

ELLIPTIC DIFFERENTIAL EQUATIONS OF DIVERGENCE FORM

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In this note, we discuss the regularity of weak solutions of elliptic equations of divergence form. Let Ω be a domain in \mathbb{R}^n . The function $u \in H^1(\Omega)$ is a *weak solution* if it satisfies

$$\int_{\Omega} a_{ij} D_i u D_j \varphi = \int_{\Omega} f \varphi \quad \text{for any } \varphi \in H_0^1(\Omega),$$

where we assume

(i) the leading coefficients $a_{ij} \in L^\infty(\Omega)$ are *uniformly elliptic*, i.e., for some positive constant λ there holds

$$a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n;$$

(ii) the nonhomogeneous term $f \in L^{\frac{2n}{n+2}}(\Omega)$. (By the Sobolev embedding, (ii) is the least assumption on f to have a meaningful equation.)

We will prove various interior regularity results concerning solutions u . Basically, there are two classes of regularity results, perturbation results and non-perturbation results. The first is based on regularity assumptions on the leading coefficients a_{ij} , which are assumed to be at least continuous. Under such assumptions, we compare solutions of the underlying equations with harmonic functions, or solutions of homogeneous equations of constant coefficients. Then, regularities of solutions depend on how close solutions are to harmonic functions or how close the leading coefficients a_{ij} are to constant coefficients. In this direction, we have Schauder estimates and $W^{2,p}$ -estimates. In this note, we only discuss the Schauder estimates. For the second class of regularity, there is no continuity assumption on the leading coefficients a_{ij} . Hence the result is not based on perturbations. The iteration methods introduced by DeGiorgi and Moser are successful in dealing with the non-perturbation situation. Results proved by them are fundamental.

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1. GROWTH OF LOCAL INTEGRALS

Let $B_R(x_0)$ be the ball in \mathbb{R}^n of radius R and center x_0 . The well-known Sobolev theorem asserts that, if $u \in W^{1,p}(B_R(x_0))$ with $p > n$, then u is Hölder continuous with exponent $\alpha = 1 - n/p$.

In the first part of this section, we prove a general result, due to S. Campanato, which characterizes Hölder continuous functions by the growth of their local integrals. This result is useful for studying regularities of solutions of elliptic differential equations. In the second part of this section, we prove a result, due to John and Nirenberg, which gives an equivalent definition of functions of bounded mean oscillations.

Let Ω be a bounded connected open set in \mathbb{R}^n and let $u \in L^1(\Omega)$. For any ball $B_r(x_0) \subset \Omega$, define

$$u_{x_0,r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u.$$

We first prove the Campanato's characterization of Hölder continuity.

Theorem 1.1. *Suppose $u \in L^2(\Omega)$ satisfies for some $\alpha \in (0, 1)$*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \leq M^2 r^{n+2\alpha} \quad \text{for any } B_r(x) \subset \Omega.$$

Then $u \in C^\alpha(\Omega)$, and for any $\Omega' \subset\subset \Omega$

$$\|u\|_{C^\alpha(\Omega')} \leq C(M + \|u\|_{L^2(\Omega)}),$$

where C is a positive constant depending only on n, α, Ω and Ω' .

Proof. Set $R_0 = \text{dist}(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$ and $0 < r_1 < r_2 \leq R_0$, we have

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq 2(|u(x) - u_{x_0,r_1}|^2 + |u(x) - u_{x_0,r_2}|^2).$$

By integrating with respect to x in $B_{r_1}(x_0)$, we obtain

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq \frac{2}{\omega_n r_1^n} \left(\int_{B_{r_1}(x_0)} |u - u_{x_0,r_1}|^2 + \int_{B_{r_2}(x_0)} |u - u_{x_0,r_2}|^2 \right),$$

and hence

$$(1) \quad |u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq C(n) M^2 r_1^{-n} (r_1^{n+2\alpha} + r_2^{n+2\alpha}).$$

For any $R \leq R_0$, with $r_1 = R/2^{i+1}$ and $r_2 = R/2^i$, we get

$$|u_{x_0,2^{-(i+1)}R} - u_{x_0,2^{-i}R}| \leq C(n) 2^{-(i+1)\alpha} M R^\alpha,$$

and therefore for any $h < k$

$$|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}| \leq \frac{C(n)}{2^{(h+1)\alpha}} M R^\alpha \sum_{i=0}^{k-h-1} \frac{1}{2^{i\alpha}} \leq \frac{C(n, \alpha)}{2^{h\alpha}} M R^\alpha.$$

This yields that $\{u_{x_0,2^{-i}R}\} \subset \mathbb{R}$ is a Cauchy sequence, hence a convergent one. Its limit $\hat{u}(x_0)$ is independent of the choice of R , since (1) can be applied with $r_1 = 2^{-i}R$ and $r_2 = 2^{-i}\bar{R}$ whenever $0 < R < \bar{R} \leq R_0$. Thus we get

$$\hat{u}(x_0) = \lim_{r \rightarrow 0} u_{x_0,r},$$

with

$$(2) \quad |u_{x_0,r} - \hat{u}(x_0)| \leq C(n, \alpha)Mr^\alpha \quad \text{for any } 0 < r \leq R_0.$$

Note that $\{u_{x,r}\}$ converges to u in $L^1(\Omega)$ as $r \rightarrow 0+$ by the Lebesgue theorem. Then $u = \hat{u}$ a.e., and (2) implies that $\{u_{x,r}\}$ converges uniformly to $u(x)$ in Ω' . Since $x \mapsto u_{x,r}$ is continuous for any $r > 0$, $u(x)$ is continuous. By (2) we get

$$|u(x)| \leq CMR^\alpha + |u_{x,R}|,$$

for any $x \in \Omega'$ and $R \leq R_0$. Hence u is bounded in Ω' with

$$\sup_{\Omega'} |u| \leq C(MR_0^\alpha + \|u\|_{L^2(\Omega)}).$$

Now we prove that u is Hölder continuous. Let $x, y \in \Omega'$ with $R = |x - y| < R_0/2$. Then

$$|u(x) - u(y)| \leq |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms on the right hand side are already estimated in (2). For the final term, we write

$$|u_{x,2R} - u_{y,2R}| \leq |u_{x,2R} - u(\zeta)| + |u_{y,2R} - u(\zeta)|.$$

Integrating with respect to ζ over $B_{2R}(x) \cap B_{2R}(y)$, which contains $B_R(x)$, yields

$$\begin{aligned} |u_{x,2R} - u_{y,2R}|^2 &\leq \frac{2}{|B_R(x)|} \left(\int_{B_{2R}(x)} |u - u_{x,2R}|^2 + \int_{B_{2R}(y)} |u - u_{y,2R}|^2 \right) \\ &\leq C(n, \alpha)M^2R^{2\alpha}. \end{aligned}$$

Therefore, we have

$$|u(x) - u(y)| \leq C(n, \alpha)M|x - y|^\alpha.$$

For $|x - y| > R_0/2$, we obtain

$$|u(x) - u(y)| \leq 2 \sup_{\Omega'} |u| \leq C \left(M + \frac{1}{R_0^\alpha} \|u\|_{L^2} \right) |x - y|^\alpha.$$

This yields the desired result. \square

The Sobolev theorem is an easy consequence of Theorem 1.1. In fact, we have the following result due to Morrey.

Corollary 1.2. *Suppose $u \in H_{loc}^1(\Omega)$ satisfies for some $\alpha \in (0, 1)$*

$$\int_{B_r(x)} |Du|^2 \leq M^2 r^{n-2+2\alpha} \quad \text{for any } B_r(x) \subset \Omega.$$

Then $u \in C^\alpha(\Omega)$, and for any $\Omega' \subset\subset \Omega$

$$|u|_{C^\alpha(\Omega')} \leq C(M + \|u\|_{L^2(\Omega)}),$$

where C is a positive constant depending only on n, α, Ω and Ω' .

Proof. By the Poincaré inequality, we obtain

$$\int_{B_r(x)} |u - u_{x,r}|^2 \leq C(n)r^2 \int_{B_r(x)} |Du|^2 \leq C(n)M^2 r^{n+2\alpha}.$$

With Theorem 1.1, we have the desired result. \square

In the rest of this section, we discuss functions of bounded mean oscillations (BMO). The following result is due to John and Nirenberg and referred to as the John-Nirenberg lemma.

Theorem 1.3. *Suppose $u \in L^1(\Omega)$ satisfies*

$$\int_{B_r(x)} |u - u_{x,r}| \leq Mr^n \quad \text{for any } B_r(x) \subset \Omega.$$

Then for any $B_r(x) \subset \Omega$

$$\int_{B_r(x)} e^{\frac{p_0}{M}|u - u_{x,r}|} \leq Cr^n,$$

where p_0 and C are positive constants depending only on n .

Remark 1.4. Functions satisfying the condition of Theorem 1.3 are called functions of bounded mean oscillations (BMO). Obviously, bounded functions are of bounded mean oscillations. In fact, L^∞ is a proper subset of BMO. The function $u(x) = \log(x)$ in $(0, 1) \subset \mathbb{R}$ is in BMO but not in L^∞ .

For convenience, we use cubes instead of balls in the proof of Theorem 1.3. We first introduce the Calderon-Zygmund decomposition.

Take the unit cube Q_0 . Cut it equally into 2^n cubes, which we take as the first generation. Do the same cutting for these small cubes to get the second generation. Continue this process. These cubes (from all generations) are called *dyadic cubes*. Any $(k+1)$ -generation cube Q arises from some k -generation cube \tilde{Q} , which is called the *predecessor* of Q .

Lemma 1.5. *Suppose $f \in L^1(Q_0)$ is nonnegative and $\alpha > |Q_0|^{-1} \int_{Q_0} f$ is a fixed constant. Then there exists a sequence of (nonoverlapping) dyadic cubes $\{Q_j\}$ in Q_0 such that*

$$f(x) \leq \alpha \quad \text{a.e. in } Q_0 \setminus \cup_j Q_j,$$

and

$$\alpha \leq \frac{1}{|Q_j|} \int_{Q_j} f dx < 2^n \alpha.$$

Proof. Cut Q_0 into 2^n dyadic cubes and keep the cube Q if $|Q|^{-1} \int_Q f \geq \alpha$. Continue cutting for others, and always keep the cube Q if $|Q|^{-1} \int_Q f \geq \alpha$ and cut the rest. Let $\{Q_j\}$ be the cubes we have kept during this infinite process. We only need to verify

$$f(x) \leq \alpha \quad \text{a.e. in } Q_0 \setminus \cup_j Q_j.$$

Let $F = Q_0 \setminus \cup_j Q_j$. For any $x \in F$, from the way we collect $\{Q_j\}$, there exists a sequence of cubes Q^i containing x such that

$$\frac{1}{|Q^i|} \int_{Q^i} f < \alpha,$$

and

$$\text{diam}(Q^i) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

The Lebesgue density theorem implies

$$f \leq \alpha \quad \text{a.e. in } F.$$

This yields the desired result. \square

Proof of Theorem 1.3. Assume $\Omega = Q_0$. We may rewrite the assumption in terms of cubes as follows

$$\int_Q |u - u_Q| < M|Q| \quad \text{for any } Q \subset Q_0.$$

We prove that there exist two positive constants c_1 and c_2 , depending only on n , such that for any $Q \subset Q_0$

$$|\{x \in Q; |u - u_Q| > t\}| \leq c_1|Q|\exp\left(-\frac{c_2}{M}t\right).$$

Then Theorem 1.3 follows easily.

Without loss of generality, we assume $M = 1$. Choose

$$\alpha > 1 \geq |Q_0|^{-1} \int_{Q_0} |u - u_{Q_0}| dx.$$

Apply the Calderon-Zygmund decomposition to $f = |u - u_{Q_0}|$. There exists a sequence of (nonoverlapping) cubes $\{Q_j^{(1)}\}_{j=1}^{\infty}$ such that

$$\alpha \leq \frac{1}{|Q_j^{(1)}|} \int_{Q_j^{(1)}} |u - u_{Q_0}| < 2^n \alpha,$$

and

$$|u(x) - u_{Q_0}| \leq \alpha \quad \text{for a.e. } x \in Q_0 \setminus \cup_{j=1}^{\infty} Q_j^{(1)}.$$

This implies

$$\sum_j |Q_j^{(1)}| \leq \frac{1}{\alpha} \int_{Q_0} |u - u_{Q_0}| \leq \frac{1}{\alpha} |Q_0|,$$

and

$$|u_{Q_j^{(1)}} - u_{Q_0}| \leq \frac{1}{|Q_j^{(1)}|} \int_{Q_j^{(1)}} |u - u_{Q_0}| dx \leq 2^n \alpha.$$

The assumption on u implies for each j

$$\frac{1}{|Q_j^{(1)}|} \int_{Q_j^{(1)}} |u - u_{Q_j^{(1)}}| dx \leq 1 < \alpha.$$

Apply the decomposition procedure above to $f = |u - u_{Q_j^{(1)}}|$ in $Q_j^{(1)}$. There exists a sequence of (nonoverlapping) cubes $\{Q_j^{(2)}\}$ in $\cup_j Q_j^{(1)}$ such that

$$\sum_{j=1}^{\infty} |Q_j^{(2)}| \leq \frac{1}{\alpha} \sum_j \int_{Q_j^{(1)}} |u - u_{Q_j^{(1)}}| \leq \frac{1}{\alpha} \sum_j |Q_j^{(1)}| \leq \frac{1}{\alpha^2} |Q_0|,$$

and

$$|u(x) - u_{Q_j^{(1)}}| \leq \alpha \quad \text{for a.e. } x \in Q_j^{(1)} \setminus \cup_j Q_j^{(2)},$$

which implies

$$|u(x) - u_{Q_0}| \leq 2 \cdot 2^n \alpha \quad \text{for a.e. } x \in Q_0 \setminus \cup_j Q_j^{(2)}.$$

Continue this process. For any integer $k \geq 1$, there exists a sequence of disjoint cubes $\{Q_j^{(k)}\}$ such that

$$\sum_j |Q_j^{(k)}| \leq \frac{1}{\alpha^k} |Q_0|,$$

and

$$|u(x) - u_{Q_0}| \leq k 2^n \alpha \quad \text{for a.e. } x \in Q_0 \setminus \cup_j Q_j^{(k)}.$$

Thus, we obtain

$$|\{x \in Q_0; |u - u_{Q_0}| > 2^n k \alpha\}| \leq \sum_{j=1}^{\infty} |Q_j^{(k)}| \leq \frac{1}{\alpha^k} |Q_0|.$$

For any t , there exists an integer k such that $t \in [2^n k \alpha, 2^n (k+1) \alpha)$. Then

$$\alpha^{-k} = \alpha \alpha^{-(k+1)} = \alpha e^{-(k+1) \log \alpha} \leq \alpha \exp\left(-\frac{\log \alpha}{2^n \alpha} t\right).$$

This implies the desired result. \square

2. SCHAUDER THEORY

In this section, we discuss the Schauder theory for elliptic differential equations of divergence form. The basic idea is to freeze the leading coefficients and then to compare solutions with harmonic functions. The regularity of solutions depends on how close solutions are to harmonic functions. To this end, we need some regularity assumption on the leading coefficients.

We first prove a result for harmonic functions, or solutions of homogeneous equations of constant coefficient.

Lemma 2.1. *Suppose $\{a_{ij}\}$ is a constant positive definite matrix with*

$$\lambda |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n,$$

for some constants $0 < \lambda \leq \Lambda$. Suppose $w \in H^1(B_r)$ satisfies

$$\int_{B_r} a_{ij} D_i w D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1(B_r).$$

Then for any $0 < \rho \leq r$

$$\int_{B_\rho} w^2 \leq C \left(\frac{\rho}{r}\right)^n \int_{B_r} w^2,$$

and

$$\int_{B_\rho} |w - w_\rho|^2 \leq C \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |w - w_r|^2,$$

where C is a positive constant depending only on n and Λ/λ .

Proof. By a simple dilation, we need only consider $r = 1$. We prove Lemma 2.1 only for $\rho \in (0, 1/2]$, since it is trivial for $\rho \in (1/2, 1]$.

First, we claim

$$|w|_{L^\infty(B_{\frac{1}{2}})}^2 + |Dw|_{L^\infty(B_{\frac{1}{2}})}^2 \leq c \int_{B_1} w^2,$$

where C is a positive constant depending only on n and Λ/λ . This implies for $\rho \in (0, 1/2]$

$$\int_{B_\rho} w^2 \leq \rho^n |w|_{L^\infty(B_{\frac{1}{2}})}^2 \leq C\rho^n \int_{B_1} w^2,$$

and

$$\int_{B_\rho} |w - w_\rho|^2 \leq \int_{B_\rho} |w - w(0)|^2 \leq \rho^{n+2} |Dw|_{L^\infty(B_{\frac{1}{2}})}^2 \leq C\rho^{n+2} \int_{B_1} w^2.$$

If w is a solution, so is $w - w_1$. With w replaced by $w - w_1$ in the above inequality, we have

$$\int_{B_\rho} |w - w_\rho|^2 \leq c\rho^{n+2} \int_{B_1} |w - w_1|^2.$$

We now prove the claim. By a rotation, we may assume (a_{ij}) is a diagonal matrix. Hence

$$\sum_{i=1}^n \lambda_i D_{ii} w = 0,$$

with $0 < \lambda \leq \lambda_i \leq \Lambda$ for $i = 1, \dots, n$. Then there exists a constant $r_0 \in (0, 1/2)$, depending only on λ and Λ , such that for any $x_0 \in B_{\frac{1}{2}}$ the rectangle

$$\left\{ x; \frac{|x_i - x_{0i}|}{\sqrt{\lambda_i}} < r_0 \right\}$$

is contained in B_1 . Consider the change of coordinates

$$x_i \mapsto y_i = \frac{x_i}{\sqrt{\lambda_i}},$$

and set

$$v(y) = w(x).$$

Then v is harmonic in $\{y; \sum_{i=1}^n \lambda_i y_i^2 < 1\}$. In the ball $\{y; |y - y_0| < r_0\}$, we use the interior estimates for harmonic functions to yield

$$|v(y_0)|^2 + |Dv(y_0)|^2 \leq C \int_{B_{r_0}(y_0)} v^2 \leq C \int_{\{\sum_{i=1}^n \lambda_i y_i^2 < 1\}} v^2,$$

where C is a positive constant depending only on n and Λ/λ . We transform back to w to get the desired estimate. \square

Note that derivatives of solutions of homogeneous equations of constant coefficients are also solutions of the same equations. So we can apply Lemma 2.1 to gradients of solutions.

Corollary 2.2. *Suppose w is as in Lemma 2.1. Then, for any $0 < \rho \leq r$*

$$\int_{B_\rho} |Dw|^2 \leq C \left(\frac{\rho}{r}\right)^n \int_{B_r} |Dw|^2,$$

and

$$\int_{B_\rho} |Dw - (Dw)_\rho|^2 \leq C \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |Dw - (Dw)_r|^2,$$

where C is a positive constant depending only on n and Λ/λ .

Now we compare any functions with harmonic functions.

Corollary 2.3. *Suppose w is as in Lemma 2.1 and u is an arbitrary H^1 -function in B_r . Then, for any $0 < \rho \leq r$*

$$\int_{B_\rho} |Du|^2 \leq C \left\{ \left(\frac{\rho}{r}\right)^n \int_{B_r} |Du|^2 + \int_{B_r} |D(u-w)|^2 \right\},$$

and

$$\begin{aligned} \int_{B_\rho(x_0)} |Du - (Du)_\rho|^2 &\leq C \left\{ \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |Du - (Du)_r|^2 \right. \\ &\quad \left. + \int_{B_r} |D(u-w)|^2 \right\}, \end{aligned}$$

where C is a positive constant depending only on n and Λ/λ .

Proof. With $v = u - w$, we have for any $0 < \rho \leq r$

$$\begin{aligned} \int_{B_\rho} |Du|^2 &\leq 2 \int_{B_\rho} |Dw|^2 + 2 \int_{B_\rho} |Dv|^2 \\ &\leq C \left(\frac{\rho}{r}\right)^n \int_{B_r} |Dw|^2 + 2 \int_{B_r} |Dv|^2 \\ &\leq C \left(\frac{\rho}{r}\right)^n \int_{B_r} |Du|^2 + C \left[1 + \left(\frac{\rho}{r}\right)^n\right] \int_{B_r} |Dv|^2, \end{aligned}$$

and

$$\begin{aligned} &\int_{B_\rho} |Du - (Du)_{x_0, \rho}|^2 \\ &\leq 2 \int_{B_\rho} |Du - (Dw)_\rho|^2 + 2 \int_{B_\rho} |Dv|^2 \\ &\leq 4 \int_{B_\rho} |Dw - (Dw)_\rho|^2 + 6 \int_{B_\rho} |Dv|^2 \\ &\leq C \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |Dw - (Dw)_r|^2 + 6 \int_{B_r} |Dv|^2 \\ &\leq C \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |Du - (Du)_r|^2 + C \left[1 + \left(\frac{\rho}{r}\right)^{n+2}\right] \int_{B_r} |Dv|^2. \end{aligned}$$

This finishes the proof. \square

The regularity of u depends on how close u is to w , the solution of the constant coefficient equation.

Now we discuss the Hölder continuity of solutions and their derivatives of elliptic differential equations of divergence form. Suppose $a_{ij} \in L^\infty(B_1)$ is uniformly elliptic in B_1 , i.e.,

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } x \in B_1, \xi \in \mathbb{R}^n.$$

In the following, we assume that a_{ij} is at least continuous and that $u \in H^1(B_1)$ satisfies

$$(*) \quad \int_{B_1} a_{ij} D_j u D_i \varphi = \int_{B_1} f \varphi,$$

for any $\varphi \in H_0^1(B_1)$.

We first prove Hölder estimates for solutions.

Theorem 2.4. *Let $u \in H^1(B_1)$ satisfy (*). Assume $a_{ij} \in C^0(\bar{B}_1)$ and $f \in L^p(B_1)$ for some $p \in (n/2, n)$. Then $u \in C^\alpha(B_1)$ with $\alpha = 2 - n/p \in (0, 1)$. Moreover,*

$$\|u\|_{C^\alpha(B_{\frac{1}{2}})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{H^1(B_1)}),$$

where C is a positive constant depending only on λ, Λ and the modulus of the continuity of a_{ij} .

The idea of the proof is to compare the solution u with harmonic functions and use a perturbation argument. An important technique is to *freeze* leading coefficients.

Proof. Set

$$\tau(r) = \sup\{|a_{ij}(x) - a_{ij}(y)|; x, y \in B_1, |x - y| \leq r\}.$$

We will prove that there exists an $R_0 \in (0, 1/2)$ such that, for any $x \in B_{1/2}$ and $r \leq R_0$,

$$\int_{B_r(x)} |Du|^2 \leq C r^{n-2+2\alpha} (\|f\|_{L^p(B_1)}^2 + \|u\|_{H^1(B_1)}^2),$$

where R_0 and C are positive constants depending only on λ, Λ and τ . To this end, we decompose u into a sum $v + w$ where w satisfies a homogeneous equation of constant coefficients and v has estimates in terms of nonhomogeneous terms.

For any $B_r(x_0) \subset B_1$, we write the equation in the form

$$\int_{B_1} a_{ij}(x_0) D_i u D_j \varphi = \int_{B_1} -(a_{ij} - a_{ij}(x_0)) D_j u D_j \varphi + \int_{B_1} f \varphi.$$

In $B_r(x_0)$, there exists a unique $w \in H^1(B_r(x_0))$ such that

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i w D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1(B_r(x_0)),$$

with $w - u \in H_0^1(B_r(x_0))$. Then, $v = u - w \in H_0^1(B_r(x_0))$ satisfies

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = \int_{B_r(x_0)} -(a_{ij} - a_{ij}(x_0)) D_j u D_j \varphi + \int_{B_r(x_0)} f \varphi,$$

for any $\varphi \in H_0^1(B_r(x_0))$. By taking the test function $\varphi = v$, we obtain with the uniform ellipticity

$$\lambda \int_{B_r(x_0)} |Dv|^2 \leq \int_{B_r(x_0)} (|(a_{ij}(x_0) - a_{ij}(x))D_i u D_j v| + \int_{B_r(x_0)} |fv|$$

For the first term involving Dv in the right hand side, we apply the Cauchy inequality. For the second term involving v , we apply the Hölder inequality and the Sobolev inequality of the form

$$\left(\int_{B_r(x_0)} v^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq C(n) \left(\int_{B_r(x_0)} |Dv|^2 \right)^{\frac{1}{2}},$$

for $v \in H_0^1(B_r(x_0))$. Therefore, we obtain

$$\int_{B_r(x_0)} |Dv|^2 \leq C \left\{ \tau^2(r) \int_{B_r(x_0)} |Du|^2 + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \right\}.$$

By the Hölder inequality, we have

$$\left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \leq \left(\int_{B_r(x_0)} |f|^p \right)^{\frac{2}{p}} r^{n-2+2\alpha},$$

where p and α are as given. Then

$$\int_{B_r(x_0)} |Dv|^2 \leq C \left\{ \tau^2(r) \int_{B_r(x_0)} |Du|^2 + r^{n-2+2\alpha} \|f\|_{L^p(B_1)}^2 \right\},$$

where C is a positive constant depending only on λ and Λ . Hence, Corollary 2.3 implies for any $B_r(x_0) \subset B_1$ and any $0 < \rho \leq r$

$$(1) \quad \int_{B_\rho(x_0)} |Du|^2 \leq C \left\{ \left[\left(\frac{\rho}{r} \right)^n + \tau^2(r) \right] \int_{B_r(x_0)} |Du|^2 + r^{n-2+2\alpha} \|f\|_{L^p(B_1)}^2 \right\}.$$

Now the result would follow if we could write $\rho^{n-2+2\alpha}$ instead of $r^{n-2+2\alpha}$ in the final term in the right hand side of (1). This is in fact true and is stated in Lemma 2.5. By Lemma 2.5, there exists an $R_0 > 0$ such that, for any $x_0 \in B_{1/2}$ and any $0 < \rho < r \leq R_0$,

$$\int_{B_\rho(x_0)} |Du|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n-2+2\alpha} \int_{B_r(x_0)} |Du|^2 + \rho^{n-2+2\alpha} \|f\|_{L^p(B_1)}^2 \right\}.$$

In particular, with $r = R_0$, we obtain for any $\rho < R_0$

$$\int_{B_\rho(x_0)} |Du|^2 \leq C \rho^{n-2+2\alpha} \left\{ \int_{B_1} |Du|^2 + \|f\|_{L^p(B_1)}^2 \right\}.$$

This is the desired estimate. \square

Lemma 2.5. *Let $\phi(t)$ be a nonnegative and nondecreasing function on $[0, R]$. Suppose for some nonnegative constants A, B, α, β with $\beta < \alpha$*

$$\phi(\rho) \leq A \left\{ \left(\frac{\rho}{r} \right)^\alpha + \varepsilon \right\} \phi(r) + Br^\beta \quad \text{for any } 0 < \rho \leq r \leq R.$$

Then for any $\gamma \in (\beta, \alpha)$, there exists a constant ε_0 such that, if $\varepsilon < \varepsilon_0$,

$$\phi(\rho) \leq C \left\{ \left(\frac{\rho}{r} \right)^\gamma \phi(r) + B\rho^\beta \right\} \quad \text{for any } 0 < \rho \leq r \leq R,$$

where ε_0 and C are positive constants depending only on A, α, β and γ . In particular,

$$\phi(r) \leq C \left\{ \frac{\phi(R)}{R^\gamma} r^\gamma + Br^\beta \right\} \quad \text{for any } 0 < r \leq R.$$

Proof. For any $\tau \in (0, 1)$ and $r < R$, we have

$$\phi(\tau r) \leq A\tau^\alpha(1 + \varepsilon\tau^{-\alpha})\phi(r) + Br^\beta.$$

Choose $\tau < 1$ such that $2A\tau^\alpha = \tau^\gamma$ and assume $\varepsilon_0\tau^{-\alpha} < 1$. Then we get for every $r < R$

$$\phi(\tau r) \leq \tau^\gamma \phi(r) + Br^\beta,$$

and therefore for any integer $k > 0$

$$\begin{aligned} \phi(\tau^{k+1}r) &\leq \tau^\gamma \phi(\tau^k r) + B\tau^{k\beta} r^\beta \\ &\leq \tau^{(k+1)\gamma} \phi(r) + B\tau^{k\beta} r^\beta \sum_{j=0}^k \tau^{j(\gamma-\beta)} \\ &\leq \tau^{(k+1)\gamma} \phi(r) + \frac{B\tau^{k\beta} r^\beta}{1 - \tau^{\gamma-\beta}}. \end{aligned}$$

Choosing k such that $\tau^{k+2}r < \rho \leq \tau^{k+1}r$, we obtain

$$\phi(\rho) \leq \frac{1}{\tau^\gamma} \left(\frac{\rho}{r} \right)^\gamma \phi(r) + \frac{B\rho^\beta}{\tau^{2\beta}(1 - \tau^{\gamma-\beta})}.$$

This implies the desired result. \square

Next, we prove Hölder estimates for gradients of solutions following the same idea used in the proof of Theorem 2.4.

Theorem 2.6. *Let $u \in H^1(B_1)$ satisfy (*). Assume $a_{ij} \in C^\alpha(\bar{B}_1)$ and $f \in L^p(B_1)$, for some $p > n$ and $\alpha = 1 - n/p \in (0, 1)$. Then $Du \in C^\alpha(B_1)$. Moreover,*

$$|Du|_{C^\alpha(B_{\frac{1}{2}})} \leq C \left\{ \|f\|_{L^p(B_1)} + \|u\|_{H^1(B_1)} \right\},$$

where C is a positive constant depending only on λ and $|a_{ij}|_{C^\alpha(B_1)}$.

Proof. We will prove that there exists an $R_0 \in (0, 1/2)$ such that, for any $x \in B_{\frac{1}{2}}$ and $r \leq R_0$,

$$\int_{B_r(x)} |Du - (Du)_{x,r}|^2 \leq Cr^{n+2\alpha} \left\{ \|f\|_{L^p(B_1)}^2 + \|u\|_{H^1(B_1)}^2 \right\},$$

where R_0 and C are positive constants depending only on λ and $|a_{ij}|_{C^\alpha(B_1)}$. As in the proof of Theorem 2.4, we decompose u into a sum $v + w$ where w satisfies a homogeneous equation of constant coefficients and v has estimates in terms of nonhomogeneous terms.

For any $B_r(x_0) \subset B_1$, we write the equation in the following form

$$\int_{B_1} a_{ij}(x_0) D_i u D_j \varphi = - \int_{B_1} (a_{ij} - a_{ij}(x_0)) D_j u D_j \varphi + \int_{B_1} f \varphi.$$

In $B_r(x_0)$, there exists a unique function $w \in H_1(B_r(x_0))$ such that

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i w D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1(B_r(x_0)),$$

with $w - u \in H_0^1(B_r(x_0))$. Then, $v = u - w \in H_0^1(B_r(x_0))$ satisfies

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = - \int_{B_r(x_0)} (a_{ij} - a_{ij}(x_0)) D_i u D_j \varphi + \int_{B_r(x_0)} f \varphi,$$

for any $\varphi \in H_0^1(B_r(x_0))$. By taking the test function $\varphi = v$, we obtain with the uniform ellipticity

$$\lambda \int_{B_r(x_0)} |Dv|^2 \leq \int_{B_r(x_0)} |(a_{ij}(x_0) - a_{ij}(x)) D_i u D_j v| + \int_{B_r(x_0)} |f v|.$$

As in the proof of Theorem 2.4, we apply the Cauchy inequality for terms involving Dv and apply the Hölder inequality and the Sobolev inequality for terms involving v . Then, we obtain

$$\int_{B_r(x_0)} |Dv|^2 \leq C \left\{ \int_{B_r(x_0)} |(a_{ij}(x_0) - a_{ij}(x)) D_i u|^2 + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \right\}.$$

By the Hölder inequality, we have for any $B_r(x_0) \subset B_1$

$$\left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \leq r^{n+2\alpha} \|f\|_{L^p(B_r(x_0))}.$$

Therefore Corollary 2.3 implies for any $0 < \rho \leq r$

$$(1) \quad \int_{B_\rho(x_0)} |Du|^2 \leq C \left\{ \left[\left(\frac{\rho}{r} \right)^n + r^{2\alpha} [a_{ij}]_{C^\alpha(B_r(x_0))} \right] \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha} \|f\|_{L^p(B_1)}^2 \right\},$$

and

$$(2) \quad \int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 + r^{2\alpha} [a_{ij}]_{C^\alpha(B_r(x_0))} \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha} \|f\|_{L^p(B_1)}^2 \right\},$$

where C is a positive constant depending only on λ and Λ .

Case 1. We first consider a special case when $a_{ij} \equiv \text{const}$. By (2), we have for any $B_r(x_0) \subset B_1$ and $0 < \rho \leq r$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 + r^{n+2\alpha} \|f\|_{L^p(B_1)}^2 \right\}.$$

By Lemma 2.5, we replace $r^{n+2\alpha}$ by $\rho^{n+2\alpha}$ to get the desired result.

Case 2. Next, we consider the general case. By (1) and (2), we have for any $B_r(x_0) \subset B_1$ and any $\rho < r$

$$(3) \quad \int_{B_\rho(x_0)} |Du|^2 \leq C \left\{ \left[\left(\frac{\rho}{r} \right)^n + r^{2\alpha} \right] \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha} \|f\|_{L^p(B_1)}^2 \right\},$$

and

$$(4) \quad \int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 + r^{2\alpha} \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha} \|f\|_{L^p(B_1)}^2 \right\}.$$

We need to estimate the integral

$$\int_{B_r(x_0)} |Du|^2.$$

Take a small $\delta > 0$. Then (3) implies

$$\int_{B_\rho(x_0)} |Du|^2 \leq C \left\{ \left[\left(\frac{\rho}{r} \right)^n + r^{2\alpha} \right] \int_{B_r(x_0)} |Du|^2 + r^{n-2\delta} \|f\|_{L^p(B_1)}^2 \right\}.$$

Hence Lemma 2.5 implies the existence of an $R_1 \in (3/4, 1)$ with $r_1 = 1 - R_1$ such that, for any $x_0 \in B_{R_1}$ and any $0 < r \leq r_1$,

$$(5) \quad \int_{B_r(x_0)} |Du|^2 \leq Cr^{n-2\delta} \left\{ \|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right\}.$$

Therefore, by substituting (5) in (4) we obtain for any $0 < \rho < r \leq r_1$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 + r^{n+2\alpha-2\delta} \left[\|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right] \right\}.$$

By Lemma 2.5 again, we have for any $x_0 \in B_{R_1}$ and any $0 < \rho < r \leq r_1$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n+2\alpha-2\delta} \int_{B_r} |Du - (Du)_{x_0, r}|^2 + \rho^{n+2\alpha-2\delta} \left[\|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right] \right\}.$$

With $r = r_1$ this implies that for any $x_0 \in B_{R_1}$ and any $0 < r \leq r_1$

$$\int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 \leq Cr^{n+2\alpha-2\delta} \left\{ \|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right\}.$$

Hence $Du \in C_{\text{loc}}^{\alpha-\delta}$ for any $\delta > 0$ small. In particular, $Du \in L_{\text{loc}}^\infty$ and

$$(6) \quad \sup_{B_{\frac{3}{4}}} |Du|^2 \leq C \left\{ \|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right\}.$$

Combining (4) and (6), we have for any $x_0 \in B_{1/2}$ and $0 < \rho < r \leq r_1$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 + r^{n+2\alpha} \left[\|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right] \right\}.$$

By Lemma 2.5 again, we have

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 \leq C \left\{ \left(\frac{\rho}{r} \right)^{n+2\alpha} \int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 + \rho^{n+2\alpha} \left[\|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right] \right\}.$$

By choosing $r = r_1$, we have for any $x_0 \in B_{1/2}$ and $r \leq r_1$

$$\int_{B_r(x_0)} |Du - (Du)_{x_0, r}|^2 \leq cr^{n+2\alpha} \left\{ \|f\|_{L^p(B_1)}^2 + \|Du\|_{L^2(B_1)}^2 \right\}.$$

This is the desired estimate. \square

3. DEGIORGI ITERATIONS

In this section, we discuss the DeGiorgi iterations for elliptic equations of the divergence form. We first prove the local boundedness of solutions and then the Hölder continuity. There is no regularity assumption on coefficients. Throughout this section, we always assume that $a_{ij} \in L^\infty(B_1)$ satisfy

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } x \in B_1, \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ .

We first prove the following boundedness result by an iteration method due to DeGiorgi.

Theorem 3.1. *Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense*

$$(*) \quad \int_{B_1} a_{ij} D_i u D_j \varphi \leq \int_{B_1} f \varphi \quad \text{for any } \varphi \in H_0^1(B_1) \text{ with } \varphi \geq 0 \text{ in } B_1.$$

If $f \in L^q(B_1)$ for some $q > n/2$, then $u^+ \in L_{loc}^\infty(B_1)$. Moreover,

$$\sup_{B_{\frac{1}{2}}} u^+ \leq C \left\{ \|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)} \right\},$$

where C is a positive constant depending only on n, λ, Λ and q .

Proof. Consider $v = (u - k)^+$ for $k \geq 0$ and $\zeta \in C_0^1(B_1)$. Set $\varphi = v\zeta^2$ as the test function. Note $v = u - k$, $Dv = Du$ a.e. in $\{u > k\}$ and $v = 0$, $Dv = 0$ a.e. in $\{u \leq k\}$. Hence, if we substitute such a φ in (*), we integrate in the set $\{u > k\}$.

By the Cauchy inequality, we have

$$\begin{aligned} \int a_{ij} D_i u D_j \varphi &= \int a_{ij} D_i u D_j v \zeta^2 + 2a_{ij} D_i u D_j \zeta v \zeta \\ &\geq \lambda \int |Dv|^2 \zeta^2 - 2\Lambda \int |Dv| |D\zeta| v \zeta \\ &\geq \frac{\lambda}{2} \int |Dv|^2 \zeta^2 - \frac{2\Lambda^2}{\lambda} \int |D\zeta|^2 v^2. \end{aligned}$$

Then, we obtain

$$\int |Dv|^2 \zeta^2 \leq C \left\{ \int v^2 |D\zeta|^2 + \int |f| v \zeta^2 \right\},$$

and hence

$$\int |D(v\zeta)|^2 \leq C \left\{ \int v^2 |D\zeta|^2 + \int |f| v \zeta^2 \right\}.$$

Recall the Sobolev inequality for $v\zeta \in H_0^1(B_1)$

$$\left(\int_{B_1} (v\zeta)^{2^*} \right)^{\frac{2}{2^*}} \leq c(n) \int_{B_1} |D(v\zeta)|^2,$$

where $2^* = 2n/(n-2)$ for $n > 2$ and $2^* > 2$ is arbitrary if $n = 2$. The Hölder inequality implies that with $\delta > 0$ small and $\zeta \leq 1$

$$\begin{aligned} \int |f| v \zeta^2 &\leq \left(\int |f|^q \right)^{\frac{1}{q}} \left(\int |v\zeta|^{2^*} \right)^{\frac{1}{2^*}} |\{v\zeta \neq 0\}|^{1 - \frac{1}{2^*} - \frac{1}{q}} \\ &\leq c(n) \|f\|_{L^q} \left(\int |D(v\zeta)|^2 \right)^{\frac{1}{2}} |\{v\zeta \neq 0\}|^{\frac{1}{2} + \frac{1}{n} - \frac{1}{q}} \\ &\leq \delta \int |D(v\zeta)|^2 + c(n, \delta) \|f\|_{L^q}^2 |\{v\zeta \neq 0\}|^{1 + \frac{2}{n} - \frac{2}{q}}. \end{aligned}$$

Therefore, we have

$$(1) \quad \int |D(v\zeta)|^2 \leq C \left\{ \int v^2 |D\zeta|^2 + \|f\|_{L^q(B_1)}^2 |\{v\zeta \neq 0\}|^{1 + \frac{2}{n} - \frac{2}{q}} \right\}.$$

To continue, we note by the Sobolev inequality

$$\begin{aligned} \int (v\zeta)^2 &\leq \left(\int (v\zeta)^{2^*} \right)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1 - \frac{2}{2^*}} \\ &\leq c(n) \int |D(v\zeta)|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}}. \end{aligned}$$

Therefore, we have

$$\int (v\zeta)^2 \leq C \left\{ \int v^2 |D\zeta|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}} + \|f\|_{L^q(B_1)}^2 |\{v\zeta \neq 0\}|^{1 + \frac{4}{n} - \frac{2}{q}} \right\}.$$

Note $q > n/2$. Hence there exists an $\varepsilon > 0$ such that

$$\int (v\zeta)^2 \leq C \left\{ \int v^2 |D\zeta|^2 |\{v\zeta \neq 0\}|^\varepsilon + \|f\|_{L^q(B_1)}^2 |\{v\zeta \neq 0\}|^{1+\varepsilon} \right\}.$$

For any fixed $0 < r < R \leq 1$, choose $\zeta \in C_0^\infty(B_R)$ such that $\zeta \equiv 1$ in B_r , and $0 \leq \zeta \leq 1$ and $|D\zeta| \leq 2(R-r)^{-1}$ in B_1 . Set

$$A(k, r) = \{x \in B_r; u \geq k\}.$$

We conclude that for any $0 < r < R \leq 1$ and $k > 0$

$$(2) \quad \int_{A(k, r)} (u - k)^2 \leq C \left\{ \frac{1}{(R-r)^2} |A(k, R)|^\varepsilon \int_{A(k, R)} (u - k)^2 + \|f\|_{L^q(B_1)}^2 |A(k, R)|^{1+\varepsilon} \right\}.$$

Next, we claim for $k = C(\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$

$$\int_{A(k, 1/2)} (u - k)^2 = 0.$$

To this end, we take any $h > k \geq k_0$ and any $0 < r < 1$. It is obvious that $A(k, r) \supset A(h, r)$. Hence we have

$$\int_{A(h, r)} (u - h)^2 \leq \int_{A(k, r)} (u - k)^2,$$

and

$$|A(h, r)| = |B_r \cap \{u - k > h - k\}| \leq \frac{1}{(h - k)^2} \int_{A(k, r)} (u - k)^2.$$

Therefore, by (2) we have for any $h > k \geq k_0$ and $1/2 \leq r < R \leq 1$

$$\begin{aligned} & \int_{A(h, r)} (u - h)^2 \\ & \leq C \left\{ \frac{1}{(R-r)^2} \int_{A(h, R)} (u - h)^2 + \|f\|_{L^q(B_1)}^2 |A(h, R)| \right\} |A(h, R)|^\varepsilon \\ & \leq C \left\{ \frac{1}{(R-r)^2} + \frac{\|f\|_{L^q(B_1)}^2}{(h-k)^2} \right\} \frac{1}{(h-k)^{2\varepsilon}} \left(\int_{A(k, R)} (u - k)^2 \right)^{1+\varepsilon}, \end{aligned}$$

or

$$(3) \quad \|(u - h)^+\|_{L^2(B_r)} \leq C \left\{ \frac{1}{R-r} + \frac{\|f\|_{L^q(B_1)}}{h-k} \right\} \frac{1}{(h-k)^\varepsilon} \|(u - k)^+\|_{L^2(B_R)}^{1+\varepsilon}.$$

Now we carry out the iteration. Set

$$\varphi(k, r) = \|(u - k)^+\|_{L^2(B_r)}.$$

For $\tau = 1/2$ and some $k > 0$ to be determined, define for $\ell = 0, 1, 2, \dots$,

$$\begin{aligned} k_\ell &= k\left(1 - \frac{1}{2^\ell}\right) \quad (\leq k), \\ r_\ell &= \tau + \frac{1}{2^\ell}(1 - \tau). \end{aligned}$$

Obviously, we have $k_0 = 0, r_0 = 1$ and

$$k_\ell - k_{\ell-1} = \frac{k}{2^\ell}, \quad r_{\ell-1} - r_\ell = \frac{1}{2^\ell}(1 - \tau).$$

Therefore, we have for $\ell = 0, 1, 2, \dots$

$$\begin{aligned} \varphi(k_\ell, r_\ell) &\leq C \left\{ \frac{2^\ell}{1 - \tau} + \frac{2^\ell \|f\|_{L^q(B_1)}}{k} \right\} \frac{2^{\varepsilon\ell}}{k^\varepsilon} [\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon} \\ &\leq \frac{C}{1 - \tau} \cdot \frac{\|f\|_{L^q(B_1)} + k}{k^{1+\varepsilon}} \cdot 2^{(1+\varepsilon)\ell} \cdot [\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon}. \end{aligned}$$

Next, we prove inductively for any $\ell = 0, 1, \dots$

$$(4) \quad \varphi(k_\ell, r_\ell) \leq \frac{\varphi(k_0, r_0)}{\gamma^\ell} \quad \text{for some } \gamma > 1.$$

Obviously, it is true for $\ell = 0$. Suppose it is true for $\ell - 1$. We write

$$[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon} \leq \left\{ \frac{\varphi(k_0, r_0)}{\gamma^{\ell-1}} \right\}^{1+\varepsilon} = \frac{\varphi(k_0, r_0)^\varepsilon}{\gamma^{\ell\varepsilon - (1+\varepsilon)}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Then, we obtain

$$\varphi(k_\ell, r_\ell) \leq \frac{C\gamma^{1+\varepsilon}}{1 - \tau} \cdot \frac{\|f\|_{L^q(B_1)} + k}{k^{1+\varepsilon}} \cdot [\varphi(k_0, r_0)]^\varepsilon \cdot \frac{2^{\ell(1+\varepsilon)}}{\gamma^{\ell\varepsilon}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Choose γ first such that $\gamma^\varepsilon = 2^{1+\varepsilon}$ and note $\gamma > 1$. Next, we need

$$\frac{C\gamma^{1+\varepsilon}}{1 - \tau} \cdot \left(\frac{\varphi(k_0, r_0)}{k} \right)^\varepsilon \cdot \frac{\|f\|_{L^q(B_1)} + k}{k} \leq 1.$$

Therefore, we choose

$$k = C_* \{ \|f\|_{L^q(B_1)} + \varphi(k_0, r_0) \},$$

for C_* large. Let $\ell \rightarrow +\infty$ in (4). We conclude

$$\varphi(k, \tau) = 0.$$

Hence we have

$$\sup_{B_{\frac{1}{2}}} u^+ \leq C_* \{ \|f\|_{L^q(B_1)} + \varphi(k_0, r_0) \}.$$

Recall $\varphi(k_0, r_0) \leq \|u^+\|_{L^2(B_1)}$. This finishes the proof by the approach due to DeGiorgi. \square

In the rest of the section, we discuss the Hölder continuity. We first discuss homogeneous equations with no lower order terms. Consider

$$Lu \equiv -D_i(a_{ij}(x)D_j u) \quad \text{in } B_1 \subset \mathbb{R}^n.$$

Definition 3.2. The function $u \in H_{loc}^1(B_1)$ is called a subsolution (supersolution) of the equation $Lu = 0$ if

$$\int_{B_1} a_{ij} D_i u D_j \varphi \leq 0 \quad (\geq 0),$$

for any $\varphi \in H_0^1(B_1)$ and $\varphi \geq 0$.

Lemma 3.3. Let $\Phi \in C_{loc}^{0,1}(\mathbb{R})$ be convex.

(i) If u is a subsolution and $\Phi' \geq 0$, then $v = \Phi(u)$ is a subsolution provided $v \in H_{loc}^1(B_1)$;

(ii) If u is a supersolution and $\Phi' \leq 0$, then $v = \Phi(u)$ is a subsolution provided $v \in H_{loc}^1(B_1)$.

Remark 3.4. If u is a subsolution, then $(u - k)^+$ is also a subsolution, where $(u - k)^+ = \max\{0, u - k\}$. In this case, $\Phi(s) = (s - k)^+$.

Proof. We only prove (i). Assume first $\Phi \in C_{loc}^2(\mathbb{R})$. Then we have $\Phi'(s) \geq 0$ and $\Phi''(s) \geq 0$. Consider a $\varphi \in C_0^1(B_1)$ with $\varphi \geq 0$. A direct calculation yields

$$\begin{aligned} \int_{B_1} a_{ij} D_i v D_j \varphi &= \int_{B_1} a_{ij} \Phi'(u) D_i u D_j \varphi \\ &= \int_{B_1} a_{ij} D_i u D_j (\Phi'(u) \varphi) - \int_{B_1} (a_{ij} D_i u D_j u) \varphi \Phi''(u) \leq 0, \end{aligned}$$

where $\Phi'(u) \varphi \in H_0^1(B_1)$ is nonnegative. In general, set $\Phi_\epsilon(s) = \rho_\epsilon * \Phi(s)$ with ρ_ϵ as the standard mollifier. Then $\Phi'_\epsilon(s) = \rho_\epsilon * \Phi'(s) \geq 0$ and $\Phi''_\epsilon(s) \geq 0$. Hence $\Phi_\epsilon(u)$ is a subsolution by what we just proved. Note $\Phi'_\epsilon(s) \rightarrow \Phi'(s)$ a.e. as $\epsilon \rightarrow 0^+$. The Lebesgue dominant convergence theorem implies the desired result. \square

We also need the following Poincaré-Sobolev inequality.

Lemma 3.5. Let ϵ be a positive constant. For any $u \in H^1(B_1)$, if

$$|\{x \in B_1; u = 0\}| \geq \epsilon |B_1|,$$

then

$$\int_{B_1} u^2 \leq C \int_{B_1} |Du|^2,$$

where C is a positive constant depending only on ϵ and n .

Proof. We prove by contradiction. If not, there exists a sequence $\{u_m\} \subset H^1(B_1)$ such that

$$|\{x \in B_1; u_m = 0\}| \geq \epsilon |B_1|,$$

and

$$\int_{B_1} u_m^2 = 1, \quad \int_{B_1} |Du_m|^2 \rightarrow 0 \text{ as } m \rightarrow \infty.$$

We may assume $u_m \rightarrow u_0$ strongly in $L^2(B_1)$ and weakly in $H^1(B_1)$, with $u_0 \in H^1(B_1)$. Clearly u_0 is a nonzero constant. Then we have

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \int_{B_1} |u_m - u_0|^2 \geq \lim_{m \rightarrow \infty} \int_{\{u_m=0\}} |u_m - u_0|^2 \\ &\geq |u_0|^2 \inf_m |\{u_m = 0\}| > 0. \end{aligned}$$

This is a contradiction. \square

Now, we begin to discuss the Hölder continuity. We first prove the following result, which is often referred to as the density theorem.

Theorem 3.6. *Suppose $\epsilon \in (0, 1)$ is a constant and u is a positive supersolution in B_2 with*

$$|\{x \in B_1; u \geq 1\}| \geq \epsilon |B_1|.$$

Then

$$\inf_{B_{\frac{1}{2}}} u \geq C,$$

where C is a positive constant depending only on ϵ , n and Λ/λ .

Proof. We may assume $u \geq \delta > 0$ and then let $\delta \rightarrow 0+$ at the end. By Lemma 3.3, $v = (\log u)^-$ is a subsolution, bounded by $\log \delta^{-1}$. Theorem 3.1 yields

$$\sup_{B_{\frac{1}{2}}} v \leq C \left(\int_{B_1} |v|^2 \right)^{\frac{1}{2}}.$$

Note $|\{x \in B_1; v = 0\}| = |\{x \in B_1; u \geq 1\}| \geq \epsilon |B_1|$. Lemma 3.5 implies

$$(1) \quad \sup_{B_{\frac{1}{2}}} v \leq C \left(\int_{B_1} |Dv|^2 \right)^{\frac{1}{2}}.$$

We will prove that the right-hand side is bounded. To this end, consider a test function $\varphi = \frac{\zeta^2}{u}$ for $\zeta \in C_0^1(B_2)$. Then we obtain

$$0 \leq \int a_{ij} D_i u D_j \left(\frac{\zeta^2}{u} \right) = - \int \zeta^2 \frac{a_{ij} D_i u D_j u}{u^2} + 2 \int \frac{\zeta a_{ij} D_i u D_j \zeta}{u},$$

and hence

$$\int \zeta^2 |D \log u|^2 \leq C \int |D \zeta|^2.$$

So for fixed $\zeta \in C_0^1(B_2)$ with $\zeta \equiv 1$ in B_1 , we have

$$\int_{B_1} |D \log u|^2 \leq C.$$

Combining with (1), we obtain

$$\sup_{B_{\frac{1}{2}}} v = \sup_{B_{\frac{1}{2}}} (\log u)^- \leq C,$$

or

$$\inf_{B_{\frac{1}{2}}} u \geq e^{-C} > 0.$$

This finishes the proof. □

The next result controls the oscillation of solutions.

Theorem 3.7. *Suppose u is a bounded solution of $Lu = 0$ in B_2 . Then*

$$\text{osc}_{B_{\frac{1}{2}}} u \leq \gamma \text{osc}_{B_1} u,$$

where $\gamma \in (0, 1)$ is a constant depending only on n and Λ/λ .

Proof. The local boundedness is proved in the previous section. Set

$$\alpha_1 = \sup_{B_1} u \quad \text{and} \quad \beta_1 = \inf_{B_1} u.$$

Consider

$$\frac{u - \beta_1}{\alpha_1 - \beta_1} \quad \text{and} \quad \frac{\alpha_1 - u}{\alpha_1 - \beta_1}.$$

Obviously, they are solutions of $Lv = 0$. Note the following equivalence

$$\begin{aligned} u \geq \frac{1}{2}(\alpha_1 + \beta_1) &\iff \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{2}, \\ u \leq \frac{1}{2}(\alpha_1 + \beta_1) &\iff \frac{\alpha_1 - u}{\alpha_1 - \beta_1} \geq \frac{1}{2}. \end{aligned}$$

Case 1. Suppose

$$|\{x \in B_1; \frac{2(u - \beta_1)}{\alpha_1 - \beta_1} \geq 1\}| \geq \frac{1}{2}|B_1|.$$

Apply Theorem 3.6 to $\frac{u - \beta_1}{\alpha_1 - \beta_1} \geq 0$ in B_1 . We have for some $C > 1$

$$\inf_{B_{\frac{1}{2}}} \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{C},$$

and hence

$$\inf_{B_{\frac{1}{2}}} u \geq \beta_1 + \frac{1}{C}(\alpha_1 - \beta_1).$$

Case 2. Suppose

$$|\{x \in B_1; \frac{2(\alpha_1 - u)}{\alpha_1 - \beta_1} \geq 1\}| \geq \frac{1}{2}|B_1|.$$

Similarly, we obtain

$$\sup_{B_{\frac{1}{2}}} u \leq \alpha_1 - \frac{1}{C}(\alpha_1 - \beta_1).$$

Now, we set

$$\alpha_2 = \sup_{B_{\frac{1}{2}}} u \quad \text{and} \quad \beta_2 = \inf_{B_{\frac{1}{2}}} u.$$

Note that $\beta_2 \geq \beta_1$ and $\alpha_2 \leq \alpha_1$. In both cases, we have

$$\alpha_2 - \beta_2 \leq \left(1 - \frac{1}{C}\right)(\alpha_1 - \beta_1).$$

This finishes the proof. \square

The DeGiorgi theorem on Hölder continuity is an easy consequence of above results.

Theorem 3.8. *Suppose u is an $H^1(B_1)$ -solution of $Lu = 0$ in B_1 . Then $u \in C^\alpha(B_1)$ for some $\alpha \in (0, 1)$ depending only on n and Λ/λ . Moreover,*

$$\|u\|_{C^\alpha(B_{\frac{1}{2}})} \leq C \|u\|_{L^2(B_1)},$$

where C is a positive constant depending only on n and Λ/λ .

In the rest of this section, we discuss the Hölder continuity of solutions of general linear equations. We need the following lemma.

Lemma 3.9. *Suppose $u \in H^1(B_r)$ satisfies*

$$\int_{B_r} a_{ij} D_i u D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1(B_r).$$

Then there exists an $\alpha \in (0, 1)$ such that for any $\rho < r$

$$\int_{B_\rho} |Du|^2 \leq C \left(\frac{\rho}{r}\right)^{n-2+2\alpha} \int_{B_r} |Du|^2,$$

where C and α are positive constants depending only on n and Λ/λ .

Proof. By a dilation, we consider $r = 1$. We restrict our consideration to the range $\rho \in (0, \frac{1}{4}]$, since it is trivial for $\rho \in (\frac{1}{4}, 1]$. We may further assume $\int_{B_1} u = 0$, since the function $u - |\mathbb{B}_1|^{-1} \int_{B_1} u$ solves the same equation. The Poincaré inequality yields

$$\int_{B_1} u^2 \leq c(n) \int_{B_1} |Du|^2.$$

Hence Theorem 3.8 implies for $|x| \leq 1/2$

$$|u(x) - u(0)|^2 \leq C |x|^{2\alpha} \int_{B_1} |Du|^2,$$

where $\alpha \in (0, 1)$ is as determined in Theorem 3.8. For any $0 < \rho \leq 1/4$, take a cut-off function $\zeta \in C_0^\infty(B_{2\rho})$ with $\zeta \equiv 1$ in B_ρ , and $0 \leq \zeta \leq 1$ and $|D\zeta| \leq 2/\rho$. Then set $\varphi = \zeta^2(u - u(0))$. Now the equation yields

$$\begin{aligned} 0 &= \int_{B_1} a_{ij} D_i u (\zeta^2 D_j u + 2\zeta D_j \zeta (u - u(0))) \\ &\geq \frac{\lambda}{2} \int_{B_{2\rho}} \zeta^2 |Du|^2 - C \sup_{B_{2\rho}} |u - u(0)|^2 \int_{B_{2\rho}} |D\zeta|^2. \end{aligned}$$

Therefore, we have

$$\int_{B_\rho} |Du|^2 \leq C \rho^{n-2} \sup_{B_{2\rho}} |u - u(0)|^2.$$

The conclusion follows easily. \square

Now, we prove the following result in the same way we proved Theorem 2.4, with Lemma 2.2 replaced by Lemma 3.9.

Theorem 3.10. *Suppose that $u \in H^1(B_1)$ satisfies*

$$\int_{B_1} a_{ij} D_j u D_i \varphi = \int_{B_1} f \varphi \quad \text{for any } \varphi \in H_0^1(B_1).$$

If $f \in L^q(B_1)$ for some $q > n/2$, then $u \in C^\alpha(B_1)$ for some $\alpha \in (0, 1)$, depending only on n, q, λ and Λ . Moreover, there exists an $R_0 \in (0, 1)$ such that for any $x \in B_{\frac{1}{2}}$ and $r \leq R_0$

$$\int_{B_r(x)} |Du|^2 \leq Cr^{n-2+2\alpha} \left\{ \|f\|_{L^q(B_1)}^2 + \|u\|_{H^1(B_1)}^2 \right\},$$

where R_0 and C are constants depending only on n, q, λ and Λ .

4. MOSER'S ITERATIONS

In this section, we discuss Moser's iterations for elliptic differential equations of divergence form.

Suppose Ω is a domain in \mathbb{R}^n . We always assume that $a_{ij} \in L^\infty(B_1)$ satisfy

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ .

We first give another proof of Theorem 3.1 by an iteration argument due to Moser.

Theorem 4.1. *Suppose that $u \in H^1(\Omega)$ is a subsolution in the following sense*

$$(*) \quad \int_{B_1} a_{ij} D_i u D_j \varphi \leq \int_{B_1} f \varphi \quad \text{for any } \varphi \in H_0^1(\Omega) \text{ with } \varphi \geq 0 \text{ in } \Omega.$$

If $f \in L^q(\Omega)$ for some $q > n/2$, then $u^+ \in L_{loc}^\infty(\Omega)$. Moreover, for any $B_R \subset \Omega$, $\theta \in (0, 1)$ and any $p > 0$

$$\sup_{B_{\theta R}} u^+ \leq C \left\{ \frac{1}{(R - \theta R)^{n/p}} \|u^+\|_{L^p(B_R)} + \|f\|_{L^q(B_R)} \right\},$$

where C is a positive constant depending only on n, λ, Λ, p and q .

Proof. We first prove for $p = 2$, $\theta = 1/2$ and $R = 1/2$. First we explain the idea. By choosing a test function appropriately, we estimate L^{p_1} norm of u in a small ball by L^{p_2} norm of u in a large ball for $p_1 > p_2$, i.e.,

$$\|u\|_{L^{p_1}(B_{r_1})} \leq C \|u\|_{L^{p_2}(B_{r_2})},$$

for $p_1 > p_2$ and $r_1 < r_2$. This is a reversed Hölder inequality. As a sacrifice, C behaves like $\frac{1}{r_2 - r_1}$. We then obtain the desired result by an iteration and a careful choice of $\{r_i\}$ and $\{p_i\}$.

For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ k + m & \text{if } u \geq m. \end{cases}$$

Then we have $D\bar{u}_m = 0$ in $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \leq \bar{u}$. Consider the test function

$$\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1),$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. A direct calculation yields

$$\begin{aligned} D\varphi &= \beta\eta^2\bar{u}_m^{\beta-1}D\bar{u}_m\bar{u} + \eta^2\bar{u}_m^\beta D\bar{u} + 2\eta D\eta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \\ &= \eta^2\bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}). \end{aligned}$$

We should emphasize that later on we will begin the iteration with $\beta = 0$. Note $\varphi = 0$ and $D\varphi = 0$ in $\{u \leq 0\}$. Hence, if we substitute such a φ in (*), we integrate in the set $\{u > 0\}$. Note also that $u^+ \leq \bar{u}$ and $\bar{u}_m^\beta \bar{u} - k^{\beta+1} \leq \bar{u}_m^\beta \bar{u}$ for $k > 0$. First, we have by the Hölder inequality

$$\begin{aligned} &\int a_{ij}D_i u D_j \varphi \\ &= \int a_{ij}D_i \bar{u}(\beta D_j \bar{u}_m + D_j \bar{u})\eta^2\bar{u}_m^\beta + 2 \int a_{ij}D_i \bar{u} D_j \eta(\bar{u}_m^\beta \bar{u} - k^{\beta+1})\eta \\ &\geq \lambda\beta \int \eta^2\bar{u}_m^\beta |D\bar{u}_m|^2 + \lambda \int \eta^2\bar{u}_m^\beta |D\bar{u}|^2 - \Lambda \int |D\bar{u}||D\eta|\bar{u}_m^\beta \bar{u}\eta \\ &\geq \lambda\beta \int \eta^2\bar{u}_m^\beta |D\bar{u}_m|^2 + \frac{\lambda}{2} \int \eta^2\bar{u}_m^\beta |D\bar{u}|^2 - \frac{2\Lambda^2}{\lambda} \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2. \end{aligned}$$

Hence, we obtain by noting $\bar{u} \geq k$

$$\begin{aligned} &\beta \int \eta^2\bar{u}_m^\beta |D\bar{u}_m|^2 + \int \eta^2\bar{u}_m^\beta |D\bar{u}|^2 \\ &\leq C \left\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int |f|\eta^2\bar{u}_m^\beta \bar{u} \right\} \\ &\leq C \left\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int f_0\eta^2\bar{u}_m^\beta \bar{u}^2 \right\}, \end{aligned}$$

where f_0 is defined by $f_0 = |f|/k$. Choose $k = \|f\|_{L^q}$ if f is not identically zero. Otherwise choose an arbitrary $k > 0$ and eventually let $k \rightarrow 0+$. Then $\|f_0\|_{L^q} \leq 1$. Set $w = \bar{u}_m^{\frac{\beta}{2}}\bar{u}$. Note

$$|Dw|^2 \leq (1 + \beta)\{\beta\bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2\}.$$

Therefore, we have

$$\int |Dw|^2 \eta^2 \leq C \left\{ (1 + \beta) \int w^2 |D\eta|^2 + (1 + \beta) \int f_0 w^2 \eta^2 \right\},$$

or

$$\int |D(w\eta)|^2 \leq C \left\{ (1 + \beta) \int w^2 |D\eta|^2 + (1 + \beta) \int f_0 w^2 \eta^2 \right\}.$$

The Hölder inequality implies

$$\int f_0 w^2 \eta^2 \leq \left(\int c_0^q \right)^{\frac{1}{q}} \left(\int (\eta w)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}} \leq \left(\int (\eta w)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}}.$$

By the interpolation inequality and the Sobolev inequality with

$$2^* = \frac{2n}{n-2} > \frac{2q}{q-1} > 2 \text{ if } q > n/2,$$

we have

$$\begin{aligned} \|\eta w\|_{L^{\frac{2q}{q-1}}} &\leq \varepsilon \|\eta w\|_{L^{2^*}} + C(n, q) \varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2} \\ &\leq \varepsilon \|D(\eta w)\|_{L^2} + C(n, q) \varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2}, \end{aligned}$$

for any small $\varepsilon > 0$. Therefore, we obtain

$$\int |D(w\eta)|^2 \leq C \left\{ (1+\beta) \int w^2 |D\eta|^2 + (1+\beta)^{\frac{2q}{2q-n}} \int w^2 \eta^2 \right\},$$

and in particular

$$\int |D(w\eta)|^2 \leq C(1+\beta)^\alpha \int (|D\eta|^2 + \eta^2) w^2,$$

where α is a positive number depending only on n and q . The Sobolev inequality then implies

$$\left(\int |\eta w|^{2\chi} \right)^{\frac{1}{\chi}} \leq C(1+\beta)^\alpha \int (|D\eta|^2 + \eta^2) w^2,$$

where $\chi = \frac{n}{n-2} > 1$ for $n > 2$ and $\chi > 1$ for $n = 2$. For any $0 < r < R \leq 1$, consider an $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \leq \frac{2}{R-r}.$$

Then, we obtain

$$\left(\int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(1+\beta)^\alpha}{(R-r)^2} \int_{B_R} w^2.$$

Recalling the definition of w , we have

$$\left(\int_{B_r} \bar{u}^{2\chi} \bar{u}_m^{\beta\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(1+\beta)^\alpha}{(R-r)^2} \int_{B_R} \bar{u}^2 \bar{u}_m^\beta.$$

Set $\gamma = \beta + 2 \geq 2$. Then we obtain

$$\left(\int_{B_r} \bar{u}_m^{\gamma\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(\gamma-1)^\alpha}{(R-r)^2} \int_{B_R} \bar{u}^\gamma,$$

provided the integral in the right hand side is bounded. By letting $m \rightarrow +\infty$ we conclude that

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \leq \left(C \frac{(\gamma-1)^\alpha}{(R-r)^2} \right)^{\frac{1}{\gamma}} \|\bar{u}\|_{L^\gamma(B_R)},$$

provided $\|\bar{u}\|_{L^\gamma(B_R)} < +\infty$, where C is a positive constant depending only on n, q, λ, Λ , and independent of γ . The above estimate suggests that we iterate, beginning with $\gamma = 2$, as $2, 2\chi, 2\chi^2, \dots$. Now we set for $i = 0, 1, 2, \dots$,

$$\gamma_i = 2\chi^i \quad \text{and} \quad r_i = \frac{1}{2} + \frac{1}{2^{i+1}}.$$

By $\gamma_i = \chi\gamma_{i-1}$ and $r_{i-1} - r_i = 1/2^{i+1}$, we have for $i = 1, 2, \dots$,

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \leq C(n, q, \lambda, \Lambda)^{\frac{i}{\chi^i}} \|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})},$$

provided $\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})} < +\infty$. Hence, by an iteration we obtain

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \leq C^{\sum \frac{i}{\chi^i}} \|\bar{u}\|_{L^2(B_1)},$$

and in particular

$$\left(\int_{B_{\frac{1}{2}}} \bar{u}^{2\chi^i} \right)^{\frac{1}{2\chi^i}} \leq C \left(\int_{B_1} \bar{u}^2 \right)^{\frac{1}{2}}.$$

Letting $i \rightarrow +\infty$, we get

$$\sup_{B_{\frac{1}{2}}} \bar{u} \leq C \|\bar{u}\|_{L^2(B_1)}$$

or

$$\sup_{B_{\frac{1}{2}}} u^+ \leq C \{ \|u^+\|_{L^2(B_1)} + k \}.$$

Recall the definition of k . This finishes the proof for $p = 2$, $\theta = 1/2$ and $R = 1$ by the approach due to Moser.

Next, we discuss the general case of Theorem 4.1. This is based on a dilation argument. Take any $R \leq 1$. Define

$$\tilde{u}(y) = u(Ry) \quad \text{for } y \in B_1.$$

It is easy to see that \tilde{u} satisfies

$$\int_{B_1} \tilde{a}_{ij} D_i \tilde{u} D_j \varphi \leq \int_{B_1} \tilde{f} \varphi \quad \text{for any } \varphi \in H_0^1(B_1) \text{ and } \varphi \geq 0 \text{ in } B_1,$$

where

$$\tilde{a}_{ij}(y) = a_{ij}(Ry), \quad \tilde{f}(y) = R^2 f(Ry) \quad \text{for any } y \in B_1.$$

We may apply what we just proved to \tilde{u} in B_1 and rewrite the result in terms of u . Hence, we obtain for $p \geq 2$

$$\sup_{B_{\frac{R}{2}}} u^+ \leq C \left\{ \frac{1}{R^{n/p}} \|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\},$$

where C is a positive constant depending only on n, λ, Λ, p and q .

The estimate in $B_{\theta R}$ can be obtained by applying the above result to $B_{(1-\theta)R}(y)$ for any $y \in B_{\theta R}$. By taking $R = 1$, we have Theorem 4.1 for any $\theta \in (0, 1)$ and $p \geq 2$.

Now we prove the statement for $p \in (0, 2)$. We showed that for any $\theta \in (0, 1)$ and $0 < R \leq 1$

$$\begin{aligned} \|u^+\|_{L^\infty(B_{\theta R})} &\leq C \left\{ \frac{1}{[(1-\theta)R]^{\frac{n}{2}}} \|u^+\|_{L^2(B_R)} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\} \\ &\leq C \left\{ \frac{1}{[(1-\theta)R]^{\frac{n}{2}}} \|u^+\|_{L^2(B_R)} + \|f\|_{L^q(B_1)} \right\}. \end{aligned}$$

For $p \in (0, 2)$, we have

$$\int_{B_R} (u^+)^2 \leq \|u^+\|_{L^\infty(B_R)}^{2-p} \int_{B_R} (u^+)^p,$$

and hence by the Hölder inequality

$$\begin{aligned} \|u^+\|_{L^\infty(B_{\theta R})} &\leq C \left\{ \frac{1}{[(1-\theta)R]^{\frac{n}{2}}} \|u^+\|_{L^\infty(B_R)}^{1-\frac{p}{2}} \left(\int_{B_R} (u^+)^p dx \right)^{\frac{1}{2}} + \|f\|_{L^q(B_R)} \right\} \\ &\leq \frac{1}{2} \|u^+\|_{L^\infty(B_R)} + C \left\{ \frac{1}{[(1-\theta)R]^{\frac{n}{p}}} \left(\int_{B_R} (u^+)^p \right)^{\frac{1}{p}} + \|f\|_{L^q(B_R)} \right\}. \end{aligned}$$

Set $h(t) = \|u^+\|_{L^\infty(B_t)}$ for $t \in (0, 1]$. Then, we obtain for any $0 < r < R \leq 1$

$$h(r) \leq \frac{1}{2} h(R) + \frac{C}{(R-r)^{\frac{n}{p}}} \|u^+\|_{L^p(B_1)} + C \|f\|_{L^q(B_1)}.$$

We apply the following lemma to get for any $0 < r < R < 1$

$$h(r) \leq \frac{C}{(R-r)^{\frac{n}{p}}} \|u^+\|_{L^p(B_1)} + C \|f\|_{L^q(B_1)}.$$

By letting $R \rightarrow 1-$, we get for any $\theta < 1$

$$\|u^+\|_{L^\infty(B_\theta)} \leq \frac{C}{(1-\theta)^{\frac{n}{p}}} \|u^+\|_{L^p(B_1)} + C \|f\|_{L^q(B_1)}.$$

This finishes the proof. \square

We need the following simple lemma.

Lemma 4.2. *Let $h(t) \geq 0$ be bounded in $[\tau_0, \tau_1]$ with $\tau_0 \geq 0$. Suppose for any $\tau_0 \leq t < s \leq \tau_1$*

$$h(t) \leq \theta h(s) + \frac{A}{(s-t)^\alpha} + B,$$

for some $\theta \in [0, 1)$. Then for any $\tau_0 \leq t < s \leq \tau_1$

$$h(t) \leq c \left\{ \frac{A}{(s-t)^\alpha} + B \right\},$$

where c is a positive constant depending only on α and θ .

Proof. Fix $\tau_0 \leq t < s \leq \tau_1$. For some $0 < \tau < 1$, we consider the sequence $\{t_i\}$ defined by

$$t_0 = t \text{ and } t_{i+1} = t_i + (1 - \tau)\tau^i(s - t).$$

Note $t_\infty = s$. By an iteration, we get

$$h(t) = h(t_0) \leq \theta^k h(t_k) + \left[\frac{A}{(1 - \tau)^\alpha} (s - t)^{-\alpha} + B \right] \sum_{i=0}^{k-1} \theta^i \tau^{-i\alpha}.$$

Choose $\tau < 1$ such that $\theta\tau^{-\alpha} < 1$, i.e., $\theta < \tau^\alpha < 1$. As $k \rightarrow \infty$, we have

$$h(t) \leq c(\alpha, \theta) \left\{ \frac{A}{(1 - \tau)^\alpha} (s - t)^{-\alpha} + B \right\}.$$

This finishes the proof. \square

Remark 4.3. If the subsolution u is bounded, we simply take the test function

$$\varphi = \eta^2(\bar{u}^{\beta+1} - k^{\beta+1}) \in H_0^1(B_1),$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$.

The next result is referred to as the weak Harnack inequality.

Theorem 4.4. *Let $u \in H^1(\Omega)$ be a nonnegative supersolution in Ω in the following sense*

$$(*) \quad \int_{\Omega} a_{ij} D_i u D_j \varphi \geq \int_{\Omega} f \varphi \quad \text{for any } \varphi \in H_0^1(\Omega) \text{ with } \varphi \geq 0 \text{ in } \Omega.$$

Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then for any $B_R \subset \Omega$, any $0 < p < n/(n - 2)$ and any $0 < \theta < \tau < 1$

$$\inf_{B_{\theta R}} u + R^{2 - \frac{n}{q}} \|f\|_{L^q(B_R)} \geq C \left(\frac{1}{R^n} \int_{B_{\tau R}} u^p \right)^{\frac{1}{p}},$$

where C is a positive constant depending only on $n, p, q, \lambda, \Lambda, \theta$ and τ .

Proof. We only prove for $R = 1$.

Step 1. We prove that the result holds for some $p_0 > 0$.

Set $\bar{u} = u + k > 0$, for some $k > 0$ to be determined and $v = \bar{u}^{-1}$. First, we derive an equation for v . For any $\varphi \in H_0^1(B_1)$ with $\varphi \geq 0$ in B_1 , consider $\bar{u}^{-2}\varphi$ as the test function in (*). We have

$$\int_{B_1} a_{ij} D_i u \frac{D_j \varphi}{\bar{u}^2} - 2 \int_{B_1} a_{ij} D_i u D_j \bar{u} \frac{\varphi}{\bar{u}^3} \geq \int_{B_1} f \frac{\varphi}{\bar{u}^2}.$$

Note $D\bar{u} = Du$ and $Dv = -\bar{u}^{-2}D\bar{u}$. Therefore, we obtain

$$\int_{B_1} a_{ij} D_j v D_i \varphi + \tilde{f} v \varphi \leq 0,$$

where

$$\tilde{f} = \frac{f}{\bar{u}}.$$

In other words, v is a nonnegative subsolution of some homogeneous equation. Choose $k = \|f\|_{L^q}$ if f is not identical zero. Otherwise, choose an arbitrary $k > 0$ and then let $k \rightarrow 0+$. Note $\|\tilde{f}\|_{L^q(B_1)} \leq 1$. Theorem 3.1 implies that for any $\tau \in (\theta, 1)$ and any $p > 0$

$$\sup_{B_\theta} \bar{u}^{-p} \leq C \int_{B_\tau} \bar{u}^{-p},$$

or,

$$\inf_{B_\theta} \bar{u} \geq C \left(\int_{B_\tau} \bar{u}^{-p} dx \right)^{-\frac{1}{p}} = C \left(\int_{B_\tau} \bar{u}^{-p} \int_{B_\tau} \bar{u}^p \right)^{-\frac{1}{p}} \left(\int_{B_\tau} \bar{u}^p \right)^{\frac{1}{p}},$$

where C is a positive constant depending only on $n, q, p, \lambda, \Lambda, \tau$ and θ .

The key point next is to show that there exists a $p_0 > 0$ such that

$$\int_{B_\tau} \bar{u}^{-p_0} \cdot \int_{B_\tau} \bar{u}^{p_0} \leq C,$$

where C is a positive constant depending only on n, q, λ, Λ and τ . We will show for any $\tau < 1$

$$(1) \quad \int_{B_\tau} e^{p_0|w|} \leq C,$$

where $w = \log \bar{u} - \beta$ with $\beta = |B_\tau|^{-1} \int_{B_\tau} \log \bar{u}$.

We have two methods to proceed:

- (i) To prove directly.
- (ii) To prove that w is BMO, i.e., for any $B_r(y) \subset B_1$,

$$\frac{1}{r^n} \int_{B_r} |w - w_{y,r}| dx \leq C.$$

Then (1) follows from Theorem 1.3 (John-Nirenberg Lemma).

We first prove (1) directly. Recall $\bar{u} = u + k \geq k > 0$. Note that

$$e^{p_0|w|} = 1 + p_0|w| + \frac{(p_0|w|)^2}{2!} + \dots + \frac{(p_0|w|)^n}{n!} + \dots.$$

Hence we need to estimate

$$\int_{B_\tau} |w|^\beta,$$

for each positive integer β .

We first derive an equation for w . Consider $\bar{u}^{-1}\varphi$ as the test function in (*). Here we need $\varphi \in L^\infty(B_1) \cap H_0^1(B_1)$ with $\varphi \geq 0$. By a direct calculation as before and by the fact $Dw = \bar{u}^{-1}D\bar{u}$, we have

$$(2) \quad \int_{B_1} a_{ij} D_i w D_j w \varphi \leq \int_{B_1} a_{ij} D_i w D_j \varphi + \int_{B_1} (-\tilde{f}\varphi)$$

for any $\varphi \in L^\infty(B_1) \cap H_0^1(B_1)$ with $\varphi \geq 0$.

Replace φ by φ^2 in (2). Then the Cauchy inequality implies

$$\int_{B_1} |Dw|^2 \varphi^2 \leq C \left\{ \int_{B_1} |D\varphi|^2 + \int_{B_1} |\tilde{f}| \varphi^2 \right\}.$$

By the Hölder inequality and the Sobolev inequality, we obtain

$$\int_{B_1} |\tilde{f}|\varphi^2 \leq \|\tilde{f}\|_{L^{n/2}} \|\varphi\|_{L^{2n/(n-2)}}^2 \leq c(n, q) \|D\varphi\|_{L^2}^2.$$

Therefore, we have

$$(3) \quad \int_{B_1} |Dw|^2 \varphi^2 \leq C \int_{B_1} |D\varphi|^2,$$

where C is a positive constant depending only on n, q, λ and Λ . Take $\varphi \in C_0^1(B_1)$ with $\varphi \equiv 1$ in B_τ . Then we obtain

$$(4) \quad \int_{B_\tau} |Dw|^2 \leq C,$$

where C is a positive constant depending only on n, q, λ, Λ and τ . Hence the Poincaré inequality implies

$$\int_{B_\tau} w^2 \leq c(n, \tau) \int_{B_\tau} |Dw|^2 \leq C,$$

since $\int_{B_\tau} w = 0$. By (3), we conclude for any $\tau' \in (\tau, 1)$

$$(5) \quad \int_{B_{\tau'}} w^2 \leq C,$$

where C is a positive constant depending only on $n, q, \lambda, \Lambda, \tau$ and τ' .

Next, we estimate $\int_{B_\tau} |w|^\beta$ for any $\beta \geq 2$. Choose $\varphi = \zeta^2 |w_m|^{2\beta} \in H_0^1(B_1) \cap L^\infty(B_1)$ with

$$w_m = \begin{cases} -m & w \leq -m \\ w & |w| < m \\ m & w \geq m. \end{cases}$$

Substitute such a φ in (2) to get

$$\begin{aligned} \int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w &\leq (2\beta) \int_{B_1} \zeta^2 a_{ij} D_i w D_j |w_m| |w_m|^{2\beta-1} \\ &+ \int_{B_1} 2\zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}. \end{aligned}$$

Note

$$a_{ij} D_i w D_j |w_m| = a_{ij} D_i w_m D_j |w_m| \leq a_{ij} D_i w_m D_j w_m \quad \text{a.e. in } B_1.$$

The Hölder inequality implies

$$\begin{aligned} (2\beta) |w_m|^{2\beta-1} &\leq \frac{2\beta-1}{2\beta} |w_m|^{2\beta} + \frac{1}{2\beta} (2\beta)^{2\beta} \\ &= \left(1 - \frac{1}{2\beta}\right) |w_m|^{2\beta} + (2\beta)^{2\beta-1}. \end{aligned}$$

We obtain

$$\begin{aligned} \int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w &\leq \left(1 - \frac{1}{2\beta}\right) \int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w_m D_j w_m \\ &\quad + (2\beta)^{2\beta-1} \int_{B_1} \zeta^2 a_{ij} D_i w_m D_j w_m \\ &\quad + \int_{B_1} 2\zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}, \end{aligned}$$

and hence

$$\begin{aligned} \int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w &\leq (2\beta)^{2\beta} \int_{B_1} \zeta^2 a_{ij} D_i w_m D_j w_m \\ &\quad + 4\beta \int_{B_1} \zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + 2\beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw|^2 &\leq C \left\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 \right. \\ &\quad \left. + \beta \int_{B_1} \zeta |w_m|^{2\beta} |Dw| |D\zeta| + \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta} \right\}. \end{aligned}$$

Note that the first term in the right hand side is bounded in (4). Applying the Cauchy inequality to the second term in the right hand side, we conclude

$$\begin{aligned} \int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw|^2 &\leq C \left\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 \right. \\ &\quad \left. + \beta^2 \int_{B_1} |w_m|^{2\beta} |D\zeta|^2 + \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta} \right\}. \end{aligned}$$

Note $Dw = Dw_m$ for $|w| < m$ and $Dw_m = 0$ for $|w| > m$. Hence we have

$$\begin{aligned} \int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw_m|^2 &\leq C \left\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 \right. \\ &\quad \left. + \beta^2 \int_{B_1} |w_m|^{2\beta} |D\zeta|^2 + \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta} \right\}. \end{aligned}$$

In the following, we write $w = w_m$ and then let $m \rightarrow +\infty$ at the end. By the Hölder inequality, we obtain

$$\begin{aligned} |D(\zeta|w|^\beta)|^2 &\leq 2|D\zeta|^2 |w|^{2\beta} + 2\beta^2 \zeta^2 |w|^{2\beta-2} |Dw|^2 \\ &\leq 2|D\zeta|^2 |w|^{2\beta} + 2\zeta^2 |Dw|^2 \left(\frac{\beta-1}{\beta} |w|^{2\beta} + \frac{1}{\beta} \beta^{2\beta} \right), \end{aligned}$$

and hence

$$\begin{aligned} \int_{B_1} |D(\zeta|w|^\beta)|^2 &\leq C \left\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw|^2 \right. \\ &\quad \left. + \beta^2 \int_{B_1} |D\zeta|^2 |w|^{2\beta} + \beta \int_{B_1} |\tilde{f}|\zeta^2 |w|^{2\beta} \right\}. \end{aligned}$$

Then the Hölder inequality implies

$$\int_{B_1} |\tilde{f}|\zeta^2 |w|^{2\beta} \leq \left(\int_{B_1} |\tilde{f}|^q \right)^{\frac{1}{q}} \left(\int_{B_1} (\zeta|w|^\beta)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}}.$$

Note $2^* = \frac{2n}{n-2} > \frac{2q}{q-1} > 2$ if $q > n/2$. By the interpolation inequality and the Sobolev inequality, we have for any small $\varepsilon > 0$

$$\begin{aligned} \|\zeta|w|^\beta\|_{L^{\frac{2q}{q-1}}} &\leq \varepsilon \|\zeta|w|^\beta\|_{L^{2^*}} + C(n, q) \varepsilon^{-\frac{n}{2q-n}} \|\zeta|w|^\beta\|_{L^2} \\ &\leq \varepsilon \|D(\zeta|w|^\beta)\|_{L^2} + C(n, q) \varepsilon^{-\frac{n}{2q-n}} \|\zeta|w|^\beta\|_{L^2}. \end{aligned}$$

Therefore, we obtain by (3)

$$\begin{aligned} \int_{B_1} |D(\zeta|w|^\beta)|^2 &\leq C \left\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw|^2 + \beta^\alpha \int_{B_1} (|D\zeta|^2 + \zeta^2) |w|^{2\beta} \right\} \\ &\leq C \left\{ (2\beta)^{2\beta} \int_{B_1} |D\zeta|^2 + \beta^\alpha \int_{B_1} (|D\zeta|^2 + \zeta^2) |w|^{2\beta} \right\}, \end{aligned}$$

for some positive constant α depending only on n and q . Apply the Sobolev inequality for $\zeta|w|^\beta \in W_0^{1,2}(\mathbb{R}^n)$ with $\chi = \frac{n}{n-2}$ to get

$$\left(\int_{B_1} \zeta^{2\chi} |w|^{2\beta\chi} \right)^{\frac{1}{\chi}} \leq C\beta^\alpha \left\{ (2\beta)^{2\beta} \int_{B_1} |D\zeta|^2 + \int_{B_1} (|D\zeta|^2 + \zeta^2) |w|^{2\beta} \right\}.$$

Choose a cut-off function ζ as follows. For $\tau \leq r < R \leq 1$, set $\zeta \equiv 1$ on B_r , $\zeta \equiv 0$ in $B_1 \setminus B_R$ and $|D\zeta| \leq \frac{2}{R-r}$. Therefore, we have

$$\left(\int_{B_r} |w|^{2\beta\chi} \right)^{\frac{1}{\chi}} \leq \frac{C\beta^\alpha}{(R-r)^2} \left\{ (2\beta)^{2\beta} + \int_{B_R} |w|^{2\beta} \right\}.$$

For some $\tau' \in (\tau, 1)$, set for any $i = 1, 2, \dots$

$$\beta_i = \chi^{i-1}, \quad \text{and} \quad r_i = \tau + \frac{1}{2^{i-1}}(\tau' - \tau).$$

Then for each $i = 1, 2, \dots$, we have

$$\left(\int_{B_{r_i}} |w|^{2\chi^i} \right)^{\frac{1}{\chi}} \leq \frac{C\chi^{(i-1)\alpha} 2^{2(i-1)}}{(\tau' - \tau)^2} \left\{ (2\chi^{i-1})^{2\chi^{i-1}} + \int_{B_{r_{i-1}}} |w|^{2\chi^{i-1}} \right\}.$$

Set

$$I_j = \|w\|_{L^{2\chi^j}(B_{r_j})}.$$

Then, we have for $j = 1, 2, \dots$,

$$I_j \leq C \overline{2\chi^j} \{2\chi^{j-1} + I_{j-1}\},$$

where C is a positive constant depending only on $n, q, \lambda, \Lambda, \tau$ and τ' . Iterating the above inequality and observing that

$$\sum_{i=0}^{\infty} \frac{i}{\chi^i} < \infty,$$

we obtain

$$I_j \leq C \sum_{i=1}^j \chi^{i-1} + CI_0,$$

or,

$$I_j \leq C\chi^j + CI_0.$$

Now for $\beta \geq 2$, there exists a j such that $2\chi^{j-1} \leq \beta < 2\chi^j$. Hence, we have

$$\begin{aligned} I_\beta(B_\tau) &\equiv \left(\int_{B_\tau} |w|^\beta \right)^{\frac{1}{\beta}} \leq CI_j \leq C\chi^j + CI_0 \\ &\leq C\beta + CI_0 \leq C_0\beta, \end{aligned}$$

since I_0 is bounded in (5). Then, we obtain for $\beta \geq 1$

$$\int_{B_\tau} |w|^\beta dx \leq C_0^\beta \beta^\beta \leq C_0^\beta e^\beta \beta!,$$

where we used the Sterling formula for the integer β . Hence, for any integer $\beta \geq 1$, we conclude

$$\int_{B_\tau} \frac{(p_0|w|)^\beta}{\beta!} \leq p_0^\beta (C_0 e)^\beta \leq \frac{1}{2^\beta},$$

by choosing $p_0 = (2C_0 e)^{-1}$. This proves that

$$\begin{aligned} \int e^{p_0|w|} &= \int 1 + p_0|w| + \frac{(p_0|w|)^2}{2!} + \dots \\ &\leq 1 + \frac{1}{2^1} + \frac{1}{2^2} + \dots \leq 2. \end{aligned}$$

We remark that the above method is elementary in nature.

Now we give the second proof of the estimate (1) by using BMO. The estimate (3) yields

$$\int_{B_1} |Dw|^2 \zeta^2 \leq C \int_{B_1} |D\zeta|^2 \quad \text{for any } \zeta \in C_0^1(B_1).$$

Then for any $B_{2r}(y) \subset B_1$, choose ζ with

$$\text{supp } \zeta \subset B_{2r}(y), \quad \zeta \equiv 1 \text{ in } B_r(y), \quad |D\zeta| \leq \frac{2}{r}.$$

We obtain

$$\int_{B_r(y)} |Dw|^2 \leq Cr^{n-2}.$$

Hence the Poincaré inequality implies

$$\begin{aligned} \frac{1}{r^n} \int_{B_r(y)} |w - w_{y,r}| &\leq \frac{1}{r^{\frac{n}{2}}} \left(\int_{B_r(y)} |w - w_{y,r}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{r^{\frac{n}{2}}} \left(r^2 \int_{B_r(y)} |Dw|^2 \right)^{\frac{1}{2}} \leq C, \end{aligned}$$

or, $w \in BMO$. Then the John-Nirenberg Lemma implies

$$\int_{B_r} e^{p_0|w|} \leq C.$$

Step 2. The result holds for any positive $p < n/(n-2)$.

We need to prove, for any $0 < r_1 < r_2 < 1$ and $0 < p_2 < p_1 < n/(n-2)$,

$$(6) \quad \left(\int_{B_{r_1}} \bar{u}^{p_1} \right)^{\frac{1}{p_1}} \leq C \left(\int_{B_{r_2}} \bar{u}^{p_2} \right)^{\frac{1}{p_2}},$$

where C is positive constant depending only on $n, q, \lambda, \Lambda, r_1, r_2, p_1$ and p_2 . A similar calculation may be found before. Here, we just point out some key steps.

Take $\varphi = \bar{u}^{-\beta} \eta^2$ for $\beta \in (0, 1)$ as the test function in (*). Then we have

$$\int_{B_1} |D\bar{u}|^2 \bar{u}^{-\beta-1} \eta^2 \leq C \left\{ \frac{1}{\beta^2} \int_{B_1} |D\eta|^2 \bar{u}^{1-\beta} + \frac{1}{\beta} \int_{B_1} \frac{|f|}{k} \eta^2 \bar{u}^{1-\beta} \right\}.$$

Set $\gamma = 1 - \beta \in (0, 1)$ and $w = \bar{u}^{\frac{\gamma}{2}}$. Then, we obtain

$$\int |Dw|^2 \eta^2 \leq \frac{C}{(1-\gamma)^\alpha} \int w^2 (|D\eta|^2 + \eta^2),$$

or

$$\int |D(w\eta)|^2 \leq \frac{C}{(1-\gamma)^\alpha} \int w^2 (|D\eta|^2 + \eta^2),$$

for some positive $\alpha > 0$. The Sobolev embedding theorem and an appropriate choice of cut-off functions imply, with $\chi = n/n-2$, for any $0 < r < R < 1$

$$\left(\int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \leq \frac{C}{(1-\gamma)^\alpha} \cdot \frac{1}{(R-r)^2} \int_{B_R} w^2,$$

or

$$\left(\int_{B_r} \bar{u}^{\gamma\chi} \right)^{\frac{1}{\gamma\chi}} \leq \left(\frac{C}{(1-\gamma)^\alpha} \frac{1}{(R-r)^2} \right)^{\frac{1}{\gamma}} \left(\int_{B_R} \bar{u}^\gamma \right)^{\frac{1}{\gamma}}.$$

This holds for any $\gamma \in (0, 1)$. Now (6) follows after finitely many iterations. \square

Now the Moser's Harnack inequality is an easy consequence of above results.

Theorem 4.5. *Let $u \in H^1(\Omega)$ be a nonnegative solution in Ω*

$$\int_{\Omega} a_{ij} D_i u D_j \varphi = \int_{\Omega} f \varphi \quad \text{for any } \varphi \in H_0^1(\Omega).$$

Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then for any $B_R \subset \Omega$,

$$\sup_{B_{\frac{R}{2}}} u \leq C \left\{ \inf_{B_{\frac{R}{2}}} u + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\},$$

where C is a positive constant depending only on n, λ, Λ and q .

The Hölder continuity follows easily from Theorem 4.5.

Corollary 4.6. *Let $u \in H^1(\Omega)$ be a solution in Ω , i.e.,*

$$\int_{\Omega} a_{ij} D_i u D_j \varphi = \int_{\Omega} f \varphi \quad \text{for any } \varphi \in H_0^1(\Omega).$$

Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then $u \in C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$ depending only on n, q, λ and Λ . Moreover, for any $B_R \subset \Omega$

$$|u(x) - u(y)| \leq C \left(\frac{|x - y|}{R} \right)^\alpha \left\{ \left(\frac{1}{R^n} \int_{B_R} u^2 \right)^{\frac{1}{2}} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\}$$

for any $x, y \in B_{\frac{R}{2}}$,

where C is a positive constant depending only on n, λ, Λ and q .

Proof. We prove the estimate for $R = 1$. Set for $r \in (0, 1)$

$$M(r) = \sup_{B_r} u \quad \text{and} \quad m(r) = \inf_{B_r} u.$$

Then $M(r) < +\infty$ and $m(r) > -\infty$. It suffices to show

$$\omega(r) \equiv M(r) - m(r) \leq C r^\alpha \{ \|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)} \} \quad \text{for any } r < \frac{1}{2}.$$

Set $\delta = 2 - \frac{n}{q}$. Apply Theorem 4.5 to $M(r) - u \geq 0$ in B_r to get

$$\sup_{B_{\frac{r}{2}}} (M(r) - u) \leq C \left\{ \inf_{B_{\frac{r}{2}}} (M(r) - u) + r^\delta \|f\|_{L^q(B_r)} \right\},$$

i.e.,

$$(1) \quad M(r) - m\left(\frac{r}{2}\right) \leq C \left\{ \left(M(r) - M\left(\frac{r}{2}\right) \right) + r^\delta \|f\|_{L^q(B_r)} \right\}.$$

Similarly, apply Theorem 4.5 to $u - m(r) \geq 0$ in B_r to get

$$(2) \quad M\left(\frac{r}{2}\right) - m(r) \leq C \left\{ \left(m\left(\frac{r}{2}\right) - m(r) \right) + r^\delta \|f\|_{L^q(B_r)} \right\}.$$

Then by adding (1) and (2) together, we obtain

$$\omega(r) + \omega\left(\frac{r}{2}\right) \leq C \left\{ \left(\omega(r) - \omega\left(\frac{r}{2}\right) \right) + r^\delta \|f\|_{L^q(B_r)} \right\},$$

or

$$\omega\left(\frac{r}{2}\right) \leq \gamma\omega(r) + Cr^\delta \|f\|_{L^q(B_r)},$$

for some $\gamma = \frac{C-1}{C+1} < 1$.

Apply Lemma 4.7 below with μ chosen such that

$$\alpha = (1 - \mu) \log \gamma / \log \tau < \mu\delta.$$

We obtain

$$\omega(\rho) \leq C\rho^\alpha \left\{ \omega\left(\frac{1}{2}\right) + \|f\|_{L^q(B_1)} \right\} \quad \text{for any } \rho \in (0, \frac{1}{2}].$$

While Theorem 4.1 implies

$$\omega\left(\frac{1}{2}\right) \leq C \{ \|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)} \}.$$

This finishes the proof. \square

Lemma 4.7. *Let ω and σ be non-decreasing functions in an interval $(0, R]$. Suppose for some $0 < \gamma, \tau < 1$*

$$\omega(\tau r) \leq \gamma\omega(r) + \sigma(r) \quad \text{for any } r \leq R.$$

Then, for any $\mu \in (0, 1)$ and $r \leq R$,

$$\omega(r) \leq C \left\{ \left(\frac{r}{R}\right)^\alpha \omega(R) + \sigma(r^\mu R^{1-\mu}) \right\},$$

where C is a positive constant depending only on γ, τ and $\alpha = (1 - \mu) \log \gamma / \log \tau$.

Proof. Fix some $r_1 \leq R$. Then for any $r \leq r_1$, we have

$$\omega(\tau r) \leq \gamma\omega(r) + \sigma(r_1),$$

since σ is nondecreasing. We now iterate this inequality to get for any positive integer k

$$\omega(\tau^k r_1) \leq \gamma^k \omega(r_1) + \sigma(r_1) \sum_{i=0}^{k-1} \gamma^i \leq \gamma^k \omega(R) + \frac{\sigma(r_1)}{1 - \gamma}.$$

For any $r \leq r_1$, we choose k such that

$$\tau^k r_1 < r \leq \tau^{k-1} r_1.$$

Hence, we have

$$\begin{aligned} \omega(r) &\leq \omega(\tau^{k-1} r_1) \leq \gamma^{k-1} \omega(R) + \frac{\sigma(r_1)}{1 - \gamma} \\ &\leq \frac{1}{\gamma} \left(\frac{r}{r_1}\right)^{\log \gamma / \log \tau} \omega(R) + \frac{\sigma(r_1)}{1 - \gamma}. \end{aligned}$$

By letting $r_1 = r^\mu R^{1-\mu}$, we obtain

$$\omega(r) \leq \frac{1}{\gamma} \left(\frac{r}{R}\right)^{(1-\mu)(\log \gamma / \log \tau)} \omega(R) + \frac{\sigma(r^\mu R^{1-\mu})}{1 - \gamma}.$$

This finishes the proof. \square

We also have the following Liouville theorem.

Corollary 4.8. *Suppose u is a solution of a homogeneous equation in \mathbb{R}^n , i.e.,*

$$\int_{\mathbb{R}^n} a_{ij} D_i u D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1(\mathbb{R}^n).$$

If u is bounded, then u is a constant.

Proof. As in the proof of Corollary 4.6, we have for some $\gamma < 1$

$$\omega(r) \leq \gamma \omega(2r) \quad \text{for any } r > 0.$$

By an iteration, we obtain

$$\omega(r) \leq \gamma^k \omega(2^k r) \leq C \gamma^k,$$

since u is bounded. Hence by letting $k \rightarrow \infty$, we conclude

$$\omega(r) = 0 \quad \text{for any } r > 0.$$

Therefore, u is constant. □

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