DEGENERATE HYPERBOLIC EQUATIONS WITH LOWER DEGREE DEGENERACY

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1. Introduction

Hyperbolic differential equation is an important class of differential equations. It is well known that the Cauchy problem is well-posed for (strictly) hyperbolic differential equations. However, it is still not known the optimal assumption to ensure the well-posedness of the Cauchy problem of degenerate hyperbolic differential equations even if the space dimension is *one*.

In [3], Colombini and Spagnolo constructed a degenerate hyperbolic equation such that the Cauchy problem is not well-posed. The equation has a simple form

$$u_{tt} - K(t)u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, T),$$

where $K(t)$ is a nonnegative function in $(0, T)$. In fact, solutions do not exist for some smooth Cauchy data even in the distribution sense. In this example, the nonnegative function $K(t)$ oscillates an infinite number of times. Such an example clearly shows that certain conditions need to be imposed on degenerate leading coefficients to ensure the well-posedness of the Cauchy problem for degenerate hyperbolic equations. In [2], Colombini, Jannelli and Spagnolo conjectured that *the well-posedness in the C^∞-category holds for the Cauchy problem of degenerate hyperbolic equations if the leading (degenerate) coefficients are analytic in time t.* Here, the analyticity in t ensures a *finite number of oscillations* in the t-direction, which excludes the example in [3].

It is well-known that the Cauchy problem of degenerate hyperbolic equations is well-posed in C^∞ if leading (degenerate) coefficients are analytic in both time and space variables with appropriate assumptions on the lower order coefficients. (See e.g. [12] and [5]). For smooth case, the Cauchy problem is well-posed for degenerate hyperbolic equations under additional conditions on oscillations of the degenerate coefficients with respect to time. Such conditions are in general imposed on *zero sets* of *time derivatives of degenerate coefficients*. In [13], those zero sets are horizontal. In [4] and [9], those zero sets are allowed to be Lipschitz or even Hölder continuous (as space variables). However, conditions in [4] and [9] are hard to verify in general for many interesting cases.

Although appropriate assumptions on time derivatives of degenerate coefficients yield affirmative results, it seems that better candidates for natural assumptions are the degenerate coefficients themselves. The reason is clear since equations are degenerate if these coefficients vanish. It seems more natural to study *zero sets* of *degenerate coefficients* instead of those of time derivatives of degenerate coefficients. In [7], the first author

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derived classical energy estimates for solutions of degenerate hyperbolic equations when degenerate coefficients do not admit complex roots.

In this paper, we continue our study of the Cauchy problem of degenerate hyperbolic equations. Consider in $\mathbb{R} \times (0,T)$ the Cauchy problem in $\mathbb{R} \times (0,T)$ of the form

$$u_{tt} - a(x,t)u_{xx} - b_0(x,t)u_t - b(x,t)u_x - c(x,t)u = f(x,t), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.2}$$

where a is a nonnegative function in $\mathbb{R} \times (0,T)$. This is a degenerate hyperbolic equation since a is allowed to vanish. We assume

$$0 \le a \le \Lambda \quad \text{in } \mathbb{R} \times [0,T], \tag{1.3}$$

and

$$|b| \le C_b\sqrt{a}, \quad \text{in } \mathbb{R} \times [0,T], \tag{1.4}$$

for positive constants Λ and C_b. The condition (1.4) is the simplest (and probably the most restricted) form of the so-called *Levi condition*. It is necessary for the well-posedness of the Cauchy problem for degenerate hyperbolic equations.

To simplify our discussion, we assume

$$a(x,t) = q(x,t)K(x,t),$$

where $q(x,t)$ is a positive smooth function and $K(x,t)$ is a nonnegative smooth function such that

$$0 < \lambda \le q \le \Lambda \quad \text{in } \mathbb{R} \times [0,T], \tag{1.5}$$

and

$$K(x,t) = t^m + c_1(x)t^{m-1} + \cdots + c_{m-1}(x)t + c_m(x). \tag{1.6}$$

Here, m is an even integer and $c_1, \cdots, c_m$ are smooth functions in $\mathbb{R}$. If a is an arbitrary function analytic in t and smooth in x, then we can always write a in this form locally by the Weierstrass preparation theorem.

In this case, the zero set seems to have a relatively simple structure. For each fixed $x \in \mathbb{R}^m$, there are m zeros for t and these zeros may be complex. In fact, it is precisely due to the presence of complex zeros that makes the discussion of the Cauchy problem difficult. If all zeros of K are real, these zeros decompose the plane into finitely many regions. We are able to derive energy estimates in each region and then patch them together to obtain global energy estimates. Refer to [7] for details. However, if complex zeros are present, we need to connect real zeros appropriately to form a decomposition. Among all possible choices, we need to choose one to obtain desired a priori estimates. A natural choice seems to be the real part of zeros. If $K(x,t)$ is assumed to be analytic in both t and x, real parts of zeros indeed decompose the plane properly, as shown in [5]. However, when the analyticity in x is absent, zero sets are complicated to analyze in the general case. The real part of zeros still works for smooth K in (1.6) with $m = 2$. In fact, it is the only known case. The real part of zeros does *not* decompose the plane properly even for $m = 4$!

In this paper, we use the zero sets of *both K and* the t-derivative of K to decompose the plane. For $m = 4$, we find a nontrivial decomposition so that energy estimates can be obtained. (The case $m = 2$ is known as mentioned earlier.) The main result is the following.

Theorem 1.1. *Let q, b_0, b, c, K be smooth functions in $\mathbb{R}\times(0,T)$ with $a = qK$ satisfying (1.4) and (1.5). Suppose K is a nonnegative function given by (1.6) for $m = 4$. Then the Cauchy problem for (1.1) admits a smooth solution in $\mathbb{R}\times(0,T)$ for any smooth function f in $\mathbb{R}\times(0,T)$. Moreover, for any $s \geq 0$*

$$\|u\|_{s,\mathbb{R}\times(0,T)} \leq C_s\big(\|u(\cdot,0)\|_{s+N,\mathbb{R}} + \|u_t(\cdot,0)\|_{s+N-1,\mathbb{R}} + \|f\|_{s+N,\mathbb{R}\times(0,T)}\big),$$

where C_s is a positive constant depending only on s, λ, Λ, C_b and the C^{s+N}-norms of q, K, b_0, b and c, and N is a nonnegative integer which is universal.

Here and thereafter, $\|\cdot\|_{s,\Omega}$ is the H^s-norm in Ω.

We expect that Theorem 1.1 holds for arbitrary nonnegative function K in (1.6) and that N depends only on m. We emphasize that it is quite delicate to decompose the plane appropriately even for $m = 4$. The obvious choice of the real part of zeros of K does not work. In fact, the decomposition has a complicated explicit expression in terms of coefficients. See (3.13) in Section 3. Last, we point out that discussions in that section can be generalized to *some* polynomials of higher order.

The paper is organized as follows. In Section 2, we recall a criterion of appropriate decompositions such that energy estimates can be obtained and hence the Cauchy problem is well-posed. In Section 3, we verify that this criterion is satisfied for K in (1.6) for $m = 4$. We also discuss briefly methods used in this paper.

2. A Criterion for Energy Estimates

In this section, we recall several results in [7] leading to energy estimates of solutions of the Cauchy problem of degenerate hyperbolic equations. See also [5].

Let $\Sigma_\pm$ be two curves given by $t = t_\pm(x) \in C^\beta(I)\cap BV(I)$, for some $\beta \in (0,1]$ and some interval $I \subset \mathbb{R}$, satisfying

$$0 \leq t_-(x) < t_+(x) \leq 1 \text{ for any } x \in I, \tag{2.1}$$

and

$$t_\pm(x) = const. \text{ for large } x \text{ if } I \text{ is an infinite interval and}$$
$$t_+(x) = t_-(x) \text{ at finite ends } x \text{ of } I.$$

Set

$$\Omega = \{(x,t);\ x \in I,\ t_-(x) < t < t_+(x)\}. \tag{2.2}$$

We note that integration by parts can be performed in Ω since $\partial\Omega$ is given by functions of bounded variations. Moreover, Sobolev embedding holds in Ω since its boundary is Hölder continuous. In fact, we have

$$H^k(\Omega) \subset L^\infty(\Omega) \quad \text{for } k > \frac{1}{2}\big(1+\frac{1}{\beta}\big).$$

Refer to [11] for details.

We consider (1.1) in Ω with the Cauchy data

$$u=\varphi,\ u_y=\psi \quad \text{on } \Sigma_-. \tag{2.3}$$

We assume

$$a(\partial_x t_\pm)^2\le \eta_0 \quad \text{on } \Sigma_\pm, \tag{2.4}$$

for some constant $\eta_0\in(0,1)$.

Lemma 2.1. *Let u be an H^2-solution of (1.1) in Ω and (1.4), (1.5) and (2.4) be assumed. Suppose there exists a nonnegative C^1 function ω in Ω satisfying*

$$\frac{\partial_t\omega}{\omega}\le c_1,\quad \frac{\partial_t\omega}{\omega}+\frac{|\partial_x\omega|}{\omega}\sqrt{a}\le c_2,\quad \frac{\partial_t(a\omega)}{a\omega}+\frac{|\partial_x\omega|}{\omega}\sqrt{a}\le c_3, \tag{2.5}$$

for some positive constants c_1, c_2 and c_3. Then for any $\mu\ge\mu_0$

$$\begin{aligned} &c_0\int_{\Sigma_+}\frac{e^{-\mu t}}{\sqrt{1+|t_+'|^2}}\,\omega(u^2+u_t^2+au_x^2)+(\mu-\mu_0)\int_\Omega e^{-\mu t}\omega(u^2+u_t^2+au_x^2)\\ &\qquad\le c_0'\int_{\Sigma_-}\frac{e^{-\mu t}}{\sqrt{1+|t_-'|^2}}\omega(u^2+u_t^2+au_x^2)+\int_\Omega e^{-\mu t}\omega f^2, \end{aligned} \tag{2.6}$$

where μ_0, c_0 and c_0' are positive constants depending only on C_b in (1.4), η_0 in (2.4), c_1, c_2 and c_3 in (2.5), the L^∞-norms of b_0 and c, and the Lipschitz norm of $\sqrt{a}$.

The proof follows from a standard process of a multiplication of (1.1) by $e^{-\eta t}u_t$ and an integration in Ω.

The difficult part in applying Lemma 2.1 is to construct ω satisfying (2.5). In some cases, ω can be constructed easily. We can take $\omega=1$ if $a_t\le C_*a$ and take $\omega=1/a$ if $a_t\ge -C_*a$ for some positive constant C_*.

The estimate (2.6) is often referred to as a weighted estimates due to the presence of ω. Such a weight can be removed if it relates to boundary appropriately. In the following, we assume

$$c_0\le\omega\le\big(t-t_-(x)\big)^{-d}\quad \text{for any } (x,t)\in\Omega, \tag{2.7}$$

where c_0 is a positive constant. As we see, the boundary $t=t_-(x)$ will be related to zeros of a. If $t=t_-(x)$ is a zero of a, then a is degenerate there. The condition (2.7) is introduced to overcome the degeneracy. The finite degree of degeneracy in (2.7) is essential.

Lemma 2.2. *Let q,b,b_0,c and K be C^d-functions in $\bar\Omega$ with $a=qK$ satisfying (1.4), (1.5) and (2.4) and u be an H^{d+3}-solution of (1.1) and (2.3) for $\varphi\in H^{d+2}(\Sigma_-)$, $\psi\in H^{d+1}(\Sigma_-)$ and $f\in H^{d+1}(\Omega)$. Suppose there exists a nonnegative ω in Ω satisfying (2.5) and (2.7). Then*

$$\|u\|_{L^2(\Omega)}\le C\big(\|\varphi\|_{H^{d+2}(\Sigma_-)}+\|\psi\|_{H^{d+1}(\Sigma_-)}+\|f\|_{H^{d+1}(\Omega)}\big), \tag{2.8}$$

where C is a positive constant depending on C_b, η_0 and the C^d-norms of a,b,b_0 and c.

In order to establish energy estimates for the Cauchy problem of degenerate hyperbolic equations, we decompose the (upper half) plane into finitely many regions such that energy estimates can be obtained in each region. Then we patch these estimates in each region together to obtain a global energy estimates.

First, we impose a basic assumption.

Assumption 2.3. *For some constant $\beta \in (0,1]$ and some integer n, there exist n functions $\varphi_1, \cdots, \varphi_n \in C^\beta(\mathbb{R}) \cap BV(\mathbb{R})$, with $\varphi_n \le \cdots \le \varphi_1$ in $\mathbb{R}$, which separate $\mathbb{R}\times\mathbb{R}$ into $n+1$ regions*

$$\begin{aligned} \Omega_1 &= \{(x,t); t > \varphi_1(x)\}, \\ \Omega_k &= \{(x,t); \varphi_k(x) < t < \varphi_{k-1}(x)\} \quad \text{for } k = 2, \cdots, n, \\ \Omega_{n+1} &= \{(x,t); t < \varphi_n(x)\}, \end{aligned}$$

such that for some constant $\eta \in (0,1)$

$$|\partial_x \varphi_k|\sqrt{a}\big|_{t=\varphi_k} \le \eta \quad \text{for } k = 1, \cdots, n, \tag{2.9}$$

and for some nonnegative constant C_*

$$\partial_t a \ge -C_* a \quad \text{in } \Omega_1 \quad \text{and} \quad \partial_t a \le C_* a \quad \text{in } \Omega_{n+1}, \tag{2.10}$$

and for $k = 2, \cdots, n$, either

$$\partial_t a \ge -C_* a \ \text{ in } \Omega_k,$$

or

$$\partial_t a \le C_* a \ \text{ in } \Omega_k,$$

or there exist a positive function ω_k in Ω_k and a nonnegative integer d_k such that in Ω_k

$$\frac{\partial_t \omega_k}{\omega_k} \le c_1, \quad \frac{\partial_t \omega_k}{\omega_k} + \frac{|\partial_x \omega_k|}{\omega_k}\sqrt{a} \le c_2, \quad \frac{\partial_t(\omega_k a)}{\omega_k a} + \frac{|\partial_x \omega_k|}{\omega_k}\sqrt{a} \le c_3, \tag{2.11}$$

and

$$c_0 \le \omega_k \le \big(t - \varphi_k(x)\big)^{-d_k} \quad \text{for any } (x,t) \in \Omega_k, \tag{2.12}$$

for some positive constants c_0, c_1, c_2 and c_3.

Remark 2.4. (i) Obviously, we have

$$\mathbb{R}\times\mathbb{R} = \bigcup_{k=0}^{n+1} \bar{\Omega}_k.$$

(ii) If ω_k is decreasing in t in Ω_k, then the first inequality in (2.11) is satisfied for $c_1 = 0$.

(iii) Let $a = qK$ where q is a positive function with a uniform positive lower bound. Then (2.10) and (2.11) can be written as

$$\partial_t K \ge 0 \text{ in } \Omega_1 \quad \text{and} \quad \partial_t K \le 0 \text{ in } \Omega_{n+1}, \tag{2.13}$$

and

$$\frac{\partial_t \omega_k}{\omega_k} \le c_1, \quad \frac{\partial_t \omega_k}{\omega_k} + \frac{|\partial_x \omega_k|}{\omega_k}\sqrt{K} \le c_2, \quad \frac{\partial_t(\omega_k K)}{\omega_k K} + \frac{|\partial_x \omega_k|}{\omega_k}\sqrt{K} \le c_3. \tag{2.14}$$

Assumption 2.3 is sufficient for energy estimates.

Theorem 2.5. *Let q, b_0, b, c, K be smooth functions in $\mathbb{R}\times(0,T)$ with $a = qK$ satisfying (1.4) and (1.5). Suppose a satisfies Assumption 2.3. Then for any smooth solution u of (1.1)-(1.2), there holds for any $s \geq 0$*

$$\|u\|_{s,\mathbb{R}^n\times(0,T)} \leq C\big(\|u_0\|_{s+N,\mathbb{R}^n} + \|u_1\|_{s+N-1,\mathbb{R}^n} + \|f\|_{s+N,\mathbb{R}^n\times(0,T)}\big),$$

where C is a positive constant depending only on s, Λ, C_b, the C^{s+N}-norms of a, b_0, b and c, and the C^β-norms and the BV-norms of $\varphi_1, \cdots, \varphi_n$ in Assumption 2.3, and N is a positive constant depending only on m.

Refer to [7] for a proof of quoted results. See also [5].

Now, we demonstrate that zero sets of K play an important role in the decomposition as in Assumption 2.3. Here, a zero, or a root, of K is referred to a function $t = t(x)$ satisfying $K(x, t(x)) = 0$.

We only consider nonnegative functions $K = K(x,t)$ given by

$$K(x,t) = t^m + c_1(x)t^{m-1} + \cdots + c_m(x), \tag{2.15}$$

where $c_1, \cdots, c_m$ are (real) smooth functions in $\mathbb{R}$. For each fixed $x \in \mathbb{R}$, $K(x,t)$ is a polynomial of t of degree m and hence has m complex roots $t_1(x), \cdots, t_m(x)$, which can be arranged to be continuous. By writing

$$K(x,t) = \prod_{i=1}^m \big(t - t_i(x)\big), \tag{2.16}$$

we have

$$\frac{\partial_t K}{K} = \sum_{i=1}^m \frac{1}{t - t_i(x)}. \tag{2.17}$$

We need to control K_t/K from above in the third inequality in (2.11). If t_i is a real root of K, then $1/(t-t_i)$ is in the expression of K_t/K. If t_i is a complex zero, i.e., $\mathrm{Im} t_i \neq 0$, then so is $\bar{t}_i$ since K has real coefficients. For such a pair, we write

$$\frac{1}{t-t_i} + \frac{1}{t-\bar{t}_i} = \frac{2\big(t - \mathrm{Re} t_i\big)}{\big(t-\mathrm{Re} t_i\big)^2 + \big(\mathrm{Im} t_i\big)^2}.$$

In this case, we need to choose an appropriate function to replace t_i and $\bar{t}_i$. It seems that $\mathrm{Re} t_i$ is an obvious choice since

$$\frac{1}{t-t_i} + \frac{1}{t-\bar{t}_i} \leq \frac{2}{t - \mathrm{Re} t_i} \quad \text{for } t > \mathrm{Re} t_i.$$

However, such a choice does not always work! The subtlety of verifying Assumption 2.3 to a large extent is to find an appropriate function other than real parts of zeros when complex zeros are present.

Next, we discuss briefly the regularity of zeros of K. The following result asserts that they are Hölder continuous if coefficients are Hölder continuous. Refer to Lemma 2.3 and Lemma 2.5 [10] for a proof.

Lemma 2.6. *Suppose* $c_1, \cdots, c_m$ *are Hölder continuous on an interval* $I \subset \mathbb{R}$ *of order* $\alpha \in (0,1]$ *and that* $t = t(x)$ *is a (complex-valued) continuous function on* I *satisfying* $K(x,t) = 0$ *on* I*. Then* t *is Hölder continuous on* I *of order* α/m*. Moreover,*

$$\|t\|_{C^{\frac{\alpha}{m}}(I)} \le 4m \sup_{1\le i\le m} \|c_i\|_{C^\alpha(I)}^{1/i}.$$

In general, the root function in Lemma 2.6 is not Lipschitz. Concerning whether it is of bounded variation, we have the following result in the simplest case. Refer to Lemma 1 in [2] for nonnegative f and the main result in [14] for the general case.

Lemma 2.7. *Suppose* f *is a* C^m*-function on* $I \subset \mathbb{R}$*. Then* $\sqrt[m]{|f|}$ *is of bounded variation on* I *and*

$$\|(\sqrt[m]{|f|})'\|_{L^1(I)} \le C(m,I)\|f\|_{C^m(I)}^{\frac{1}{m}}, \tag{2.18}$$

where $C(m,I)$ *is a positive constant depending only on* m *and* I*.*

Lemma 2.7 implies that the curve $y = \sqrt[m]{|f|}$ has a finite arc length over I with the length bounded by $C(m,I)\|f\|_{C^m(I)}^{1/m}$.

3. Nonnegative Polynomials of Degree Four

The main goal in this section is to verify that Assumption 2.3 is satisfied by nonnegative degree four polynomials.

We start with an arbitrary nonnegative polynomials of degree m in (2.15). Obviously, m must be even since $K \ge 0$. We begin with the simplest case when $m = 2$. Suppose K is a quadratic polynomial in t given by

$$K(x,t) = t^2 + c_1(x)t + c_2(x),$$

where c_1 and c_2 are smooth functions on $\mathbb{R}$. Then $\partial_t K = 2t + c_1$, and hence

$$\partial_t K > 0 \text{ if } t > -\frac{c_1}{2} \quad \text{and} \quad \partial_t K < 0 \text{ if } t < -\frac{c_1}{2}.$$

Note that the curve $t = -c_1/2$ is smooth. Hence Assumption 2.3 holds for such a K with

$$\Omega_1 = \{(x,t); t > -\frac{1}{2}c_1(x)\}, \quad \Omega_2 = \{(x,t); t < -\frac{1}{2}c_1(x)\}.$$

Since $K \ge 0$, then $4c_2 \ge c_1^2$. Hence $K(x,t) = 0$ is satisfied by

$$t(x) = \frac{1}{2}\big(-c_1(x) \pm i\sqrt{4c_2(x) - c_1^2(x)}\big).$$

Hence the real part of zeros of K is exactly $-c_1/2$.

Next we consider a degree four polynomial K in t given by

$$K(x,t) = t^4 + c_1(x)t^3 + c_2(x)t^2 + c_3(x)t + c_4(x) \quad \text{in } \mathbb{R}\times\mathbb{R}.$$

We prove that Assumption 2.3 holds for such a K.

Lemma 3.1. *There exist two functions $t = \varphi_1(x)$, $t = \varphi_2(x)$ in $C^{1/4}(\mathbb{R}) \cap BV(\mathbb{R})$, with $\varphi_2 \le \varphi_1$ on $\mathbb{R}$, such that*

$$\begin{aligned} (i) \qquad & \partial_t K \ge 0 \quad \text{for any } t > \varphi_1(x), \\ & \partial_t K \le 0 \quad \text{for any } t < \varphi_2(x), \\ (ii) \qquad & \frac{\partial_t K}{K} \le \frac{4}{t - \varphi_2(x)} \quad \text{for } \varphi_2(x) < t < \varphi_1(x), \\ (iii) \qquad & |\partial_x \varphi_2| \sqrt{K}\big|_{\varphi_2 \le t \le \varphi_1} \le \kappa, \\ & |\partial_x \varphi_1| \sqrt{K}\big|_{t = \varphi_1} \le \kappa, \end{aligned}$$

for some $\kappa \in (0, 1)$.

With Lemma 3.1, we decompose $\mathbb{R} \times \mathbb{R}$ by

$$\Omega_1 = \{t > \varphi_1\}, \quad \Omega_2 = \{\varphi_2 < t < \varphi_1\}, \quad \Omega_3 = \{t < \varphi_2\}.$$

In Ω_2, we take

$$\omega_2 = (t - \varphi_2)^{-N},$$

for N sufficiently large depending only on κ.

Before proving Lemma 3.1, we first explain through some straightforward calculation why real parts of zeros do not decompose the plane properly. In other words, we cannot take φ_i as real parts of zeros of K.

First, by changing $t \to t - c_1/4$, we may assume $c_1 = 0$. Hence K is given by

$$K(x,t) = t^4 + c_2(x)t^2 + c_3(x)t + c_4(x) \quad \text{in } \mathbb{R} \times \mathbb{R}. \tag{3.1}$$

Then the zeros of K are given by $a \pm b_1 i$ and $-a \pm b_2 i$, with $a \ge 0$. Then

$$K = [(t-a)^2 + b_1^2][(t+a)^2 + b_2^2]. \tag{3.2}$$

A simple comparison with (3.1) yields

$$\begin{aligned} -2a^2 + b_1^2 + b_2^2 &= c_2, \\ 2a(b_1^2 - b_2^2) &= c_3, \\ (a^2 + b_1^2)(a^2 + b_2^2) &= c_4. \end{aligned} \tag{3.3}$$

With $K \ge 0$, we have

$$c_4 \ge 0, \quad 2\sqrt{c_4} + c_2 \ge 0. \tag{3.4}$$

In fact, by taking $t = 0$ in (3.1), we have $c_4 \ge 0$. Next, by taking $t = -\mathrm{sgn}(c_3) \cdot \sqrt[4]{c_4}$ in (3.1), we have

$$2c_4 + c_2 c_4^{\frac{1}{2}} \ge |c_3| c_4^{\frac{1}{4}}.$$

This implies $2\sqrt{c_4} + c_2 \ge 0$.

We first consider a special case when K has two real zeros a and $-a$. In this case, with (3.3), we have $c_3 = 0$, $c_2 = -2\sqrt{c_4}$ and

$$a = \sqrt[4]{c_4}.$$

This implies for $-a \le t \le a$

$$\sqrt{K} = (a-t)(t+a) \le 4a^2 = 4\sqrt{c_4},$$

and hence

$$|\partial_x a|\sqrt{K}\big|_{|t|\le a} \le c_4^{-\frac{1}{4}}|\partial_x c_4| \le 2\sqrt[4]{c_4}|\partial_x \sqrt{c_4}|,$$

which is bounded. Hence, Lemma 3.1(iii) is satisfied if c_4 is small for $\varphi_1 = \sqrt[4]{c_4}$ and $\varphi_2 = -\sqrt[4]{c_4}$.

When complex zeros are present, it is quite complicated. We consider the case $c_3 = 0$ for an illustration. A straightforward calculation shows that (3.4) is sufficient for $K \ge 0$ in the case when $c_3 = 0$. In fact, for $c_3 = 0$, we have for $c_4 \ge 0$, $|c_2| \le 2\sqrt{c_4}$,

$$K = \big((t + \frac{1}{2}\sqrt{2\sqrt{c_4} - c_2})^2 + \frac{1}{4}(2\sqrt{c_4} + c_2)\big)\big((t - \frac{1}{2}\sqrt{2\sqrt{c_4} - c_2})^2 + \frac{1}{4}(2\sqrt{c_4} + c_2)\big),$$

and for $c_4 \ge 0$, $c_2 \ge 2\sqrt{c_4}$,

$$K = \big(t^2 + \frac{c_2}{2} + \sqrt{\frac{c_2^2}{4} - c_4}\big)\big(t^2 + \frac{c_2}{2} - \sqrt{\frac{{c_2}^2}{4} - c_4}\big).$$

Then, the zero of K is given by

$$t = \begin{cases} \pm\frac{1}{2}\sqrt{2\sqrt{c_4} - c_2} \pm \frac{1}{2}i\sqrt{2\sqrt{c_4} + c_2} & \text{if } |c_2| \le 2\sqrt{c_4}, \\ \pm i\sqrt{\frac{c_2}{2} \pm \sqrt{\frac{c_2^2}{4} - c_4}} & \text{if } c_2 \ge 2\sqrt{c_4}, \end{cases}$$

and hence

$$\mathrm{Re}t = \begin{cases} \pm\frac{1}{2}\sqrt{2\sqrt{c_4} - c_2} & \text{if } |c_2| \le 2\sqrt{c_4}, \\ 0 & \text{if } c_2 \ge 2\sqrt{c_4}. \end{cases}$$

A simple calculation shows that Lemma 3.1(iii) is *not* satisfied for the real part of zeros $\varphi_1 = \mathrm{Re}t$ and $\varphi_2 = -\mathrm{Re}t$!

Now we start to prove Lemma 3.1. As we have explained before, we need to include all real zeros in the decomposition and to extend these functions appropriately.

Suppose $t = a$ is a real zero of K. Since $K \ge 0$, then $t = a$ is also a zero of $\partial_t K$. In other words, a real zero of K is a common zero of K and $\partial_t K$. Hence we may consider a linear combination of K and $\partial_t K$. Note

$$\begin{aligned} K =& t^4 + c_2 t^2 + c_3 t + c_4, \\ \partial_t K =& 4t^3 + 2c_2 t + c_3. \end{aligned}$$

Consider

$$H(t) = t\partial_t K - K = 3t^4 + c_2 t^2 - c_4. \tag{3.5}$$

Then H has two real zeros

$$\alpha = \sqrt{\frac{\sqrt{c_2^2 + 12c_4} - c_2}{6}}, \quad -\alpha = -\sqrt{\frac{\sqrt{c_2^2 + 12c_4} - c_2}{6}}. \tag{3.6}$$

In the following, we set $\varphi_1 = \alpha$ and $\varphi_2 = -\alpha$, and verify that (i)-(iii) in Lemma 3.1 are satisfied.

We emphasize again that a zero of K is referred to an expression $t = t(x)$ such that $K(x, t(x)) = 0$. All c_i, α and zeros of K are functions of $x \in \mathbb{R}$.

We first discuss the regularity of α.

Lemma 3.2. *Let α be as in (3.6). Then $\alpha \in C^{1/4}(\mathbb{R}) \cap BV(\mathbb{R})$.*

Proof. First, we note that α is a continuous zero of H in (3.5). Then, $\alpha \in C^{1/4}(\mathbb{R})$ by Lemma 2.6. Next, we calculate α'. Here, the derivative is taken with respect to x. By

$$3\alpha^4 + c_2\alpha^2 - c_4 = 0,$$

we get

$$2\alpha' = -\frac{\alpha}{6\alpha^2 + c_2} c_2' + \frac{1}{\alpha(6\alpha^2 + c_2)} c_4'.$$

By the expression for α in (3.6), we obtain $6\alpha^2 + c_2 = \sqrt{c_2^2 + 12c_4}$, and hence

$$2\alpha' = -\frac{\alpha}{\sqrt{c_2^2 + 12c_4}} c_2' + \frac{1}{\alpha\sqrt{c_2^2 + 12c_4}} c_4'.$$

With $\pm c_2 \le |c_2| \le \sqrt{c_2^2 + 12c_4}$ and the explicit expression of α in (3.6), it is easy to see

$$\frac{\alpha}{6\alpha^2 + c_2} = \frac{1}{\sqrt{6}} \frac{\sqrt{\sqrt{c_2^2 + 12c_4} - c_2}}{\sqrt{c_2^2 + 12c_4}} \le \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[4]{c_2^2 + 12c_4}},$$

and

$$\frac{1}{\alpha\sqrt{c_2^2 + 12c_4}} = \frac{\sqrt{6}}{\sqrt{\sqrt{c_2^2 + 12c_4} - c_2} \cdot \sqrt{c_2^2 + 12c_4}} \le \frac{1}{\sqrt{c_4}\sqrt[4]{c_2^2 + 12c_4}}.$$

Hence, we obtain

$$|\alpha'| \le \frac{1}{2\sqrt{3}} \cdot \frac{1}{\sqrt[4]{c_2^2 + 12c_4}} |c_2'| + \frac{1}{2} \cdot \frac{1}{\sqrt{c_4}\sqrt[4]{c_2^2 + 12c_4}} |c_4'|. \tag{3.7}$$

Then

$$|\alpha'| \le \frac{1}{2\sqrt{3}} \cdot \frac{1}{\sqrt{|c_2|}} |c_2'| + \frac{1}{2\sqrt[4]{12}} \cdot \frac{1}{\sqrt[4]{c_4^3}} |c_4'| \le |(\sqrt{|c_2|})'| + |(\sqrt[4]{c_4})'|.$$

Therefore by Lemma 2.7, we get for any interval $I \subset \mathbb{R}$

$$\int_I |\alpha'| \le C(I)\big(\|c_2\|_{C^2(I)}^{\frac{1}{2}} + \|c_4\|_{C^4(I)}^{\frac{1}{4}}\big).$$

Hence, α is of bounded variation on I. □

We note that α is in fact a $C^{1/2}$-function by the explicit expression in (3.6).

Lemma 3.3. *Let $K \geq 0$ be as in (3.1) with roots $a \pm b_1 i$ and $-a \pm b_2 i$, and α as in (3.6). Then*

$$|a| \leq \alpha \leq \min\{\sqrt[4]{c_4}, \sqrt{a^2 + b_i^2}\} \quad \text{for } i = 1, 2, \tag{3.8}$$

$$K_t > 0 \text{ for } t > \alpha \text{ and } K_t < 0 \text{ for } t < -\alpha, \tag{3.9}$$

$$\frac{\partial_t K}{K} \leq \frac{4}{t + \alpha} \quad \text{for any } |t| \leq \alpha, \tag{3.10}$$

$$|\alpha'|\sqrt{K} \leq \sqrt[4]{K}\big(|c_2'| + |(\sqrt{c_4})'|\big) \quad \text{for any } |t| \leq \alpha. \tag{3.11}$$

Note that the expression in the parenthesis in the right hand side of (3.11) is bounded since $c_4 \geq 0$. Therefore, $|\alpha'|\sqrt{K}\big|_{|t|\leq\alpha}$ is small if K is small.

Proof. (1) First by $H(\alpha) = 0$, we have

$$0 \leq K(\alpha) + K(-\alpha) = 2(\alpha^4 + c_2\alpha^2 + c_4) = -4\alpha^4 + 4c_4.$$

This implies $|\alpha| \leq \sqrt[4]{c_4}$. In fact, it can be proved directly with the explicit expression of α in (3.6). We simply note by a straightforward calculation that

$$\sqrt{\frac{-c_2 + \sqrt{c_2^2 + 12c_4}}{6}} \leq \sqrt[4]{c_4} \;\Leftrightarrow\; 0 \leq c_2 + 2\sqrt{c_4}.$$

Next, by (3.2) and (3.3), we have

$$a^4 + (b_1^2 + b_2^2)a^2 + b_1^2 b_2^2 = c_4,$$

or

$$a^4 + (c_2 + 2a^2)a^2 + b_1^2 b_2^2 = c_4,$$

or

$$3a^4 + c_2 a^2 + b_1^2 b_2^2 = c_4.$$

Hence, we get

$$3a^4 + c_2 a^2 \leq c_4.$$

Note, by $H(\alpha) = 0$,

$$3\alpha^4 + c_2\alpha^2 = c_4.$$

By taking a difference, we obtain

$$(a^2 - \alpha^2)(3a^2 + 3\alpha^2 + c_2) \leq 0.$$

With $c_4 = \alpha^2(3\alpha^2 + c_2) \geq 0$ or $3a^2 + c_2 = a^2 + b_1^2 + b_2^2 \geq 0$, we obtain $a^2 - \alpha^2 \leq 0$.

To prove another upper bound of α, we simply note for $i = 1, 2$

$$\begin{aligned}
\alpha^2 \leq a^2 + b_i^2 \Leftrightarrow \frac{-c_2 + \sqrt{c_2^2 + 12c_4}}{6} &\leq a^2 + b_i^2 \\
\Leftrightarrow \sqrt{c_2^2 + 12c_4} &\leq c_2 + 6(a^2 + b_i^2) \\
\Leftrightarrow c_4 &\leq c_2(a^2 + b_i^2) + 3(a^2 + b_i^2)^2 \\
\Leftrightarrow a^2 + b_2^2 &\leq c_2 + 3(a^2 + b_i^2) \\
\Leftrightarrow 0 &\leq 4b_i^2,
\end{aligned}$$

where we used expressions of c_2 and c_4 in (3.3).

(2) By (3.2), we have

$$\frac{\partial_t K}{K} = \frac{2(t-a)}{(t-a)^2+b_1^2} + \frac{2(t+a)}{(t+a)^2+b_2^2}.$$

Then (3.9) follows easily. By (3.8), we have $\alpha^2 \le a^2 + b_i^2$ for $i = 1, 2$. Then an elementary calculation shows

$$\begin{aligned} \frac{t-a}{(t-a)^2+b_1^2} &\le \frac{1}{t+\alpha} \quad \text{for any } t \in (a,\alpha), \\ \frac{t+a}{(t+a)^2+b_2^2} &\le \frac{1}{t+\alpha} \quad \text{for any } t \in (-a,\alpha). \end{aligned}$$

This implies (3.10).

In the above proof, we showed that each term in $\partial_t K/K$ satisfies the required estimate. In the following, we provide another proof to control $\partial_t K/K$ itself. We consider

$$\frac{\partial_t K}{K} - \frac{4}{t+\alpha} = \frac{(t+\alpha)\partial_t K - 4K}{(t+\alpha)K}.$$

Set

$$g(t) = (t+\alpha)\partial_t K - 4K.$$

We need to prove $g(t) \le 0$ for $|t| \le \alpha$. Note

$$\begin{aligned} g(-\alpha) &= -4K(-\alpha) < 0, \\ g(\alpha) &= 2\alpha\partial_t K(\alpha) - 4K(\alpha) = 2H(\alpha) - 2K(\alpha) = -2K(\alpha) < 0. \end{aligned}$$

To proceed, we write

$$g(t) = 4\alpha t^3 - 2c_2 t^2 + 2c_2\alpha t - 3c_3 t + c_3\alpha - 4c_4.$$

By setting $t = s\alpha$ and $h(s) = g(s\alpha)$, with $|s| \le 1$, we have

$$h(s) = 4s^3\alpha^4 + (2s - 2s^2)c_2\alpha^2 + (1-3s)c_3\alpha - 4c_4.$$

With $K(\alpha) = \alpha^4 + c_2\alpha^2 + c_3\alpha + c_4$, we get

$$h(s) = (4s^3 + 3s - 1)\alpha^4 + (-2s^2 + 5s - 1)c_2\alpha^2 + (3s-5)c_4 + (1-3s)K(\alpha).$$

By $H(\alpha) = 3\alpha^4 + c_2\alpha^2 - c_4 = 0$, we have $c_2\alpha^2 = -3\alpha^4 + c_4$ and hence

$$\begin{aligned} h(s) &= (4s^3 + 6s^2 - 12s + 2)\alpha^4 + (-2s^2 + 8s - 6)c_4 + (1-3s)K(\alpha) \\ &= 2(s-1)\big[(2s^2 + 5s - 1)\alpha^4 - 2(s-3)c_4\big] + (1-3s)K(\alpha). \end{aligned}$$

We consider $1/3 < s < 1$ first. Note that $2s^2 + 5s - 1 > 0$ for $1/3 < s < 1$. Hence, we obtain $h(s) \le 0$ for $1/3 < s < 1$. Next, we consider $-1 < s < 1/3$. Note that

$$K(\alpha) + K(-\alpha) = -4\alpha^4 + 4c_4.$$

We therefore obtain

$$\begin{aligned} h(s) &= (4s^3 + 6s^2 - 2)\alpha^4 + (-2s^2 - 4s - 2)c_4 - (1-3s)K(-\alpha) \\ &= 2(s+1)^2\big[(2s-1)\alpha^4 - 2c_4\big] - (1-3s)K(-\alpha). \end{aligned}$$

Hence $h(s) \le 0$ for $-1 < s < 1/3$. This finishes the proof.

(3) We first claim for $|t| \le \alpha$

$$K \le 4c_4. \tag{3.12}$$

By replacing t by $\text{sgn}(c_3)|t|$ in $K \ge 0$, we get

$$|c_3 t| \le t^4 + c_2 t^2 + c_4.$$

Hence,

$$K \le t^4 + c_2 t^2 + |c_3 t| + c_4 \le 2(t^4 + c_2 t^2 + c_4).$$

If $c_2 > 0$, we have for $|t| \le \alpha$

$$K \le 2(\alpha^4 + c_2\alpha^2 + c_4) = 4(c_4 - \alpha^4) \le 4c_4.$$

If $c_2 \le 0$, we have for $|t| \le \alpha$

$$K \le 2(\alpha^4 + c_4) \le 4c_4,$$

where we used $\alpha \le \sqrt[4]{c_4}$. This proves (3.12). By (3.7), we have for $|t| \le \alpha$

$$|\alpha'| \le \frac{1}{\sqrt[4]{c_2^2 + 12c_4}}\big(|c_2'| + |(\sqrt{c_4})'|\big),$$

and hence with (3.12)

$$|\alpha'|\sqrt[4]{K} \le \frac{\sqrt[4]{4c_4}}{\sqrt[4]{c_2^2 + 12c_4}}\big(|c_2'| + |(\sqrt{c_4})'|\big) \le |c_2'| + |(\sqrt{c_4})'|.$$

This prove (3.11). □

Note that Lemma 3.1 follows easily from Lemma 3.3. We should point out that there may be other ways to extend real zeros of K. The point is whether we can construct functions other than H in (3.5) so that their zeros satisfy Lemma 3.1.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is based on a standard regularization process. Instead of (1.1), we consider a modified equation

$$u_{tt} - q(K + \varepsilon)u_{xx} - b_0 u_t - b u_x - cu = f.$$

This is a (strictly) hyperbolic differential equation for $\varepsilon > 0$. Hence there exists a smooth solution u_ε with the given initial values. When we apply Lemma 3.1 to $K + \varepsilon$, the resulting φ_1 and φ_2 have $C^{1/4}$-norms and BV-norms independent of ε although φ_1 and φ_2 themselves depend on ε. Hence, with Theorem 2.5, we get energy estimates for u_ε uniform in ε. With $\varepsilon \to 0+$, we obtain the desired result. □

To end this section, we discuss briefly methods used in this paper. As pointed out in the introduction, the key point in deriving energy estimates is to decompose the plane appropriately. The decomposition should be related to zero sets of leading degenerate coefficients so that the degeneracy can be overcome by appropriately constructed weight functions. The decomposition of the plane and the construction of weight functions are intrinsically related.

In this paper, we only discussed a special case when the leading degenerate coefficients are polynomials in time variable of degree 4. Although zeros of degree 4 polynomials can be expressed explicitly, our decomposition is not given by this expression. We need to drop complex zeros and substitute by appropriate functions.

Let $K(x,t)$ be a nonnegative function given by

$$K(x,t) = t^4 + c_1(x)t^3 + c_2(x)t^2 + c_3(x)t + c_4(x).$$

With the help of (3.6), we note that an appropriate decomposition is given by

$$\begin{aligned}\varphi_1(x) &= -\frac{1}{4}c_1 + \frac{1}{\sqrt{6}}\big((c_2^2 - 3c_1c_3 + 12c_4)^{\frac{1}{2}} - c_2 + \frac{3}{8}c_1^2\big)^{\frac{1}{2}},\\ \varphi_2(x) &= -\frac{1}{4}c_1 - \frac{1}{\sqrt{6}}\big((c_2^2 - 3c_1c_3 + 12c_4)^{\frac{1}{2}} - c_2 + \frac{3}{8}c_1^2\big)^{\frac{1}{2}}.\end{aligned} \tag{3.13}$$

This is by no means a trivial decomposition.

It is expected that the main result in this paper should hold for arbitrary polynomials in t. However, it is unlikely that appropriate decompositions can be expressed explicitly in terms of coefficients. In fact, when complex zeros are absent, no explicit expressions are used in [7]. In fact, algebraic properties of zeros in terms of coefficients were employed.

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