mu_0)\int_{D_\infty} e^{-\mu t}(z_x^2+z_{xt}^2)\\ \le& c_0\left\{\int_{\{t=0\}}(z_x^2+z_{xt}^2)+\int_{D_\infty} e^{-\mu t}(z^2+z_x^2+f_x^2)\right\}.\end{aligned}\tag{3.41}$$

We finish the proof of $(3.30)_1$ by substituting (3.40) and (3.41) in (3.39).

To prove $(3.30)_{s+1}$ for any $s \geq 0$, we consider the equation satisfied by $\partial_x^s u$ and get an estimate for $\partial_t\partial_x^s u$ and $\partial_x^{s+1}u$ similarly. Other $(s+1)$-derivatives can be obtained from equation (3.22). We omit details, since it is similar to the corresponding part in the proof of Theorem 3.1. □

4. ITERATIONS

For any smooth function K in a neighborhood of $0 \in \mathbb{R}^2$ with $K(0) = 0$ and $\nabla K(0) \neq 0$ as in Theorem 1.1, the implicit function theorem implies that $K^{-1}(0)$ in a neighborhood of $0 \in \mathbb{R}^2$ consists a (single) smooth curve σ through 0. Moreover, there exists a vector field X in a neighborhood of $0 \in \mathbb{R}^2$ which is orthogonal to σ such that $\nabla_X K > 0$. In this case, K vanishes at order 1 across σ.

We now consider a general case. A smooth function K *changes its sign monotonically across a curve through* $0 \in \mathbb{R}^2$ if K is given by $K = \eta \circ \varphi$, where

$$\eta = \eta(t) \text{ is a smooth strictly increasing function with } \eta(0) = 0, \tag{4.1}$$

and φ is a smooth function in a neighborhood of $0 \in \mathbb{R}^2$ satisfying

$$\varphi(0) = 0, \quad \nabla\varphi(0) \neq 0. \tag{4.2}$$

It is easy to see that $K^{-1}(0)$ in a neighborhood of $0 \in \mathbb{R}^2$ consists a (single) smooth curve σ through 0 and that there exists a vector field X in a neighborhood of $0 \in \mathbb{R}^2$ which is orthogonal to σ such that $\nabla_X K \geq 0$. We note that K is allowed to vanish up to an infinite order along σ.

Theorem 4.1. *Suppose g is a smooth metric in a neighborhood of $0 \in \mathbb{R}^2$ such that its Gauss curvature K changes sign monotonically across a smooth curve through 0. Then g, restricted to some neighborhood of 0, admits a smooth isometric embedding in $\mathbb{R}^3$.*

Obviously, Theorem 1.1 is a special case of Theorem 4.1.

In the rest of this section, we will prove Theorem 4.1. We use the same setup as that at the beginning of Section 2. We denote by $\tilde{g}$ temporarily the metric in Theorem 4.1 with Gauss curvature $\tilde{K}$. By an appropriate rotation and (4.1)-(4.2), we may assume $\tilde{K}$ satisfies (2.34) and (2.35) in Corollary 2.2. To proceed, we use Nash-Moser iterations to construct a smooth solution w of $\mathcal{F}(w) = 0$ in (2.3) separately in $\{\tilde{K} > 0\}$ and $\{\tilde{K} < 0\}$ such that w is smooth across $\{\tilde{K} = 0\}$. Theorem 3.1 and Theorem 3.2 play important roles in the iteration process in this section. As noted at the beginning of the previous section, there is a smallness assumption (3.7) in Theorem 3.1. Such an assumption makes it complicated to construct an iteration sequence in the elliptic region. We need to maintain all but finitely many linearized equations in this iteration satisfy the smallness assumption when a regularity level is fixed. Otherwise, there is no hope to obtain C^∞-solutions. This requires us to modify linearized equations derived in Section 2 carefully. The modified linear operators in the elliptic region are constructed in (4.10). The last term in (4.10) is introduced to satisfy the smallness assumption (4.13), which is simply (3.7) in the present setting. Compare (4.13) with the modified linear operator (4.25) in the hyperbolic region, especially the last terms.

We first consider the transform

$$x \mapsto \big(x_1, \hat{\varphi}(\varepsilon^2 x)\big)$$

as in (2.38). In the following, we solve $\mathcal{F}(w) = 0$ in (2.40) in the new coordinate, which is still denoted by x for brevity. We note in the new coordinate

$$K = \eta(\varepsilon x_2).$$

We also write $\mathcal{F}(w)$ in the form

$$\mathcal{F}(w) = F(\varepsilon, x, w, Dw, D^2 w), \tag{4.3}$$

where F is smooth in $\sqrt{\varepsilon}, x, w, Dw, D^2 w$.

To proceed, we set

$$\Omega_\pm = \{x; |x_1| < 1,\ 0 < \pm x_2 < 1\}.$$

In the following, we consider a family of operators $S_\ell : \Omega_\pm \to \Omega_\pm$ satisfying for any $s_1, s_2 \geq 0$

$$\|S_\ell u\|_{s_1} \leq C_{s_1 s_2} \|u\|_{s_2}, \qquad \text{if } s_1 \leq s_2, \tag{4.4}$$

$$\|S_\ell u\|_{s_1} \leq C_{s_1 s_2} \nu_\ell^{s_1 - s_2} \|u\|_{s_2}, \qquad \text{if } s_1 \geq s_2, \tag{4.5}$$

$$\|S_\ell u - u\|_{s_1} \leq C_{s_1 s_2} \nu_\ell^{s_1 - s_2} \|u\|_{s_2}, \qquad \text{if } s_1 \leq s_2, \tag{4.6}$$

where the constant $C_{s_1 s_2}$ depends only on s_1, s_2, independent of ℓ. We choose

$$\nu_\ell = \nu^{\tau^\ell} \text{ for any } \ell \geq 0,$$

where $\nu > 1$ and $\tau \in (1, 2)$ are constants to be fixed. Obviously, we have for any $\ell \geq 0$

$$\nu_{\ell+1} = \nu_\ell^\tau. \tag{4.7}$$

Now we begin to solve $\mathcal{F}(w) = 0$ in Ω. We first do this in Ω_+. In the first part of this section, all functions are defined in Ω_+ and all norms are taken in Ω_+.

Set $w_0 = 0$. We construct w_ℓ in Ω_+ by an induction on ℓ as follows. Suppose $w_0, w_1, \cdots, w_\ell$ have been chosen. For ρ_ℓ to be determined, we define

$$w_{\ell+1} = w_\ell + S_\ell \rho_\ell. \tag{4.8}$$

By the Taylor expansion, we have

$$\begin{aligned} \mathcal{F}(w_{\ell+1}) =& \mathcal{F}(w_\ell) + \mathcal{F}'(w_\ell)(S_\ell \rho_\ell) + Q(w_\ell; S_\ell \rho_\ell) \\ =& \mathcal{F}(w_\ell) + \mathcal{F}'(w_\ell)\rho_\ell + \mathcal{F}'(w_\ell)(S_\ell - 1)\rho_\ell + Q(w_\ell; S_\ell \rho_\ell), \end{aligned}$$

where $Q(w_\ell; S_\ell\rho_\ell)$ is the quadratic error. We apply Corollary 2.2 to w_ℓ to get a transform $T : x \mapsto y$ as in (2.7) such that we have in y-coordinates

$$\begin{aligned}\mathcal{F}'(w_\ell)\rho_\ell =&a_{22}\partial_{y_2y_2}\rho_\ell + a_{11}K\partial_{y_1y_1}\rho_\ell + b_1K\partial_{y_1}\rho_\ell + b_2\partial_{y_2}\rho_\ell \\ &+ \tilde{a}_{11}\mathcal{F}(w_\ell)\partial_{y_1y_1}\rho_\ell + \big(\tilde{b}_{10}\mathcal{F}(w_\ell) + \tilde{b}_{11}\partial_{y_1}\mathcal{F}(w_\ell)\big)\partial_{y_1}\rho_\ell.\end{aligned} \tag{4.9}$$

We need to modify the last term in the right hand side of (4.9), as it involves a derivative of $\mathcal{F}(w_\ell)$. For w_ℓ, we set

$$\theta_\ell^+ = |\mathcal{F}(w_\ell)|_{C^1},$$

and

$$\begin{aligned}\mathcal{L}^+(w_\ell)\rho_\ell =&a_{22}\partial_{y_2y_2}\rho_\ell + a_{11}(K + C\theta_\ell^+)\partial_{y_1y_1}\rho_\ell + b_1(K + C\theta_\ell^+)\partial_{y_1}\rho_\ell \\ &+ b_2\partial_{y_2}\rho_\ell + \tilde{b}_{11}(1 - S_\ell)\big(\partial_{y_1}\mathcal{F}(w_\ell)\big)\partial_{y_1}\rho_\ell,\end{aligned} \tag{4.10}$$

for some positive constant C. We note that $1 - S_\ell$ in the last term is introduced to satisfy the smallness assumption in Theorem 3.1. Then, we obtain

$$\begin{aligned}\mathcal{F}(w_{\ell+1}) =&\mathcal{F}(w_\ell) + \mathcal{L}^+(w_\ell)\rho_\ell + \big(\tilde{a}_{11}\mathcal{F}(w_\ell) - Ca_{11}\theta_\ell^+\big)\partial_{y_1y_1}\rho_\ell \\ &+ \big(\tilde{b}_{10}\mathcal{F}(w_\ell) - Cb_1\theta_\ell^+\big)\partial_{y_1}\rho_\ell + \tilde{b}_{11}S_\ell\big(\partial_{y_1}\mathcal{F}(w_\ell)\big)\partial_{y_1}\rho_\ell \\ &+ \mathcal{F}'(w_\ell)(S_\ell - 1)\rho_\ell + Q(w_\ell; S_\ell\rho_\ell).\end{aligned}$$

By Theorem 3.1, we have a smooth solution ρ_ℓ to the following problem

$$\begin{aligned}&\mathcal{L}^+(w_\ell)\rho_\ell = -\mathcal{F}(w_\ell) \quad \text{in } T(\Omega_+), \\ &\rho_\ell = 0 \quad \text{on } \{y_2 = 0\} \cap \partial\big(T(\Omega_+)\big).\end{aligned} \tag{4.11}$$

Then we obtain

$$\begin{aligned}\mathcal{F}(w_{\ell+1}) =&\big(\tilde{a}_{11}\mathcal{F}(w_\ell) - Ca_{11}\theta_\ell^+\big)\partial_{y_1y_1}\rho_\ell + \big(\tilde{b}_{10}\mathcal{F}(w_\ell) - Cb_1\theta_\ell^+\big)\partial_{y_1}\rho_\ell \\ &+ \tilde{b}_{11}S_\ell\big(\partial_{y_1}\mathcal{F}(w_\ell)\big)\partial_{y_1}\rho_\ell + \mathcal{F}'(w_\ell)(S_\ell - 1)\rho_\ell + Q(w_\ell; S_\ell\rho_\ell).\end{aligned} \tag{4.12}$$

Now we examine the condition (3.7). We note by Sobolev embedding and (4.6)

$$\begin{aligned}&s\big|(1 - S_\ell)\big(\partial_{y_1}\mathcal{F}(w_\ell)\big)\big|_{C^1} \le cs\|(1 - S_\ell)\big(\partial_{y_1}\mathcal{F}(w_\ell)\big)\|_3 \\ \le&cs\nu_\ell^{-1}\|\partial_{y_1}\mathcal{F}(w_\ell)\|_4 \le cs\nu_\ell^{-1}\|\mathcal{F}(w_\ell)\|_5 \le cs\nu_\ell^{-1}\big(1 + \|w_\ell\|_7\big).\end{aligned}$$

By the dependence of a_{11}, a_{22}, b_1, b_2 on w_ℓ in (4.9), we obtain the following: there exists an ε_0 such that, for any integers s and ℓ with

$$s\nu_\ell^{-1} < \varepsilon_0, \tag{4.13}$$

there holds

$$\|\rho_\ell\|_s \le c_s\big(\|f_\ell\|_{s+d} + (\|w_\ell\|_{s+d} + 1)\|f_\ell\|_d\big), \tag{4.14}$$

provided

$$\|w_\ell\|_{d_0} \le 1,$$

where d and d_0 are integers satisfying

$$d \ge 7, \quad d_0 \ge 7. \tag{4.15}$$

We should note that (4.14) was proved to be true in y-coordinates. It is easy to see that it holds in x-coordinates. We also require

$$d_0 \geq d + 2. \tag{4.16}$$

For the existence of smooth solutions in Ω_+, we may in fact take $d = d_0 = 7$ in (4.17). However, we need a large d_0 here in order to construct a smooth solution in Ω_- later. Next, we note $\nu_\ell^{-1} \leq \nu^{-\tau} \leq \nu^{-1}$, as $\tau > 1$. Hence, s satisfies (4.13) for any $\ell \geq 1$ if

$$s < \varepsilon_0 \nu. \tag{4.17}$$

On the other hand, for any fixed s, (4.13) always holds for large ℓ.

In the next result, we denote $\rho_{-1} = \nu_{-1} = 0$.

Lemma 4.2. *There exist constants $c_* \geq 1$, $\tilde{s} \geq d + 2$ and $\mu_* \in (0, 1)$, independent of j, such that if*

$$\|\mathcal{F}(w_0)\|_{\tilde{s}-2} \leq \mu_*, \tag{4.18}$$

the following inequalities hold for any $j \geq 0$

$$\begin{aligned} &(Q_1)_j \qquad && \|\rho_{j-1}\|_{d_0} \leq c_* \nu^a \nu_{j-1}^{-a} \|\mathcal{F}(w_0)\|_{\tilde{s}-2}; \\ &(Q_2)_j && \|w_j\|_{d_0} \leq 1; \\ &(Q_3)_j && \|\mathcal{F}(w_j)\|_d \leq \nu^b \nu_j^{-b} \|\mathcal{F}(w_0)\|_{\tilde{s}-2}. \end{aligned}$$

Lemma 4.2 is based on the following two results.

Lemma 4.3. *Suppose (4.18) and $(Q_2)_\ell$ are valid. Then for any s and ℓ satisfying (4.13), there holds*

$$\|\rho_\ell\|_s \leq c_s (\|\mathcal{F}(w_0)\|_{s+d} + \|w_\ell\|_{s+d+2}). \tag{4.19}$$

Lemma 4.4. *Suppose (4.18) and $(Q_2)_j$ are valid, for $0 \leq j \leq \ell$. Then there exists an $\nu = \nu(\tilde{s}) > 1$, depending only on $\tilde{s}$, such that there holds for any $d + 2 \leq s \leq \tilde{s}$*

$$\|w_{\ell+1}\|_s \leq \nu_{\ell+1}^{\frac{d+2}{\tau-1}+1} \|\mathcal{F}(w_0)\|_{s-2}. \tag{4.20}$$

Moreover, for any $s \geq d + 2$, there exists an $\ell(s)$ such that there holds for any $\ell \geq \ell(s)$

$$\|w_{\ell+1}\|_s + \|\mathcal{F}(w_0)\|_{s-2} \leq c_s^{\ell+1} \nu_{\ell+1}^{\frac{d+2}{\tau-1}} \big(\|w_{\ell(s)}\|_s + \|\mathcal{F}(w_0)\|_{s-2}\big). \tag{4.21}$$

Lemmas 4.2-4.4 correspond to Lemmas 7.4.2-7.4.4 in [4]. The only difference lies between (4.21) and (7.4.16) in [4]. We note that (4.21) holds only for any $\ell \geq \ell(s)$. In fact, $\ell(s)$ can be chosen to satisfy

$$(s - d - 2)\nu_{\ell(s)}^{-1} < \varepsilon_0.$$

This is related to (4.13). The proof of Lemmas 4.2-4.4 is similar to that of Lemma 7.4.2-7.4.4 in [4] and hence is omitted.

Now we are ready to prove Theorem 4.1 in Ω_+.

Proof of Theorem 4.1 in Ω_+. Proceeding similarly as in the proof of Theorem 7.4.1 in [4], we have that w_ℓ converges to some w in $H^s(\Omega_+)$ for any s and $F(w) = 0$ in Ω_+. In fact, for any fixed $s \geq d_0$, we take an appropriate $s^* > \max\{s, \tilde{s} - m - d\}$ and conclude that

$$\|\rho_\ell\|_s \leq c(\nu, s) c_{s^*}^\ell \, \nu^{-\frac{b}{2}\tau^\ell} \big(\|\mathcal{F}(w_0)\|_{s^*+d} + \|w_{\ell^*}\|_{s^*+d+2}\big),$$

for any $\ell \geq \ell(s^* + d + 2)$ as in Lemma 4.4. We refer to the proof of Theorem 7.4.1 in [4] for details. □

By examining the proof of $(Q_2)_j$, we have

$$\|w\|_{d_0, \Omega_+} \leq C\mu_*.$$

Set

$$\Sigma = \{(x_1, 0); |x_1| < 1\}.$$

We extend w in Ω_+ to $\bar{w}_0$ in Ω_- to form a smooth function in $\Omega_+ \cup \Sigma \cap \Omega_-$ with

$$\|\bar{w}_0\|_{d_0, \Omega_-} \leq C\mu_*.$$

By taking ε smaller in (2.40) if necessary, we then have

$$\|\mathcal{F}(\bar{w}_0)\|_{d_0 - 2, \Omega_-} \leq C\mu_*. \tag{4.22}$$

Now we use a similar method to solve $\mathcal{F}(w) = 0$ in Ω_-. We construct $\bar{w}_\ell$ in Ω_- by an induction on ℓ as follows. Suppose $\bar{w}_0, \bar{w}_1, \cdots, \bar{w}_\ell$ have been chosen. For $\bar{\rho}_\ell$ to be chosen, we define

$$\bar{w}_{\ell+1} = \bar{w}_\ell + S_\ell \bar{\rho}_\ell. \tag{4.23}$$

By the Taylor expansion, we have

$$\mathcal{F}(\bar{w}_{\ell+1}) = \mathcal{F}(\bar{w}_\ell) + \mathcal{F}'(\bar{w}_\ell)\bar{\rho}_\ell + \mathcal{F}'(\bar{w}_\ell)(S_\ell - 1)\bar{\rho}_\ell + Q(\bar{w}_\ell; S_\ell\bar{\rho}_\ell),$$

where $Q(\bar{w}_\ell; S_\ell\bar{\rho}_\ell)$ is the quadratic error. We apply Corollary 2.2 to $\bar{w}_\ell$ to get a transform $x \mapsto y$ as in (2.7) such that we have in y-coordinates

$$\begin{aligned}\mathcal{F}'(\bar{w}_\ell)\bar{\rho}_\ell =& a_{22}\partial_{y_2y_2}\bar{\rho}_\ell + a_{11}K\partial_{y_1y_1}\bar{\rho}_\ell + b_1K\partial_{y_1}\bar{\rho}_\ell + b_2\partial_{y_2}\bar{\rho}_\ell \\ &+ \tilde{a}_{11}\mathcal{F}(\bar{w}_\ell)\partial_{y_1y_1}\bar{\rho}_\ell + \big(\tilde{b}_{10}\mathcal{F}(\bar{w}_\ell) + \tilde{b}_{11}\partial_{y_1}\mathcal{F}(\bar{w}_\ell)\big)\partial_{y_1}\bar{\rho}_\ell.\end{aligned} \tag{4.24}$$

For $\bar{w}_\ell$, we set

$$\theta_\ell^- = |\mathcal{F}(\bar{w}_\ell)|_{C^1},$$

and

$$\begin{aligned}\mathcal{L}^-(\bar{w}_\ell)\bar{\rho}_\ell =& a_{22}\partial_{y_2y_2}\bar{\rho}_\ell + a_{11}(K - C\theta_\ell^-)\partial_{y_1y_1}\bar{\rho}_\ell + b_1K\partial_{y_1}\bar{\rho}_\ell \\ &+ b_2\partial_{y_2}\bar{\rho}_\ell + \tilde{b}_{11}\partial_{y_1}\mathcal{F}(\bar{w}_\ell)\partial_{y_1}\bar{\rho}_\ell,\end{aligned} \tag{4.25}$$

for some positive constant C. By $K \leq 0$ in Ω_- and Theorem 3.2, we have a smooth $\bar{\rho}_\ell$ in Ω_- satisfying

$$\begin{aligned}&\mathcal{L}^-(\bar{w}_\ell)\bar{\rho}_\ell = -\mathcal{F}(\bar{w}_\ell) \quad \text{in } \Omega_-, \\ &\bar{\rho}_\ell = 0, \ \partial_2\bar{\rho}_\ell = 0 \quad \text{on } \Sigma.\end{aligned} \tag{4.26}$$

We may modify S_ℓ so that we have $S_\ell\bar{\rho}_\ell = 0$ and $\partial_2(S_\ell\bar{\rho}_\ell) = 0$ on Σ. The rest is similar and hence omitted.

Proof of Theorem 4.1. As in the proof of Theorem 4.1 in Ω_+, we may find a smooth solution $\bar{w}$ of $\mathcal{F}(\bar{w}) = 0$ in Ω_- if d_0 in (4.22) is sufficiently large. We note that $d_0 - 2$ in (4.22) plays the same role as $\tilde{s} - 2$ in (4.18). Next, we note $\bar{w} = w$ and $\partial_2 \bar{w} = \partial_2 w$ on Σ by (4.26), where w is the smooth solution to $\mathcal{F}(w) = 0$ in Ω_+ established before. Hence $\bar{w}$ is a C^1 extension of w to Ω_-. By (2.40), we have $\partial_{22} w = \partial_{22} \bar{w}$ on Σ, and hence $\bar{w}$ is a smooth extension of w to Ω_-. □

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