

# DIRICHLET PROBLEMS OF MONGE-AMPÈRE EQUATIONS

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This note presents a detailed and self-contained discussion of the Dirichlet problem of real Monge-Ampère equations in strictly convex domains and complex Monge-Ampère equations in strongly pseudo-convex domains. Sections 1.1 and 1.2 follow [2] and [3] respectively, while Sections 2.1, 2.2 and 2.3 are based on [5], [4] and [1] respectively.

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## 1. GLOBAL $C^2$ -ESTIMATES FOR MONGE-AMPÈRE EQUATIONS

In this chapter, we use the method of continuity to discuss the solvability of Monge-Ampère equations. We will derive a priori estimates for solutions and its derivatives up to the second order.

**1.1. Real Monge-Ampère Equations.** Suppose  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ . The Monge-Ampère operator  $M$  is defined by

$$M(u) = \det(u_{ij}),$$

for any  $u \in C^\infty(\Omega)$ . Obviously,  $M(u) \geq 0$  if  $u$  is convex, and  $M(u) > 0$  if  $u$  is strictly convex.

For strictly convex function  $u$ , it is convenient to introduce

$$F(D^2u) \equiv \log \det(u_{ij}).$$

We claim

$$\begin{aligned} F_{ij} &\equiv \frac{\partial F}{\partial u_{ij}} = u^{ij}, \\ F_{ij,kl} &\equiv \frac{\partial^2 F}{\partial u_{ij} \partial u_{kl}} = -u^{ik}u^{jl}, \end{aligned}$$

where  $(u^{ij})$  denotes the inverse of the Hessian matrix  $(u_{ij})$ . To see this, we denote by  $A = \{A^{ij}\}$  the cofactor matrix of the Hessian matrix  $H = (u_{ij})$ , i.e.,  $A = (\det H)H^{-1}$ . For a fixed  $i = 1, \dots, n$ , we expand the determinant according to the  $i$ -th row,

$$\det D^2u = A^{il}u_{il} + \dots + A^{in}u_{in}.$$

Then it is easy to see

$$\frac{\partial F}{\partial u_{ij}} = \frac{1}{\det D^2u} \cdot A^{ij} = u^{ij}.$$

Next, for fixed  $i, j = 1, \dots, n$ , we have

$$u^{ik}u_{jk} = \delta_j^i.$$

Differentiating with respect to  $u_{pq}$ , we get

$$(u^{ik})_{u_{pq}}u_{jk} + u^{ik}(u_{jk})_{u_{pq}} = 0.$$

Multiplying  $u^{jl}$  and summing over  $j$ , we have

$$(u^{il})_{u_{pq}} = (u^{ik})_{u_{pq}}u_{jk}u^{jl} = -u^{ik}u^{jl}(u_{jk})_{u_{pq}} = -u^{iq}u^{pl},$$

or

$$\frac{\partial u^{ij}}{\partial u_{kl}} = -u^{il}u^{kj}.$$

Hence we obtain

$$\frac{\partial^2 F}{\partial u_{ij} \partial u_{kl}} = \frac{\partial u^{ij}}{\partial u_{kl}} = -u^{il}u^{kj}.$$

We now show that  $F$  is a concave function of its argument, the positive definite matrices  $D^2u = (u_{ij})$ . This means

$$\frac{\partial^2 F}{\partial u_{ij} \partial u_{kl}} m_{ij} m_{kl} \leq 0,$$

for any symmetric matrices  $M = (m_{ij})$ . To see this, we diagonalize the matrix  $(u_{ij})$ . Then  $(u^{ij})$  is a diagonal matrix  $\text{diag}(\lambda^1, \dots, \lambda^n)$ , with  $\lambda^i > 0, i = 1, \dots, n$ . Hence, we have

$$\frac{\partial^2 F}{\partial u_{ij} \partial u_{kl}} m_{ij} m_{kl} = -u^{il} u^{kj} m_{ij} m_{kl} = -\lambda^i \lambda^j m_{ij}^2 \leq 0.$$

Before discussing further Monge-Ampère equations, we give a simple result on positive matrices, which will be needed later. If  $H = (u_{ij})$  is a positive matrix, then there holds

$$|u_{ij}| \leq \frac{1}{2}(u_{ii} + u_{jj}).$$

This can be seen easily as follows. Since  $H$  is positive, any  $2 \times 2$  diagonal minor has positive determinant. This implies

$$u_{ij}^2 \leq u_{ii} u_{jj}.$$

Then Cauchy inequality implies the desired result.

Now we return to the Monge-Ampère equation

$$\det(u_{ij}) = f.$$

We write it as

$$F(D^2u) = \log \det(u_{ij}) = \log f,$$

for strictly convex  $u$ .

Suppose  $\partial$  is an arbitrary directional derivative in  $\mathbb{R}^n$ . Differentiating the equation with  $\partial$ , we obtain

$$u^{ij} \partial u_{ij} = \partial \log f.$$

This leads to the linear differential operator

$$L \equiv u^{ij} \partial_{ij}.$$

Since  $u$  is strictly convex,  $L$  is elliptic. We get

$$L(\partial u) = \partial \log f.$$

Differentiate again with  $\partial$ . We obtain

$$L(\partial^2 u) - u^{il} u^{kj} \partial u_{ij} \partial u_{kl} = \partial^2 \log f,$$

or

$$L(\partial^2 u) = u^{il} u^{kj} \partial u_{ij} \partial u_{kl} + \partial^2 \log f.$$

The first term in the right side is positive, since  $u$  is strictly convex. Then we get

$$L(\partial^2 u) \geq \partial^2 \log f.$$

We now use the method of continuity to discuss the Dirichlet problem in a strictly convex domain  $\Omega \subset \mathbb{R}^n$ . We consider

$$(*) \quad \begin{aligned} \det u_{ij} &= f(x) \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

where  $f \in C^\infty(\bar{\Omega})$ ,  $f > 0$  in  $\bar{\Omega}$  and  $\varphi \in C^\infty(\partial\Omega)$ .

Let  $u^0 \in C^\infty(\bar{\Omega})$  be a strictly convex function with  $u^0 = \varphi$  on  $\partial\Omega$ . We may easily find such a function  $u^0$  satisfying, in addition,

$$f^0 \equiv \det(u_{ij}^0) \geq f \quad \text{in } \Omega.$$

Such a condition will be needed later in the a priori estimate of  $u$  itself.

For each  $t \in [0, 1]$ , we intend to find a strictly convex solution  $u^t \in C^{2+\alpha}(\bar{\Omega})$  of

$$(*)_t \quad \begin{aligned} \det(u_{ij}^t) &= tf + (1-t)f^0 \quad \text{in } \Omega \\ u^t &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

We set

$$I = \{t \in [0, 1]; (*)_t \text{ has a strictly convex solution } u^t \in C^{2+\alpha}(\bar{\Omega})\}.$$

Obviously  $0 \in I$ , since  $(*)_0$  has a solution  $u^0$ . Now we prove  $I$  is open. We rewrite the equation in  $(*)_t$  as

$$G(u, t) = \det(u_{ij}) - tf - (1-t)f^0.$$

We find the Frèchet derivative of  $G$  at  $u$  given by

$$G_u v = \det(u_{ij}) u^{ij} \partial_{ij} v.$$

Since  $u$  is a strictly convex  $C^{2+\alpha}$  function,  $G_u$  is a uniformly elliptic linear operator with  $C^\alpha$  coefficients. By the classical Schauder theory,  $G_u$  is an invertible operator with any fixed boundary condition. Suppose  $t_0 \in I$ , i.e.,  $G(u^{t_0}, t_0) = 0$  for some strictly convex function  $u^{t_0} \in C^{2+\alpha}(\bar{\Omega})$ . By the implicit function theorem, for any  $t$  close to  $t_0$ , there is a unique  $u^t \in C^{2+\alpha}(\bar{\Omega})$ , close to  $u^{t_0}$  in  $C^{2+\alpha}$ -norm, satisfying  $G(u^t, t) = 0$ . Obviously,  $u^t$  is strictly convex for  $t$  close to  $t_0$ . Hence  $t \in I$  for all such  $t$ , and therefore  $I$  is open.

If we can establish the a priori estimate

$$|u^t|_{2+\alpha} \leq K, \quad \text{independent of } t,$$

it follows that  $I$  is also closed, by Arzela-Ascoli Theorem. Hence  $I$  is the whole unit interval. The function  $u^1$  is then our desired solution of  $(*)$ .

We will derive a priori estimates

$$|u|_{2+\alpha} \leq K$$

for solutions of  $(*)$ , which apply to solutions of  $(*)_t$ , for the constant  $K$  depending only on  $\Omega$ , the  $C^3$ -norm  $|f|_3$  of  $f$ ,  $\max f^{-1}$ , and  $|\varphi|_4$ .

We will derive the  $C^{2+\alpha}$  estimates in two steps. In the first step, we derive the  $C^2$  estimates

$$|u|_2 \leq K_2.$$

In the second step, we derive the  $C^\alpha$  estimates for  $D^2 u$

$$|D^2 u|_\alpha \leq K_{2,\alpha}.$$

The  $C^2$  estimate is based on the maximum principle. For the  $C^\alpha$  estimate of  $D^2u$ , we will derive it for solutions of the general elliptic equation of the concave type.

**Remark 1.1.** It follows from the standard elliptic theory that if  $f \in C^l(\bar{\Omega})$ ,  $\varphi \in C^{k+\alpha}(\partial\Omega)$  for integers  $k \geq 4$  and  $l \geq \max(3, k - 2 + \alpha)$ , then there is a solution  $u \in C^{k+\alpha}(\bar{\Omega})$ .

The uniqueness follows from the following comparison principle.

**Lemma 1.2.** *If  $\Omega \subset \mathbb{R}^n$  is a bounded domain and  $u, v \in C^2(\bar{\Omega})$  are convex functions satisfying*

$$\begin{aligned} \det(u_{ij}) &\geq \det(v_{ij}) \text{ in } \Omega, \\ u &\leq v \text{ on } \partial\Omega, \end{aligned}$$

then  $u \leq v$  in  $\Omega$ .

*Proof.* First, we assume  $u$  is strictly convex in  $\bar{\Omega}$ . Then

$$\begin{aligned} \det(u_{ij}) - \det(v_{ij}) &= \int_0^1 \frac{d}{dt} \det[(tu + (1-t)v)_{ij}] dt \\ &= \sum_{i,j} \int_0^1 a^{ij}(t) dt (u - v)_{ij} \geq 0, \end{aligned}$$

where  $a^{ij}(t)$  are cofactors of  $(tu + (1-t)v)_{ij}$ . The strict convexity of  $u$  implies that  $u - v$  is a subsolution of some uniformly elliptic equation. Hence  $u - v$  attains its maximum on  $\partial\Omega$ , which gives the desired result.

If  $u$  is only convex, we consider

$$u_\varepsilon = u + \varepsilon(|x|^2 - \max_{\partial\Omega} |x|^2)$$

for some positive  $\varepsilon$ . By comparing  $u_\varepsilon$  with  $v$  and letting  $\varepsilon \rightarrow 0$ , we get the result.  $\square$

**Theorem 1.3.** *Suppose that  $\Omega \subset \mathbb{R}^n$  is a strictly convex domain in  $\mathbb{R}^n$  with a smooth boundary and that  $u, f, \varphi$  are smooth functions in  $\bar{\Omega}$  such that  $u$  is strictly convex and  $f$  is positive in  $\bar{\Omega}$ . If  $u$  satisfies*

$$(*) \quad \begin{aligned} \det(u_{ij}) &= f \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

then there holds

$$|u|_2 \leq K,$$

where  $K$  is a positive constant depending only on  $\Omega$ , the  $C^3$ -norm  $|f|_3$  of  $f$ ,  $\max f^{-1}$ , and  $|\varphi|_4$ .

*Proof.* Let  $u^0 \in C^\infty(\bar{\Omega})$  be a strictly convex function which equals  $\varphi$  on  $\partial\Omega$  and satisfies

$$f^0 = \det(u_{ij}^0) \geq f \text{ in } \Omega.$$

We divide the proof into four steps:

Step 1. The estimate  $|u|$  in  $\Omega$ ;

Step 2. The estimate  $|\nabla u|$  in  $\Omega$ ;

Step 3. The estimate  $|D^2u|$  on  $\partial\Omega$ ;

Step 4. The estimate  $|D^2u|$  in  $\Omega$ .

Now, we carry out each step.

Step 1. Since  $u$  is convex, we have

$$u \leq \max_{\partial\Omega} \varphi.$$

Furthermore, it follows by Lemma 1.2

$$(1) \quad u^0 \leq u.$$

Here, we used the fact  $\det(u_{ij}^0) \geq f$ . Thus we have

$$(2) \quad |u| \leq K_0.$$

Step 2. Since  $u$  is convex,  $|Du|$  takes its maximum on the boundary. Since the tangential derivatives are known, it suffices to estimate the exterior normal derivative  $u_\nu$  on  $\partial\Omega$ . Note the convex function  $u$  is subharmonic. By (1) and the maximum principle, we have

$$u^0 \leq u \leq h \quad \text{in } \Omega,$$

where  $h$  is the harmonic function in  $\Omega$  which equals  $\varphi$  on  $\partial\Omega$ . Thus

$$(3) \quad h_\nu \leq u_\nu \leq u_\nu^0 \quad \text{on } \partial\Omega.$$

Therefore, we obtain

$$(4) \quad |Du| \leq K_1, \quad \text{on } \partial\Omega \text{ and hence in } \bar{\Omega}.$$

The constant  $K_1$  depends on  $|u^0|_1$ .

Step 3. We estimate the second derivatives of  $u$  on the boundary with the aid of suitable barrier functions. Recall the linearized operator

$$L = u^{ij} \partial_{ij}.$$

If we take the logarithms of both sides of equation (\*) and differentiate with respect to  $x_k$ , we have

$$(5) \quad Lu_k = u^{ij} u_{ijk} = (\log f)_k.$$

Note

$$\begin{aligned} L(x_l u_k) &= u^{ij} \partial_{ij}(x_l u_k) = u^{ij} \partial_i(\delta_{jl} u_k + x_l u_{jk}) \\ &= u^{ij} (\delta_{jl} u_{ik} + \delta_{il} u_{jk} + x_l u_{ijk}) = u^{il} u_{ik} + u^{lj} u_{jk} + x_l u^{ij} u_{ijk} \\ &= 2\delta_k^l + x_l (\log f)_k. \end{aligned}$$

Hence we obtain

$$(6) \quad L(x_l u_k - x_k u_l) = (x_l \partial_k - x_k \partial_l) \log f.$$

This simply reflects the fact that the operator  $x_l \partial_k - x_k \partial_l$  is an angular derivative (on  $|x| = \text{constant}$ ) and the expression  $\det(u_{ij})$  is invariant under the rotation of coordinates.

Consider any boundary point; without loss of generality, we may take it to be the origin and the  $x_n$ -axis to be the interior normal. Then, near the origin,  $\partial\Omega$  is represented by

$$(7) \quad x_n = \rho(x') = \frac{1}{2}B_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3),$$

where  $x' = (x_1, \dots, x_{n-1})$ . In the summation, Greek letters  $\alpha, \beta$  etc. go from 1 to  $n-1$ . On  $\partial\Omega$ , we have

$$u - \varphi = 0,$$

or

$$(u - \varphi)(x', \rho(x')) = 0 \quad \text{for small } x'.$$

Recall  $\varphi$  is defined in  $\bar{\Omega}$ . So we get by differentiating with respect to  $x_\alpha$  and then  $x_\beta$

$$(\partial_\alpha + \rho_\alpha \partial_n)(u - \varphi) = 0 \quad \text{on } \partial\Omega,$$

and

$$(\partial_\beta + \rho_\beta \partial_n)(\partial_\alpha + \rho_\alpha \partial_n)(u - \varphi) = 0 \quad \text{on } \partial\Omega.$$

Note  $\partial_\alpha \rho(0) = 0$  and  $\partial_{\alpha\beta} \rho(0) = B_{\alpha\beta}$ . Hence at 0 we obtain

$$\partial_{\alpha\beta}(u - \varphi)(0) + B_{\alpha\beta} \partial_n(u - \varphi)(0) = 0.$$

Since  $|Du| \leq K_1$  on  $\partial\Omega$ , we obtain

$$(8) \quad |\partial_{\alpha\beta} u(0)| \leq C \quad \text{for } \alpha, \beta = 1, \dots, n-1.$$

Now we establish, in addition, the estimate

$$(9) \quad \sum_{\alpha, \beta < n} u_{\alpha\beta}(0) \xi_\alpha \xi_\beta \geq C_0 > 0,$$

for any unit vector  $\xi = (\xi_1, \dots, \xi_{n-1})$ . Without loss of generality, we assume  $\xi_1 = 1$ . We will prove

$$(9') \quad u_{11}(0) \geq C_0 > 0.$$

We suppose, furthermore, that

$$(10) \quad u(0) = 0, u_\alpha(0) = 0 \quad \text{for } \alpha = 1, \dots, n-1.$$

To prove (9'), we make use of a more carefully constructed barrier function and redo the proof of the lower bound of (3). Recall we have (7) on  $\partial\Omega$ .

Let  $\tilde{u} = u - \lambda x_n$  with  $\lambda$  so chosen that

$$\frac{\partial^2}{\partial x_1^2} \tilde{u}(x', \rho(x')) = 0 \quad \text{at } 0,$$

i.e.,

$$(11) \quad u_{11}(0) + \tilde{u}_{x_n}(0) \rho_{11}(0) = 0.$$

Note  $\tilde{u}$  still satisfies  $\det \tilde{u}_{ij} = f$ . We claim

$$(12) \quad \tilde{u}|_{\partial\Omega} \leq \sum_{1 < j \leq n} a_{ij} x_1 x_j + C \left( \sum_{1 < \beta < n} x_\beta^2 + x_n^2 \right).$$

To prove (12), we consider the Taylor expansion of  $\tilde{u}(x', \rho(x'))$ . In view of (7) and (10), there is no linear terms. For quadratic terms of  $x_\alpha x_\beta$ , there is no  $x_1^2$  term by the definition of  $\tilde{u}$ . Hence the quadratic part of  $\tilde{u}$  can be written as

$$\sum_{1 < \alpha < n} a_{1\alpha} x_1 x_\alpha + \sum_{1 < \alpha, \beta < n} a_{\alpha\beta} x_\alpha x_\beta,$$

which is a part of the right side in (12). Now we consider the cubic terms and higher order terms. By (7), there holds on  $\partial\Omega$

$$x_1^2 = \frac{2x_n}{B_{11}} - \sum_{(\alpha, \beta) \neq (1, 1)} \frac{B_{\alpha\beta}}{B_{11}} x_\alpha x_\beta + O(|x'|^3).$$

Hence we have

$$x_1^3 = \frac{2}{B_{11}} x_1 x_n - \sum_{(\alpha, \beta) \neq (1, 1)} \frac{B_{\alpha\beta}}{B_{11}} x_\alpha x_\beta x_1 + O(|x'|^4).$$

Note  $x_1 x_n$  term goes to the first sum in the right side of (12). The rest of the cube terms in  $\tilde{u}(x', \rho(x'))$  have the form  $x_1^2 x_\alpha$ ,  $x_1 x_\alpha x_\beta$  and  $x_\alpha x_\beta x_\gamma$ . For any  $1 < \alpha, \beta < n$ , we have

$$x_1^2 x_\alpha \leq \frac{1}{2}(x_\alpha^2 + x_1^4),$$

and

$$x_1 x_\alpha x_\beta \leq \frac{1}{2}(x_\alpha^2 + x_\beta^2).$$

For the fourth order term, we note for  $i \geq 2$ ,

$$x_1^4 + |x_1^3 x_i| \leq \sum_{1 < \alpha < n} x_\alpha^2 + x_n^2.$$

Therefore, (12) is proved for  $x \in \partial\Omega$  close to the origin. By the strict convexity of  $\partial\Omega$ , we have  $x_n \geq a > 0$  for any  $x \in \partial\Omega$  away from the origin. We may simply choose  $C$  large enough so that (12) holds there.

Now we consider a barrier function

$$\begin{aligned} h &= -\varepsilon x_n + \delta |x|^2 + \frac{1}{2B} \sum_{1 < j \leq n} (a_{1j} x_1 + B x_j)^2 \\ &= -\varepsilon x_n + \delta |x|^2 + \frac{1}{2B} \sum_{1 < j \leq n} a_{1j}^2 x_1^2 + \sum_{1 < j \leq n} a_{1j} x_1 x_j + \frac{B}{2} \sum_{1 < j \leq n} x_j^2. \end{aligned}$$

On  $\partial\Omega$ , for  $x$  close to the origin, we require

$$-\varepsilon x_n + \delta |x|^2 \geq 0,$$

which is equivalent to

$$x_n \leq C \frac{\delta}{\varepsilon} |x'|^2.$$

This can be achieved by taking  $\varepsilon \ll \delta$ . For  $x \in \partial\Omega$  not close to the origin,  $x_n^2$  obviously controls  $-x_n$ . Hence by taking  $B \gg C$  and  $\varepsilon \ll \delta$  for  $\delta$  to be chosen, we obtain

$$h \geq \tilde{u} \text{ on } \partial\Omega.$$

Next

$$\begin{aligned} D^2h &= 2\delta I + \begin{pmatrix} \frac{1}{B} \sum_{1 < j \leq n} a_{1j}^2 & a_{12} & \cdots & a_{1n} \\ a_{12} & B & & \\ \vdots & & \ddots & \\ a_{1n} & & \cdots & B \end{pmatrix} \\ &= \begin{pmatrix} 2\delta + \frac{1}{B} \sum_{1 < j \leq n} a_{1j}^2 & a_{12} & \cdots & a_{1n} \\ a_{12} & 2\delta + B & & \\ \vdots & & \ddots & \\ a_{1n} & & & 2\delta + B \end{pmatrix}. \end{aligned}$$

A straightforward calculation shows

$$\det(h_{ij}) = 2\delta \left( 2\delta + B + \frac{1}{B} \sum_{1 < j \leq n} a_{1j}^2 \right) (2\delta + B)^{n-2}.$$

In fact, eigenvalues of  $h_{ij}$  are given by

$$2\delta, 2\delta + B + \frac{1}{B} \sum_{1 < j \leq n} a_{1j}^2, 2\delta + B, \dots, 2\delta + B.$$

Hence  $h$  is strictly convex in  $\bar{\Omega}$  and satisfies

$$\det h_{ij} < f \text{ in } \Omega,$$

if we choose  $\delta$  small. Thus  $h$  is an upper barrier for  $\tilde{u}$ . By Lemma 1.2, there holds

$$\tilde{u} \leq h.$$

Since  $\tilde{u}(0) = h(0) = 0$ , we get

$$\tilde{u}_n(0) \leq h_n(0) = -\varepsilon.$$

By (11), we obtain

$$u_{11}(0) = -\tilde{u}_n(0)\rho_{11}(0) \geq \varepsilon \rho_{11}(0).$$

Hence (9') and (9) are proved. The constant  $C_0$  depends only on  $\max f$ ,  $\max f^{-1}$ ,  $\Omega$  and  $|\varphi|_{C^4}$ .

Next we estimate the mixed derivative  $u_{\alpha n}(0)$ . Consider the vector field (directional derivative)

$$T = \partial_\alpha + \sum_{\beta < n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta).$$

In view of (6), we have

$$L(Tu) = T(\log f).$$

This implies

$$L(T(u - \varphi)) = LTu - L(T\varphi) = T(\log f) - u^{ij} \partial_{ij}(T\varphi).$$

Since  $(u^{ij})$  is positive definite, we get

$$(13) \quad |L(T(u - \varphi))| \leq C(1 + \sum_i u^{ii}).$$

On  $\partial\Omega$  close to the origin, we have

$$(\partial_\alpha + \rho_\alpha \partial_n)(u - \varphi) = 0 \quad \text{for } \alpha = 1, \dots, n-1,$$

or,

$$|(u - \varphi)_\alpha + (u - \varphi)_n B_{\alpha\beta} x_\beta| \leq C|x|^2.$$

This implies

$$(14) \quad |T(u - \varphi)| \leq C|x|^2 \text{ on } \partial\Omega.$$

As before, we first prove (14) for any  $x \in \partial\Omega$  close to the origin. Then it is trivially true for  $x_n \geq a > 0$ .

Now, we consider a barrier function of the form

$$w = -a|x|^2 + bx_n,$$

for suitable positive constants  $a, b$  as a barrier function. First we have

$$Lw = -2a \sum u^{ii},$$

and hence for  $a$  large

$$\begin{aligned} |L(T(u - \varphi))| + Lw &\leq -2a \sum u^{ii} + C(1 + \sum u^{ii}) \\ &\leq -a \sum u^{ii} + C. \end{aligned}$$

By the theorem of the arithmetic and geometric mean, we get

$$\frac{1}{n} \sum u^{ii} \geq (\det u^{ij})^{\frac{1}{n}} = f^{-\frac{1}{n}}.$$

Choosing  $a$  large further, we then obtain

$$|L(T(u - \varphi))| + Lw \leq 0 \quad \text{in } \Omega.$$

By (14), we note

$$|T(u - \varphi)| \leq w \text{ on } \partial\Omega \iff (C + a)|x|^2 \leq bx_n.$$

Since  $\Omega$  is strictly convex, we may then choose  $b$  so large that

$$|T(u - \varphi)| \leq w \text{ on } \partial\Omega.$$

By the maximum principle, we obtain

$$|T(u - \varphi)| \leq w \text{ in } \Omega.$$

By taking  $x' = 0$ , dividing by  $x_n$  and then letting  $x_n \rightarrow 0$ , we get

$$|\partial_n T(u - \varphi)| \leq b \text{ at } 0,$$

or

$$|\partial_{\alpha n}(u - \varphi)(0) - \sum_{\beta < n} B_{\alpha\beta} \partial_\beta(u - \varphi)(0)| \leq b.$$

Thus we obtain

$$|u_{\alpha n}(0)| \leq C.$$

Finally, we use equation (\*) to estimate  $u_{nn}(0)$ . Note that

$$f(0) = \det u_{ij}(0) = \sum A^{ni}(0)u_{in}(0).$$

By the estimates already established, we see that the first  $(n-1)$  terms in the sum are bounded so that

$$A^{nn}(0)u_{nn}(0) \leq C.$$

By (9), we have a bound from below for  $A^{nn}(0)$ . Hence, we obtain

$$u_{nn}(0) \leq C.$$

Having an upper bound for all the eigenvalues of  $H = (u_{ij})$ , we obtain also a lower bound for each, since their product equals  $f$ . Thus we have

$$u_{nn}(0) \geq C_0 > 0.$$

Step 4. We estimate the second derivatives in  $\bar{\Omega}$ . We write the equation in the form

$$(15) \quad F(D^2u) \equiv \log \det (u_{ij}) = \log f.$$

By differentiating (15) twice with respect to  $x_r$ , we have by the concavity established before that

$$Lu_{rr} \geq (\log f)_{rr} \geq -nC,$$

for some constant  $C$  depending only on  $f$ . Since  $Lu = n$ , we get

$$L(u_{rr} + Cu) \geq 0,$$

and hence  $u_{rr} + Cu$  achieves its maximum on the boundary. Therefore, we conclude

$$u_{rr} \leq K \text{ in } \Omega.$$

Since  $(u_{ij})$  is positive definite, we have  $u_{ii} > 0$  and

$$|u_{ij}| \leq K \text{ in } \Omega.$$

This finishes the proof. □

**Remark 1.4.** Since the eigenvalues of  $D^2u$  are bounded from above by  $K$  and their product is equal to  $f$ , we obtain a positive lower bound for each one. Therefore, the linearized operator  $L$  is uniformly elliptic. Let  $T$  be any constant directional derivative  $T = \sum c_j \partial_j$  with  $\sum c_j^2 = 1$ . We have

$$\begin{aligned} L(T^2u) &= u^{ik}u^{jl} T u_{ij} T u_{kl} + T^2(\log f) \\ &\geq C_0 \sum_{i,j} |T u_{ij}|^2 - C, \end{aligned}$$

where  $C_0$  and  $C$  are positive constants under control. In particular, one has a positive lower bound for  $C_0$ .

**1.2. Complex Monge-Ampère Equations.** Let  $z_1, \dots, z_n$  be complex coordinates in  $\mathbb{C}^n$ , with  $x_j = \Re(z_j)$  and  $y_j = \Im(z_j)$ . If  $u$  is a smooth function in an open subset of  $\mathbb{C}^n$ , we define  $u_j$  and  $u_{\bar{j}}$  by

$$\begin{aligned} u_j &= \partial_j u = \partial_{z_j} u = \frac{1}{2} (\partial_{x_j} u - \sqrt{-1} \partial_{y_j} u), \\ u_{\bar{j}} &= \partial_{\bar{j}} u = \partial_{\bar{z}_j} u = \frac{1}{2} (\partial_{x_j} u + \sqrt{-1} \partial_{y_j} u). \end{aligned}$$

Hence, we have

$$\partial_{j\bar{k}} u = \frac{1}{4} (\partial_{x_j x_k} u + \partial_{y_j y_k} u) + \frac{\sqrt{-1}}{4} (\partial_{x_j y_k} u - \partial_{x_k y_j} u),$$

and in particular

$$\partial_{j\bar{j}} u = \frac{1}{4} (\partial_{x_j x_j} u + \partial_{y_j y_j} u) = \frac{1}{4} \Delta_j u.$$

If  $u$  is a real-valued function,  $\partial_{j\bar{k}} u = \overline{\partial_{k\bar{j}} u}$ . In other words, the  $n \times n$  matrix  $(u_{i\bar{j}})$  is a Hermitian matrix.

**Definition 1.5.** A real-valued smooth function  $u$  is pluri-subharmonic if  $(u_{i\bar{j}})$  is non-negative, and  $u$  is strictly pluri-subharmonic if  $(u_{i\bar{j}})$  is positive definite.

Now we recall some terminology for complex matrices. For convenience, we use  $a_{i\bar{j}}$  to denote the  $(i, j)$  entry in an  $n \times n$  matrix  $(a_{i\bar{j}})$ . A matrix  $(a_{i\bar{j}})$  is Hermitian if  $a_{i\bar{j}} = \overline{a_{j\bar{i}}}$ . For a Hermitian matrix, the quadratic form

$$a_{i\bar{j}} t_i \bar{t}_j \quad \text{for } t \in \mathbb{C}^n$$

is real. If we write

$$a_{i\bar{j}} = b_{ij} + \sqrt{-1} c_{ij},$$

then  $(b_{ij})$  is a real symmetric matrix and  $(c_{ij})$  is a real skew symmetric matrix. With  $t_i = \xi_i + \sqrt{-1} \eta_i$ , we have

$$a_{i\bar{j}} t_i \bar{t}_j = b_{ij} (\xi_i \xi_j + \eta_i \eta_j) + c_{ij} (\xi_i \eta_j - \eta_i \xi_j).$$

A Hermitian matrix  $(a_{i\bar{j}})$  is positive definite if

$$a_{i\bar{j}} t_i \bar{t}_j > 0 \quad \text{for any } t \in \mathbb{C}^n.$$

For a positive definite matrix  $(a_{i\bar{j}})$ , we have obviously  $a_{i\bar{i}} > 0$ . Moreover, any diagonal  $2 \times 2$  minors have positive determinants. Hence, we have

$$|a_{i\bar{j}}|^2 \leq a_{i\bar{i}} \cdot a_{j\bar{j}}.$$

This implies

$$|a_{i\bar{j}}| \leq \frac{1}{2} (a_{i\bar{i}} + a_{j\bar{j}}).$$

Suppose  $(a_{i\bar{j}})$  is a Hermitian matrix. Define

$$L = a_{i\bar{j}} \partial_{i\bar{j}}.$$

This is a linear operator in  $\mathbb{R}^{2n}$  with real coefficients. If  $(a_{i\bar{j}})$  is positive definite, then  $L$  is elliptic. To see this, recall

$$\partial_{i\bar{j}} = \frac{1}{4} ((\partial_{x_i x_j} + \partial_{y_i y_j}) + \sqrt{-1}(\partial_{x_i y_j} - \partial_{y_i x_j})).$$

With  $a_{i\bar{j}} = b_{ij} + \sqrt{-1}c_{ij}$ , then we have

$$L = a_{i\bar{j}} \partial_{i\bar{j}} = \frac{1}{4} (b_{ij}(\partial_{x_i x_j} + \partial_{y_i y_j}) - c_{ij}(\partial_{x_i y_j} - \partial_{y_i x_j})).$$

For any  $\xi \in \mathbb{R}^n$  (corresponding to  $x$ ) and  $\eta \in \mathbb{R}^n$  (corresponding to  $y$ ), we get

$$b_{ij}(\xi_i \xi_j + \eta_i \eta_j) - c_{ij}(\xi_i \eta_j - \xi_j \eta_i) = a_{i\bar{j}}(\xi_i - \sqrt{-1}\eta_i)(\xi_j + \sqrt{-1}\eta_j) = a_{i\bar{j}} \bar{t}_i t_j.$$

It is positive for nonzero  $(\xi, \eta) \in \mathbb{R}^{2n}$ .

Now we define the complex Monge-Ampère operator by

$$M(u) = \det(u_{i\bar{j}}).$$

This is positive if  $u$  is strictly pluri-subharmonic. As in the real case, it is convenient to introduce

$$F(D^2 u) = \log \det(u_{i\bar{j}}).$$

We then have

$$\begin{aligned} F_{u_{i\bar{j}}} &= u^{i\bar{j}}, \\ F_{u_{i\bar{j}} u_{k\bar{l}}} &= -u^{i\bar{l}} u^{k\bar{j}}, \end{aligned}$$

where  $(u^{i\bar{j}})$  denotes the inverse of the Hessian matrix  $(u_{i\bar{j}})$ . To see this, we denote by  $A = \{A^{i\bar{j}}\}$  the cofactor matrix of the Hessian matrix  $H = (u_{i\bar{j}})$ , i.e.,  $A = (\det H)H^{-1}$ . Note for each fixed  $i = 1, \dots, n$ ,

$$\det(u_{i\bar{j}}) = A^{i\bar{1}} u_{i\bar{1}} + \dots + A^{i\bar{n}} u_{i\bar{n}}.$$

Then it is easy to see

$$\frac{\partial F}{\partial u_{i\bar{j}}} = \frac{1}{\det(u_{i\bar{j}})} A^{i\bar{j}} = u^{i\bar{j}}.$$

Next, for each fixed  $i, j = 1, \dots, n$ , we have

$$u^{i\bar{k}} u_{j\bar{k}} = \delta_j^i.$$

Differentiating with respect to  $u_{p\bar{q}}$ , we get

$$(u^{i\bar{k}})_{u_{p\bar{q}}} u_{j\bar{k}} + u^{i\bar{k}} (u_{j\bar{k}})_{u_{p\bar{q}}} = 0.$$

Multiplying  $u^{j\bar{l}}$  and summing over  $j$ , we obtain

$$(u^{i\bar{l}})_{u_{p\bar{q}}} = (u^{i\bar{k}})_{u_{p\bar{q}}} u_{j\bar{k}} u^{j\bar{l}} = -u^{i\bar{k}} u^{j\bar{l}} (u_{j\bar{k}})_{u_{p\bar{q}}} = -u^{i\bar{q}} u^{p\bar{l}},$$

or

$$\frac{\partial(u^{i\bar{j}})}{\partial u_{k\bar{l}}} = -u^{i\bar{l}} u^{k\bar{j}}.$$

Hence we have

$$\frac{\partial^2 F}{\partial u_{i\bar{j}} \partial u_{k\bar{l}}} = \frac{\partial(u^{i\bar{j}})}{\partial u_{k\bar{l}}} = -u^{i\bar{l}} u^{k\bar{j}}.$$

Therefore, for any Hermitian matrix  $M = (m_{i\bar{j}})$ , we obtain

$$\frac{\partial^2 F}{\partial u_{i\bar{j}} \partial u_{k\bar{l}}} m_{i\bar{j}} m_{k\bar{l}} \leq 0.$$

Now we consider the following equation for a strictly pluri-subharmonic function  $u$

$$\det(u_{i\bar{j}}) = f.$$

As in the real case, we rewrite it as

$$F(D^2u) \equiv \log \det(u_{i\bar{j}}) = \log f.$$

Suppose  $\partial$  is an arbitrary directional derivative in  $\mathbb{R}^{2n}$ . Differentiating the equation with  $\partial$ , we obtain

$$u^{i\bar{j}} \partial u_{i\bar{j}} = \partial \log f.$$

This leads to the linear differential operator

$$L = u^{i\bar{j}} \partial_{i\bar{j}}.$$

Since  $u$  is strictly pluri-subharmonic,  $L$  is elliptic. We get

$$L(\partial u) = \partial \log f.$$

Differentiating again with  $\partial$ , we get

$$L(\partial^2 u) - u^{i\bar{l}} u^{k\bar{j}} \partial u_{i\bar{j}} \partial u_{k\bar{l}} = \partial^2 \log f,$$

or

$$L(\partial^2 u) = u^{i\bar{l}} u^{k\bar{j}} \partial u_{i\bar{j}} \partial u_{k\bar{l}} + \partial^2 \log f.$$

The first term in the right side is positive. Hence, we conclude

$$L(\partial^2 u) \geq \partial^2 \log f.$$

We now look at the change of the Monge-Ampère operator under a holomorphic change of variables. Suppose  $w = w(z)$  is a holomorphic change of variables. Then, we have

$$u_{z_i} = u_{w_k} \frac{\partial w_k}{\partial z_i},$$

and

$$u_{z_i \bar{z}_j} = u_{w_k \bar{w}_l} \frac{\partial \bar{w}_l}{\partial \bar{z}_j} \frac{\partial w_k}{\partial z_i} = \left( \frac{\partial \bar{w}_l}{\partial z_j} \right) u_{w_k \bar{w}_l} \frac{\partial w_k}{\partial z_i}.$$

Therefore, we obtain

$$\det(u_{z_i \bar{z}_j}) = |\det(w_z)|^2 \det(u_{w_k \bar{w}_l}).$$

Now we prove that the comparison principle holds as in the real case.

**Lemma 1.6.** *Suppose  $\Omega \subset \mathbb{C}^n$  is a bounded domain. If  $u, v \in C^2(\bar{\Omega})$  are pluri-subharmonic satisfying*

$$\begin{aligned} \det(u_{i\bar{j}}) &\geq \det(v_{i\bar{j}}) \quad \text{in } \Omega \\ u &\leq v \quad \text{on } \partial\Omega, \end{aligned}$$

then  $u \leq v$  in  $\Omega$ .

*Proof.* We first assume that  $u$  is strictly pluri-subharmonic. As in the real case, we have

$$\begin{aligned} \det(u_{i\bar{j}}) - \det(v_{i\bar{j}}) &= \int_0^1 \frac{d}{dt} \det[(tu + (1-t)v)_{i\bar{j}}] dt \\ &= \sum \int_0^1 B^{i\bar{j}}(t) dt (u - v)_{i\bar{j}} \geq 0, \end{aligned}$$

where  $(B^{i\bar{j}}(t))$  are cofactors of  $(tu + (1-t)v)_{i\bar{j}}$ . Hence  $u - v$  attains its maximum on  $\partial\Omega$ , which gives the desired result.

In general, we consider  $u_\varepsilon = u + \varepsilon(|z|^2 - \max_{\partial\Omega} |z|^2)$  and compare  $u_\varepsilon$  and  $v$ .  $\square$

Now we discuss the Dirichlet problem.

A domain  $\Omega \subset \mathbb{C}^n$  with a smooth boundary  $\partial\Omega$  is called strongly pseudo-convex if there exists a smooth strictly pluri-subharmonic function  $r$  defined in  $\bar{\Omega}$  such that  $r < 0$  in  $\Omega$  and  $r = 0, dr \neq 0$  on  $\partial\Omega$ .

We now study the Dirichlet problem for the complex Monge-Ampère equation

$$\begin{aligned} \det(u_{i\bar{j}}) &= f(z) \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

Here  $f$  is a smooth positive function for  $z \in \bar{\Omega}$ . Assuming  $\Omega$  is a bounded strongly pseudo-convex domain with a smooth boundary, we intend to solve for a strictly pluri-subharmonic solution  $u \in C^\infty(\bar{\Omega})$ .

As in the real case, we reduce the solvability to  $C^{2+\alpha}$  a priori estimates. We will derive the  $C^2$  estimates in the rest of the section and study the  $C^\alpha$ -estimates of the second derivatives in the next chapter.

**Theorem 1.7.** *Let  $u$  be a smooth strictly pluri-subharmonic function in the strongly pseudo-convex domain  $\bar{\Omega}$  satisfying*

$$\begin{aligned} \det(u_{i\bar{j}}) &= f \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

for some  $f > 0$  in  $\bar{\Omega}$  and smooth function  $\varphi$  on  $\partial\Omega$ . Then there holds in  $\bar{\Omega}$

$$|u|_2 \leq K,$$

where  $K$  depends on  $\Omega, \max f^{-1}, |f|_3$  and  $|\varphi|_4$ .

*Proof.* We first assume that  $\varphi$  has been extended inside  $\Omega$  so as to be strictly pluri-subharmonic and to satisfy

$$\det(\varphi_{i\bar{j}}) > \max_{\bar{\Omega}} f(z).$$

This can be done by simply extending  $\varphi$  arbitrarily in  $\Omega$  and then replacing  $\varphi$  by  $\varphi + Cr$  for  $C$  large. Here  $r$  is the defining function for  $\Omega$ :  $r$  is strictly pluri-subharmonic in  $\Omega$ ,  $r < 0$  in  $\Omega$  and  $r = 0, dr \neq 0$  on  $\partial\Omega$ .

Step 1. Since a pluri-subharmonic function achieves its maximum on the boundary, we have

$$u \leq \max_{\partial\Omega} \varphi.$$

With  $\varphi$  extended as indicated, we see that

$$\det(u_{i\bar{j}}) < \det(\varphi_{i\bar{j}}).$$

It follows by Lemma 1.6 that

$$(1) \quad u \geq \varphi \quad \text{in } \Omega.$$

We conclude

$$(2) \quad |u|_0 \leq K_0.$$

Step 2. We derive an estimate of the first derivatives of  $u$  on the boundary. By (1) and the maximum principle, we have

$$\varphi \leq u \leq h \quad \text{in } \Omega,$$

where  $h$  is the harmonic function in  $\Omega$  which equals  $\varphi$  on  $\partial\Omega$ . Thus, we get

$$(3) \quad |Du(z)| \leq \max\{|D\varphi(z)|, |Dh(z)|\} \quad \text{for } z \in \partial\Omega.$$

Step 3. We estimate the first derivatives of  $u$  in  $\Omega$ . As in the real case, we write the equation in the form

$$(4) \quad F(D^2u) \equiv \log \det(u_{i\bar{j}}) = \log f.$$

Then  $(F_{u_{i\bar{j}}}) = (u^{i\bar{j}})$  is the inverse of the matrix  $(u_{i\bar{j}})$ , and  $F_{u_{i\bar{j}}u_{k\bar{l}}} = -u^{i\bar{l}}u^{k\bar{j}}$ . Denote the linearized operator of  $F$  at  $u$  by

$$L = u^{i\bar{j}} \partial_{i\bar{j}}.$$

Then  $L$  is elliptic, since  $u$  is strictly pluri-subharmonic.

By  $\det(u^{i\bar{j}}) = f^{-1}$  and the inequality for arithmetic and geometric means, we get

$$(5) \quad \frac{1}{n} \sum u^{i\bar{i}} \geq f^{-\frac{1}{n}}.$$

If  $D$  is a real, constant coefficient operator of the form

$$D = \sum (a_j \partial_{x_j} + b_j \partial_{y_j}) \quad \text{with } \sum (a_j^2 + b_j^2) = 1,$$

we have

$$(6) \quad LDu = f^{-1} Df.$$

Consider the function

$$w = \pm Du + e^{\lambda|z|^2}.$$

Note

$$L(e^{\lambda|z|^2}) = e^{\lambda|z|^2} (\lambda \sum u^{i\bar{i}} + \lambda^2 u^{i\bar{j}} \bar{z}_i z_j) \geq n e^{\lambda|z|^2} \lambda f^{-\frac{1}{n}}.$$

Hence, we get

$$Lw \geq \pm \frac{Df}{f} + n\lambda e^{\lambda|z|^2} f^{-\frac{1}{n}} \geq 0,$$

if we choose  $\lambda$  large enough. The maximum principle implies

$$\max_{\Omega} |Du| \leq \max_{\partial\Omega} |Du| + C.$$

With (3), we obtain

$$(7) \quad |u|_1 \leq K_1.$$

Step 4. We estimate the second derivatives at any point  $P \in \partial\Omega$ . Choose coordinates  $z_1, \dots, z_n$  with the origin at  $P$  such that  $r_{z_\alpha}(0) = 0$  for  $\alpha < n$ ,  $r_{y_n}(0) = 0$  and  $r_{x_n}(0) = -1$ . We recall that  $r$  is strictly pluri-subharmonic near 0. The following notation is convenient. Set  $t_1 = x_1, t_2 = y_1, \dots, t_{2n-3} = x_{n-1}, t_{2n-2} = y_{n-1}, t_{2n-1} = y_n = t, t' = (t_1, \dots, t_{2n-1})$ .

Writing the Taylor expansion of  $r$  up to second order, we obtain

$$r = \Re(-z_n + \sum a_{ij} z_i z_j) + \sum b_{i\bar{j}} z_i \bar{z}_j + O(|z|^3).$$

Note  $\{a_{ij}\}$  has a uniform bound, independent of the point  $P \in \partial\Omega$ . Introducing new coordinates of the form

$$\begin{aligned} z'_k &= z_k, & k &= 1, \dots, n-1, \\ z'_n &= z_n - \sum a_{ij} z_i z_j, \end{aligned}$$

we can write

$$r = -\Re(z'_n) + \sum c_{i\bar{j}} z'_i \bar{z}'_j + O(|z|^3).$$

Furthermore, at points where the Jacobian of the transformation does not vanish, a function is pluri-subharmonic with respect to  $\{z_j\}$  if and only if it is pluri-subharmonic with respect to  $\{z'_j\}$ , and  $\det(u_{i\bar{j}})$  differs from  $\det(u_{z'_i \bar{z}'_j})$  by a factor of the absolute value squared of the Jacobian. Thus, we drop the primes and assume, in a neighborhood of 0,  $r$  is of the form

$$(8) \quad r = -\Re(z_n) + \sum c_{i\bar{j}} z_i \bar{z}_j + O(|z|^3).$$

Since  $r$  is strictly pluri-subharmonic,  $(c_{i\bar{j}})$  is positive definite. In a neighborhood of 0,  $\partial\Omega$  can be expressed as

$$(9) \quad x_n = \rho(t') = \sum_{i,j < 2n} b_{ij} t_i t_j + O(|t'|^3),$$

where  $(b_{ij})$  is a (real) positive definite matrix. To see this, we simply note that

$$\sum_{i,j < 2n} b_{ij} t_i t_j = \sum_{i,j=1}^n c_{i\bar{j}} z_i \bar{z}_j,$$

for

$$z_1 = t_1 + \sqrt{-1}t_2, \dots, z_{n-1} = t_{2n-3} + \sqrt{-1}t_{2n-2}, z_n = \sqrt{-1}t_{2n-1}.$$

We first estimate  $u_{t_i t_j}(0)$  for  $i, j \leq 2n - 1$ . On  $\partial\Omega$ , we have

$$u - \varphi = 0,$$

or

$$(u - \varphi)(t', \rho(t')) = 0 \quad \text{for small } t'.$$

Recall  $\varphi$  is defined in  $\bar{\Omega}$ . We get by differentiating with respect to  $t_i$  and then  $t_j$ ,

$$(\partial_{t_i} + \rho_{t_i} \partial_{x_n})(u - \varphi) = 0 \quad \text{on } \partial\Omega,$$

and

$$(\partial_{t_j} + \rho_{t_j} \partial_{x_n})(\partial_{t_i} + \rho_{t_i} \partial_{x_n})(u - \varphi) = 0 \quad \text{on } \partial\Omega.$$

Note  $\partial_{t_i} \rho(0) = 0$  and  $\partial_{t_i t_j} \rho(0) = b_{ij}$ . Hence at 0, we obtain

$$\partial_{t_i t_j}(u - \varphi)(0) + b_{ij} \partial_n(u - \varphi)(0) = 0.$$

By (7), we obtain

$$(10) \quad |\partial_{t_i t_j} u(0)| \leq C_1 \quad \text{for any } i, j \leq 2n - 1.$$

Next, we estimate the mixed derivatives  $u_{t_i x_n}(0)$  for  $i \leq 2n - 1$ . We define  $T_i$  in a neighborhood of 0 by

$$(11) \quad T_i = \frac{\partial}{\partial t_i} - \frac{r_{t_i}}{r_{x_n}} \frac{\partial}{\partial x_n}, \quad i = 1, \dots, 2n - 1;$$

then  $T_i r = 0$  and hence  $T_i$  is a tangential vector to  $\partial\Omega$ . By  $u - \varphi = 0$  on  $r = 0$ , we have  $T_i(u - \varphi) = 0$  on  $r = 0$ . We can also see it by a direct calculation. Note

$$(u - \varphi)(t', x_n(t')) = 0.$$

Hence, we obtain

$$\frac{\partial}{\partial t_i}(u - \varphi) + \frac{\partial(u - \varphi)}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i} = 0.$$

Since  $r = 0$  gives  $x_n = x_n(t')$ , we get

$$\frac{\partial r}{\partial t_i} + \frac{\partial r}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i} = 0,$$

and then

$$\frac{\partial x_n}{\partial t_i} = -\frac{r_{t_i}}{r_{x_n}}.$$

Therefore on  $\partial\Omega$ , we conclude

$$\frac{\partial(u - \varphi)}{\partial t_i} - \frac{r_{t_i}}{r_{x_n}} \frac{\partial(u - \varphi)}{\partial x_n} = 0.$$

We set

$$w = \pm T_i(u - \varphi) + (u_t - \varphi_t)^2,$$

and for  $\varepsilon > 0$ ,

$$S_\varepsilon = \{z \in U; r(z) \leq 0, x_n \leq \varepsilon\},$$

where  $U$  is a neighborhood of the origin. We show that for suitable  $\varepsilon > 0$

- (a)  $Lw \geq L(Ax_n - B|z|^2)$  in  $S_\varepsilon$ , for  $B$  sufficiently large;
- (b)  $w \leq Ax_n - B|z|^2$  on  $\partial S_\varepsilon$ , for  $A$  sufficiently large.

To proceed, we set  $a = -\frac{r_{t_i}}{r_{x_n}}$ . Then, we have

$$T_i u = u_{t_i} + a u_{x_n},$$

and

$$\begin{aligned} L(T_i u) &= u^{p\bar{q}} \partial_{p\bar{q}}(u_{t_i} + a u_{x_n}) \\ &= u^{p\bar{q}} \partial_{p\bar{q}} u_{t_i} + a u^{p\bar{q}} \partial_{p\bar{q}} u_{x_n} + u^{p\bar{q}} a_p u_{x_n \bar{q}} + u^{p\bar{q}} a_{\bar{q}} u_{x_n p} + u^{p\bar{q}} a_{p\bar{q}} u_{x_n}, \end{aligned}$$

which yields

$$(12) \quad L(T_i u) = T_i \log f + u^{p\bar{q}} a_p u_{x_n \bar{q}} + u^{p\bar{q}} a_{\bar{q}} u_{x_n p} + u^{p\bar{q}} a_{p\bar{q}} u_{x_n}.$$

Observe that  $u^{p\bar{q}} u_{n\bar{q}} = \delta_n^p$  and that

$$\frac{\partial}{\partial z_n} = \frac{1}{2} \left( \frac{\partial}{\partial x_n} - \sqrt{-1} \frac{\partial}{\partial t} \right),$$

or

$$\frac{\partial}{\partial x_n} = 2 \frac{\partial}{\partial z_n} + \sqrt{-1} \frac{\partial}{\partial t},$$

so that

$$u_{x_n \bar{q}} = 2u_{n\bar{q}} + \sqrt{-1} u_{t\bar{q}}.$$

Thus the second term on the right side in (12) can be written as

$$u^{p\bar{q}} a_p u_{x_n \bar{q}} = 2a_p u^{p\bar{q}} u_{n\bar{q}} + \sqrt{-1} u^{p\bar{q}} a_p u_{t\bar{q}} = 2a_n + \sqrt{-1} u^{p\bar{q}} a_p u_{t\bar{q}}.$$

Here the derivatives of  $a$  are bounded. We also have

$$|u^{p\bar{q}} a_p u_{t\bar{q}}| \leq (u^{p\bar{q}} a_p a_{\bar{q}})^{\frac{1}{2}} (u^{p\bar{q}} u_{tp} u_{t\bar{q}})^{\frac{1}{2}}.$$

This follows from the Cauchy inequality and the fact that the positive matrix  $(u_{p\bar{q}})$  induces an inner product in  $\mathbb{C}^n$ . Therefore

$$|u^{p\bar{q}} a_p u_{x_n \bar{q}}| \leq C + C \left( \sum u^{i\bar{i}} \right)^{\frac{1}{2}} (u^{p\bar{q}} u_{tp} u_{t\bar{q}})^{\frac{1}{2}}.$$

A similar estimate holds for the third term on the right side of (12), while the fourth term is  $O(\sum u^{i\bar{i}})$ . Therefore, we have

$$|L(T_i u)| \leq C + C \sum u^{i\bar{i}} + C \left( \sum u^{i\bar{i}} \right)^{\frac{1}{2}} (u^{p\bar{q}} u_{pt} u_{\bar{q}t})^{\frac{1}{2}},$$

and hence

$$|LT_i(u - \varphi)| \leq C + C \sum u^{i\bar{i}} + C \left( \sum u^{i\bar{i}} \right)^{\frac{1}{2}} (u^{p\bar{q}} u_{pt} u_{\bar{q}t})^{\frac{1}{2}}.$$

Further, we get

$$\begin{aligned} L(u_t - \varphi_t)^2 &= u^{p\bar{q}} \partial_{p\bar{q}}(u_t - \varphi_t)^2 = 2u^{p\bar{q}} \partial_p((u_t - \varphi_t)(u_{t\bar{q}} - \varphi_{t\bar{q}})) \\ &= 2u^{p\bar{q}}(u_{pt} - \varphi_{pt})(u_{t\bar{q}} - \varphi_{t\bar{q}}) + 2u^{p\bar{q}}(u_t - \varphi_t)(u_{tp\bar{q}} - \varphi_{tp\bar{q}}) \\ &= 2(I + II). \end{aligned}$$

Note

$$I = u^{p\bar{q}} u_{pt} u_{t\bar{q}} - u^{p\bar{q}} u_{t\bar{q}} \varphi_{pt} - u^{p\bar{q}} u_{pt} \varphi_{t\bar{q}} + u^{p\bar{q}} \varphi_{pt} \varphi_{t\bar{q}}.$$

For the second term, we use Cauchy inequality as before

$$|u^{p\bar{q}} u_{t\bar{q}} \varphi_{pt}| \leq (u^{p\bar{q}} \varphi_{pt} \varphi_{\bar{q}t})^{\frac{1}{2}} (u^{p\bar{q}} u_{t\bar{q}} u_{tp})^{\frac{1}{2}}.$$

The third and fourth terms are handled in the same way. For  $II$ , we write

$$II = (u_t - \varphi_t)(u^{p\bar{q}}u_{tp\bar{q}} - u^{p\bar{q}}\varphi_{tp\bar{q}}) = (u_t - \varphi_t)(\partial_t \log f - u^{p\bar{q}}\varphi_{tp\bar{q}}).$$

Therefore, we obtain

$$L(u_t - \varphi_t)^2 \geq 2u^{p\bar{q}}u_{pt}u_{t\bar{q}} - C(u^{i\bar{i}})^{\frac{1}{2}}(u^{p\bar{q}}u_{pt}u_{q\bar{t}})^{\frac{1}{2}} - Cu^{i\bar{i}} - C,$$

and hence

$$\begin{aligned} Lw &= \pm LT_i(u - \varphi) + L(u_t - \varphi_t)^2 \\ &\geq 2u^{p\bar{q}}u_{pt}u_{q\bar{t}} - C(u^{i\bar{i}})^{\frac{1}{2}}(u^{p\bar{q}}u_{pt}u_{q\bar{t}})^{\frac{1}{2}} - Cu^{i\bar{i}} - C \\ &\geq -C \sum u^{i\bar{i}} - C. \end{aligned}$$

Next, we note

$$L(Ax_n - B|z|^2) = -Bu^{p\bar{q}}\partial_{p\bar{q}}|z|^2 = -B \sum u^{i\bar{i}}.$$

To prove (a), we need

$$(B - C) \sum u^{i\bar{i}} \geq C.$$

This can be achieved by (5) and choosing  $B$  sufficiently large. To prove (b), consider first  $\partial S_\varepsilon \cap \partial\Omega$ . By (8) or (9), we have for some  $a > 0$

$$x_n \geq a|z|^2.$$

Since  $u(t', \rho(t')) = \varphi(t', \rho(t'))$ , we obtain

$$|u_t - \varphi_t| = |\rho_t||u_{x_n} - \varphi_{x_n}| \leq C|t'| \leq C|z| \leq Cx_n^{\frac{1}{2}}.$$

Taking  $A$  large, we obtain (b) on  $\partial\Omega \cap \partial S_\varepsilon$ . Note (b) is trivially true on  $\partial S_\varepsilon \cap \Omega$  if  $A$  is sufficiently large.

By the maximum principle, we obtain

$$\pm T_i(u - \varphi) + (u_t - \varphi_t)^2 \leq Ax_n - B|z|^2 \quad \text{in } S_\varepsilon,$$

or

$$\pm T_i(u - \varphi) \leq Ax_n - B|z|^2 \quad \text{in } S_\varepsilon.$$

This implies

$$|(T_i u)_{x_n}(0)| \leq A + |(T_i \varphi)_{x_n}(0)|.$$

Note

$$(T_i u)_{x_n} = u_{t_i x_n} - \left( \frac{r_{t_i}}{r_{x_n}} \right)_{x_n} u_{x_n} - \frac{r_{t_i}}{r_{x_n}} u_{x_n x_n}.$$

Hence we get

$$(13) \quad |u_{t_i x_n}(0)| \leq C_2.$$

Using the special coordinates above again, we establish the estimate

$$(14) \quad |u_{x_n x_n}(0)| \leq C.$$

Since we already proved

$$(15) \quad |u_{t_i t_j}(0)|, |u_{t_i x_n}(0)| \leq C, \quad i, j \leq 2n - 1,$$

it suffices to prove

$$(16) \quad |u_{n\bar{n}}(0)| \leq C.$$

We may solve the equation  $\det(u_{i\bar{j}}) = f$  for  $u_{n\bar{n}}(0)$ , and see that (16) follows from (15) provided the  $(n-1) \times (n-1)$  matrix  $(u_{z_\alpha \bar{z}_\beta}(0))$  satisfies

$$(17) \quad (u_{z_\alpha \bar{z}_\beta}(0))_{\alpha, \beta < n} \geq c_1 I_{n-1},$$

for some positive constant  $c_1 > 0$ . After a subtraction of a linear function, we may assume  $\varphi_{t_j}(0) = 0, j \leq 2n-1$ . For (17), it suffices to prove

$$\sum_{\alpha, \beta < n} u_{z_\alpha \bar{z}_\beta}(0) \xi_\alpha \bar{\xi}_\beta \geq c_1 |\xi|^2.$$

In the following, we take  $\xi = (1, 0, \dots, 0) \in \mathbb{C}^{n-1}$  and prove

$$u_{1\bar{1}}(0) \geq c_1.$$

Let  $\tilde{u} = u - \lambda x_n$ , with  $\lambda$  so chosen that

$$\left( \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} \right) \tilde{u}(t_1, \dots, t_{2n-1}, \rho(t_1, \dots, t_{2n-1})) = 0 \quad \text{at } 0.$$

Note

$$\begin{aligned} \left( \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} \right) \tilde{u}(t', \rho(t')) &= \sum_{i=1}^2 \frac{\partial}{\partial t_i} (\tilde{u}_{t_i} + \tilde{u}_{x_n} \rho_{t_i}) \\ &= \sum_{t=1}^2 (\tilde{u}_{t_i t_i} + \tilde{u}_{x_n t_i} \rho_{t_i} + (\tilde{u}_{x_n t_i} + \tilde{u}_{x_n x_n} \rho_{t_i}) \rho_{t_i} + \tilde{u}_{x_n} \rho_{t_i t_i}). \end{aligned}$$

This implies at 0

$$(18) \quad u_{1\bar{1}}(0) + \tilde{u}_{x_n}(0) \rho_{1\bar{1}}(0) = u_{1\bar{1}}(0) + (u_{x_n}(0) - \lambda) \rho_{1\bar{1}}(0) = 0.$$

Now we claim close to the origin

$$(19) \quad \tilde{u}|_{\partial\Omega} \leq \Re p(z) + \Re \sum_{j=2}^n a_j z_1 \bar{z}_j + C \sum_{j=2}^n |z_j|^2,$$

where  $p$  is a holomorphic cubic polynomial without linear terms and  $a_2, \dots, a_n$  are complex numbers.

To prove (19), we consider the Taylor expansion of  $\tilde{u}(t', \rho(t'))$ . Obviously, there is no linear terms. For the quadratic terms, we consider  $t_i t_j, 1 \leq i, j \leq 2$  and  $t_i t_j, 1 \leq i \leq 2, 3 \leq j \leq 2n-1$ . Since

$$\left( \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} \right) \tilde{u}(0) = 0,$$

the linear combination of  $t_i t_j, 1 \leq i, j \leq 2$ , can be written as

$$a(t_1^2 - t_2^2) + bt_1 t_2,$$

which can be put into the form

$$\Re[(\alpha - \sqrt{-1}\beta)z_1^2],$$

for some complex numbers  $\alpha$  and  $\beta$ . Next, a linear combination of  $t_1 t_{2j-1}, t_2 t_{2j-1}, t_1 t_{2j}, t_2 t_{2j}$ ,  $2 \leq j \leq n-1$ , can be written as

$$\Re((a + \sqrt{-1}b)z_1 z_j) + \Re((c + \sqrt{-1}d)z_1 \bar{z}_j).$$

In fact, we have

$$\begin{aligned} \Re((a + \sqrt{-1}b)z_1 z_j) &= a(t_1 t_{2j-1} - t_2 t_{2j}) - b(t_1 t_{2j} + t_2 t_{2j-1}), \\ \Re((c + \sqrt{-1}d)z_1 \bar{z}_j) &= c(t_1 t_{2j-1} + t_2 t_{2j}) + d(t_1 t_{2j} - t_2 t_{2j-1}). \end{aligned}$$

Last a linear combination of  $t_1 t$  and  $t_2 t$  can be written as

$$\Re((a + \sqrt{-1}b)z_1 t),$$

for some complex numbers  $a$  and  $b$ . Now we consider the cubic terms. We consider two cases:

Case 1. cubic in  $(t_1, t_2)$ .

Case 2. quadratic in  $(t_1, t_2)$  and linear in  $t_j, 3 \leq j \leq 2n-1$ .

For Case 1, we notice that any real homogeneous cubic in  $(t_1, t_2)$  admits a unique decomposition

$$(20) \quad \Re(\alpha(t_1 + \sqrt{-1}t_2)^3 + \beta(t_1 - \sqrt{-1}t_2)(t_1 + \sqrt{-1}t_2)^2) = \Re(\alpha z_1^3 + \beta z_1 |z_1|^2),$$

for some complex numbers  $\alpha$  and  $\beta$ . This can be seen easily by expanding (20) in terms of  $t_1, t_2$  and noticing that complex numbers  $\alpha, \beta$  provide four real parameters and any real homogeneous cubic in  $(t_1, t_2)$  has at most four coefficients. Hence we have proved that

$$(21) \quad \begin{aligned} \tilde{u}|_{\partial\Omega} &\leq \Re \sum_{j=2}^{n-1} a_j z_1 \bar{z}_j + \Re(\alpha z_1 y_n) + \Re(\beta z_1 |z_1|^2) + \Re p(z_1, \dots, z_n) \\ &+ \text{cubic terms such that } (t_1, t_2) \text{ appears at most quadratically} \\ &+ \text{fourth order terms,} \end{aligned}$$

for some complex numbers  $\alpha, \beta$  and  $a_1, \dots, a_{n-1}$ . Now we focus on  $z_1 |z_1|^2$ . By (9), there holds on  $\partial\Omega$

$$x_n = \sum_{i,j \leq 2n-1} b_{ij} t_i t_j + O(|t'|^3).$$

We may assume  $(b_{ij})_{i,j \leq 2n-2}$  is diagonal. Note that the coefficients of  $t_1^2$  and  $t_2^2$  are same and given by  $\rho_{1\bar{1}}(0) > 0$ . Hence on  $\partial\Omega$  there holds

$$(22) \quad |z_1|^2 = t_1^2 + t_2^2 = ax_n + \text{quadratic of } t' + O(|t'|^3),$$

where there is no quadratic expression of  $(t_1, t_2)$  in the quadratic part of  $t'$ . By adjusting  $a_j$  and  $p$  in (21) appropriately, we get

$$\begin{aligned} \tilde{u}|_{\partial\Omega} &\leq \Re \sum_{j=2}^{n-1} a_j z_1 \bar{z}_j + \Re(\alpha z_1 y_n) + \Re(\beta z_1 x_n) + \Re p(z_1, \dots, z_{n-1}) \\ &\quad + \text{quadratic terms in } t_3, t_4, \dots, t_{2n-1} \\ &\quad + \text{cubic terms such that } (t_1, t_2) \text{ appears at most quadratically} \\ &\quad + \text{fourth order terms.} \end{aligned}$$

Obviously

$$\Re(\alpha z_1 y_n + \beta z_1 x_n) = \Re(az_1 z_n + bz_1 \bar{z}_n),$$

for some complex numbers  $a, b$ . Applying Cauchy inequality to cubic terms in Case 2, we obtain

$$\tilde{u}|_{\partial\Omega} \leq \Re p(z) + \Re \sum_{j=2}^n a_j z_1 \bar{z}_j + C \sum_{j=2}^n |z_j|^2 + C|z|^4.$$

With (22), it is easy to see

$$|z|^4 \leq C \sum_{j=2}^n |z_j|^2.$$

Hence (19) is proved.

Let  $\tilde{\tilde{u}} = \tilde{u} - \Re p(z)$  and observe that  $\tilde{\tilde{u}}$  satisfies

$$\det \tilde{\tilde{u}}_{i\bar{j}} = \det u_{i\bar{j}} = f(z).$$

We set again  $S_\varepsilon = \{z \in \bar{\Omega}; 0 \leq x_n \leq \varepsilon\}$ . Let

$$\begin{aligned} h(z) &= -\delta_0 x_n + \delta_1 |z|^2 + \frac{1}{2B} \sum_{j=z}^n |a_j z_1 + Bz_j|^2 \\ &= -\delta_0 x_n + \delta_1 |z|^2 + \frac{1}{2B} \sum_{1 < j \leq n} |a_j|^2 |z_1|^2 + \Re \sum_{j=2}^n a_j z_1 \bar{z}_j + \frac{B}{2} \sum_{j=2}^n |z_j|^2. \end{aligned}$$

On  $\partial\Omega \cap \partial S_\varepsilon$ , we require

$$-\delta_0 x_n + \delta_1 |z|^2 \geq 0,$$

or

$$x_n \leq \frac{\delta_1}{\delta_0} |z|^2.$$

This can be achieved by taking  $\delta_0 \ll \delta_1$ . Then we have

$$\tilde{\tilde{u}} \leq h \quad \text{on } \partial\Omega \cap \partial S_\varepsilon,$$

if  $B \geq 2C$ . On  $\partial S_\varepsilon \cap \Omega$ ,  $\tilde{\tilde{u}} \leq h$  holds obviously if  $B$  is chosen sufficiently large. This is because  $B|z_n|^2 \geq B\varepsilon^2$  there. Therefore by taking  $B \gg C$  and  $\delta_0 \ll \delta_1$  for  $\delta_1$  to be fixed later, we obtain

$$\tilde{\tilde{u}} \leq h \quad \text{on } \partial S_\varepsilon.$$

Next, as in the real case, we get

$$(h_{i\bar{j}}) = \begin{pmatrix} \delta_1 + \frac{1}{B} \sum_{j=2}^n |a_j|^2 & a_2 & \cdots & a_n \\ \bar{a}_2 & 2\delta_1 + B & & \\ \vdots & & \ddots & \\ \bar{a}_n & & & 2\delta_1 + B \end{pmatrix}.$$

The eigenvalues of  $(h_{i\bar{j}})$  are given by

$$2\delta_1, 2\delta_1 + B + \frac{1}{B} \sum_{j=2}^n |a_j|^2, 2\delta_1 + B, \dots, 2\delta_1 + B.$$

This implies  $h$  is strictly pluri-subharmonic and

$$\det h_{i\bar{j}} = 2\delta_1(2\delta_1 + B)^{n-2} \left( 2\delta_1 + B + \frac{1}{B} \sum_{j=2}^n |a_j|^2 \right).$$

Therefore, choosing  $\delta_1$  small, we have

$$\det h_{i\bar{j}} < f \text{ in } \Omega.$$

Thus  $h$  is an upper barrier for  $\tilde{u}$ . By Lemma 1.6, we obtain

$$\tilde{u} \leq h \text{ in } S_\varepsilon.$$

Since  $\tilde{u}(0) = h(0) = 0$ , we get

$$\tilde{u}_n \leq h_n(0) = -\delta_0.$$

Then (18) implies

$$u_{1\bar{1}}(0) = -\tilde{u}_n(0)\rho_{1\bar{1}}(0) \geq \delta_0\rho_{1\bar{1}}(0).$$

Therefore, we conclude

$$|D^2u|_0 \leq K \text{ on } \partial\Omega.$$

Step 5. We estimate  $D^2u$  in  $\Omega$ . For any real constant coefficient operator  $D$  of the form

$$D = \sum (a_j \partial_{x_j} + b_j \partial_{y_j}) \text{ with } \sum (a_j^2 + b_j^2) = 1,$$

we have

$$LD^2u \geq D^2 \log f \geq -C.$$

As in Step 3, we obtain for large  $\lambda$

$$L(D^2u + e^{\lambda|z|^2}) \geq 0.$$

By the maximum principle, we conclude

$$\max_{\bar{\Omega}} D^2u \leq \max_{\partial\Omega} D^2u + C \leq K.$$

With the upper bound for every  $D^2u$  and the lower bound  $u_{x_i x_i} + u_{y_i y_i} \geq 0$ , we will prove the bound for any second derivatives.

First, by taking  $D = \partial_{x_i}$  and  $D = \partial_{y_i}$ , we have

$$\partial_{x_i x_i} u \leq K \text{ and } \partial_{y_i y_i} u \leq K.$$

With  $\partial_{x_i x_i} u + \partial_{y_i y_i} u \geq 0$ , we obtain

$$|\partial_{x_i x_i} u|, |\partial_{y_i y_i} u| \leq K.$$

Next, by taking  $D = \frac{1}{\sqrt{2}}(\partial_{x_i} \pm \partial_{y_i})$ , we get

$$D^2 u = \frac{1}{2}(\partial_{x_i x_i} u \pm 2\partial_{x_i y_i} u + \partial_{y_i y_i} u) \leq K,$$

and hence

$$|\partial_{x_i y_i} u| \leq K.$$

Last, for  $i \neq j$ , by taking  $D = \frac{1}{\sqrt{2}}(\partial_{x_i} \pm \partial_{y_j})$  and  $D = \frac{1}{\sqrt{2}}(\partial_{x_j} \pm \partial_{y_i})$ , we obtain

$$\frac{1}{2}(\partial_{x_i x_i} u \pm 2\partial_{x_i y_j} u + \partial_{y_j y_j} u) \leq K,$$

$$\frac{1}{2}(\partial_{x_j x_j} u \pm 2\partial_{x_j y_i} u + \partial_{y_i y_i} u) \leq K.$$

Adding these two inequalities yields

$$\pm \partial_{x_i y_j} u \pm \partial_{x_j y_i} u \leq 2K.$$

We remark that there are four inequalities here. It is then easy to see

$$|\partial_{x_i y_j} u|, |\partial_{x_j y_i} u| \leq 2K.$$

This finishes the proof. □

## 2. HÖLDER ESTIMATES FOR SECOND DERIVATIVES

In this section, we derive the Hölder estimates for second derivatives of solutions of fully nonlinear elliptic equations under the key assumption that the functional is concave with respect to the Hessian matrices. This restriction still enables us to cover the equations of Monge-Ampère type. To be specific, we consider a smooth function

$$F : \mathcal{S} \longrightarrow \mathbb{R},$$

where  $\mathcal{S}$  is the space of  $n \times n$  symmetric matrices. For any  $M = (m_{ij}) \in \mathcal{S}$ , we denote

$$F_{ij}(M) = \frac{\partial F}{\partial m_{ij}}(M),$$

$$F_{ij,kl}(M) = \frac{\partial^2 F}{\partial m_{ij} \partial m_{kl}}(M).$$

Suppose a smooth function  $u$  in an open set in  $\mathbb{R}^n$  satisfies

$$F(D^2u) = f(x),$$

for some smooth function  $f$  defined in the same open set. For any unit vector  $\gamma \in \mathbb{R}^n$ , we differentiate the equation with respect to  $\gamma$  to get

$$F_{ij}u_{ij\gamma} = f_\gamma,$$

where  $F_{ij}$  is evaluated at  $(D^2u(x))$ . This leads to the introduction of the linear operator

$$L = F_{ij}\partial_{ij}.$$

The above calculation shows  $u_\gamma$  satisfies

$$Lu_\gamma = f_\gamma.$$

Now we differentiate with respect to  $\gamma$  again to get

$$F_{ij}u_{ij\gamma\gamma} + F_{ij,kl}u_{ij\gamma}u_{kl\gamma} = f_{\gamma\gamma},$$

or

$$Lu_{\gamma\gamma} + F_{ij,kl}u_{ij\gamma}u_{kl\gamma} = f_{\gamma\gamma}.$$

The fully nonlinear equation  $F(D^2u) = f$  is elliptic if the matrix  $(F_{ij})$  is positive definite. In other words, the linearized operator  $L$  is elliptic. The function  $F(M)$  is concave with respect to  $M$  if there holds for any  $M = (m_{ij}) \in \mathcal{S}$

$$F_{ij,kl}m_{ij}m_{kl} \leq 0.$$

If  $F$  is concave at  $D^2u(x)$  for any  $x$ , then  $u_{\gamma\gamma}$  satisfies

$$Lu_{\gamma\gamma} \geq f_{\gamma\gamma}.$$

In this chapter, we prove the Hölder estimates for the second derivatives of the solutions to the fully nonlinear elliptic equation of the concave type.

**2.1. Harnack Inequality.** In this section, we derive Harnack inequality and its corollaries. These results will be needed in later sections. We will focus on a priori estimates instead of regularity. Our assumptions are more than what we need.

Suppose  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  and consider a linear elliptic operator  $L$  in  $\Omega$

$$L \equiv a_{ij}(x)D_{ij},$$

where coefficients  $a_{ij}$  are at least continuous in  $\Omega$ . The ellipticity means that the coefficient matrix  $A = (a_{ij})$  is positive definite everywhere in  $\Omega$ . We assume that  $L$  is uniformly elliptic in the following sense

$$\lambda I \leq (a_{ij}) \leq \Lambda I,$$

where  $\lambda$  and  $\Lambda$  are two positive constants. This means that all eigenvalues are between  $\lambda$  and  $\Lambda$ . We also set  $D = \det(A)$  and  $D^* = D^{\frac{1}{n}}$  so that  $D^*$  is the geometric mean of the eigenvalues of  $A$ . Obviously the uniform ellipticity implies

$$\lambda \leq D^* \leq \Lambda.$$

Before stating the Alexandroff maximum principle, we first introduce the concept of contact sets. For  $u \in C^2(\Omega)$  we define

$$\Gamma^+ = \{y \in \Omega; u(x) \leq u(y) + Du(y) \cdot (x - y), \text{ for any } x \in \Omega\}.$$

The set  $\Gamma^+$  is called the upper contact set of  $u$ , and Hessian matrix  $D^2u = (D_{ij}u)$  is nonpositive on  $\Gamma^+$ . In fact, the upper contact set can also be defined for continuous function  $u$  by the following

$$\Gamma^+ = \{y \in \Omega; u(x) \leq u(y) + p \cdot (x - y), \text{ for any } x \in \Omega \text{ and some } p = p(y) \in \mathbb{R}^n\}.$$

Clearly,  $u$  is concave if and only if  $\Gamma^+ = \Omega$ . If  $u \in C^1(\Omega)$ , then  $p(y) = Du(y)$  and any support hyperplane must then be a tangent plane to the graph.

Now we consider the equation of the following form

$$Lu = f \quad \text{in } \Omega$$

for some  $f \in C(\Omega)$ . In fact, it suffices to assume that  $a_{ij}$  are bounded and measurable,  $f \in L^n(\Omega)$  and  $u \in W^{2,n}(\Omega)$ . So the equation holds almost everywhere.

**Theorem 2.1.** *Suppose  $u \in C(\bar{\Omega}) \cap C^2(\Omega)$  satisfies  $Lu \geq f$  in  $\Omega$ . Then there holds*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \frac{d}{n\omega_n^{\frac{1}{n}}} \| \frac{f^-}{D^*} \|_{L^n(\Gamma^+)},$$

where  $\Gamma^+$  is the upper contact set of  $u$ ,  $d = \text{diam}(\Omega)$  and  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ .

*Proof.* It suffices to prove

$$(1) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{\frac{1}{n}}} \left( \int_{\Gamma^+} |\det D^2u| \right)^{\frac{1}{n}}.$$

In fact, for the positive definite matrix  $A = (a_{ij})$  we have

$$\det(-D^2u) \leq \frac{1}{D} \left( \frac{-a_{ij}D_{ij}u}{n} \right)^n \quad \text{on } \Gamma^+.$$

This implies Theorem 2.1.

Next we proceed to prove (1). Without loss of generality we assume  $u \leq 0$  on  $\partial\Omega$ . Set  $\Omega^+ = \{u > 0\}$ . By the area-formula for  $Du$  in  $\Gamma^+ \cap \Omega^+ \subset \Omega$ , we have

$$(2) \quad |Du(\Gamma^+ \cap \Omega^+)| \leq \int_{\Gamma^+ \cap \Omega^+} |\det(D^2u)|,$$

where  $|\det(D^2u)|$  is the Jacobian of the map  $Du : \Omega \rightarrow \mathbb{R}^n$ . In fact we may consider  $\chi_\varepsilon = Du - \varepsilon \text{Id} : \Omega \rightarrow \mathbb{R}^n$ . Then  $D\chi_\varepsilon = D^2u - \varepsilon I$ , which is negative definite in  $\Gamma^+$ . Hence by the change of variable formula, we have

$$|\chi_\varepsilon(\Gamma^+ \cap \Omega^+)| = \int_{\Gamma^+ \cap \Omega^+} |\det(D^2u - \varepsilon I)|,$$

which implies (2) if we let  $\varepsilon \rightarrow 0$ .

Now we claim  $B_{\tilde{M}}(0) \subset Du(\Gamma^+ \cap \Omega^+)$ , i.e., for any  $a \in \mathbb{R}^n$  with  $|a| < \tilde{M}$  there exists  $x \in \Gamma^+ \cap \Omega^+$  such that  $a = Du(x)$ . Here  $\tilde{M} = \sup u/d$ .

We may assume  $u$  attains its maximum  $m > 0$  at  $0 \in \Omega$ , i.e.,

$$u(0) = m = \sup_{\Omega} u.$$

Consider an affine function for  $|a| < m/d (\equiv \tilde{M})$

$$L(x) = m + a \cdot x.$$

Then  $L(x) > 0$  for any  $x \in \Omega$  and  $L(0) = m$ . Since  $u$  assumes its maximum at 0, then  $Du(0) = 0$ . Hence there exists an  $x_1$  close to 0 such that  $u(x_1) > L(x_1) > 0$ . Note that  $u \leq 0 < L$  on  $\partial\Omega$ . Hence there exists an  $\tilde{x} \in \Omega$  such that  $Du(\tilde{x}) = DL(\tilde{x}) = a$ . Now we may translate vertically the plane  $y = L(x)$  to the highest such position, i.e., the whole surface  $y = u(x)$  lies below the plane. Clearly at such point, the function  $u$  is positive.  $\square$

**Remark 2.2.** The integral domain  $\Gamma^+$  can be replaced by

$$\Gamma^+ \cap \{x \in \Omega; u(x) > \sup_{\partial\Omega} u^+\}.$$

**Remark 2.3.** There is no assumption on uniform ellipticity in Theorem 2.1.

Next we discuss the Calderon-Zygmund decomposition. First we introduce some terminology.

Take the unit cube  $Q_1$ . Cut it equally into  $2^n$  cubes, which we take as the first generation. Do the same cutting for these small cubes to get the second generation. Continue this process. These cubes (from all generations) are called *dyadic cubes*. Any  $(k+1)$ -generation cube  $Q$  comes from some  $k$ -generation cube  $\tilde{Q}$ , which is called the *predecessor* of  $Q$ .

**Lemma 2.4.** *Suppose  $f \in L^1(Q_1)$  is nonnegative and  $\alpha > |Q_1|^{-1} \int_{Q_1} f$  is a fixed constant. Then there exists a sequence of (non-overlapping) dyadic cubes  $\{Q^j\}$  in  $Q_1$  such that*

$$f(x) \leq \alpha \text{ a.e. in } Q_1 \setminus \cup_j Q^j,$$

and

$$\alpha \leq \frac{1}{|Q^j|} \int_{Q^j} f dx < 2^n \alpha.$$

*Proof.* Cut  $Q_1$  into  $2^n$  dyadic cubes and keep the cube  $Q$  if  $|Q|^{-1} \int_Q f \geq \alpha$ . For others keep cutting and always keep the cube  $Q$  if  $|Q|^{-1} \int_Q f \geq \alpha$  and cut the rest. Let  $\{Q^j\}$  be the cubes we have kept during this infinite process. We only need to verify that

$$f(x) \leq \alpha \text{ a.e. in } Q_1 \setminus \cup_j Q^j.$$

Let  $F = Q_1 \setminus \cup_j Q^j$ . For any  $x \in F$ , from the way we collect  $\{Q^j\}$ , there exists a sequence of cubes  $\tilde{Q}^i$  containing  $x$  such that

$$\frac{1}{|\tilde{Q}^i|} \int_{\tilde{Q}^i} f < \alpha$$

and

$$\text{diam}(\tilde{Q}^i) \rightarrow 0 \text{ as } i \rightarrow \infty.$$

By Lebesgue density theorem, this implies

$$f \leq \alpha \text{ a.e. in } F.$$

This finishes the proof.  $\square$

Now we give a simple consequence of Calderon-Zygmund decomposition.

**Lemma 2.5.** *Suppose measurable sets  $A \subset B \subset Q_1$  have the following properties*

(i)  $|A| < \delta$  for some  $\delta \in (0, 1)$ ;

(ii) for any dyadic cube  $Q$ ,  $|A \cap Q| \geq \delta|Q|$  implies  $\tilde{Q} \subset B$  for the predecessor  $\tilde{Q}$  of  $Q$ .

Then there holds

$$|A| \leq \delta|B|.$$

*Proof.* Apply the Calderon-Zygmund decomposition (Lemma 2.4) to  $f = \chi_A$ . We obtain, by the assumption (i), a sequence of dyadic cubes  $\{Q^j\}$  such that

$$A \subset \cup_j Q^j \text{ except for a set of measure zero,}$$

$$\delta \leq \frac{|A \cap Q^j|}{|Q^j|} < 2^n \delta,$$

and

$$\frac{|A \cap \tilde{Q}^j|}{|\tilde{Q}^j|} < \delta,$$

for any predecessor  $\tilde{Q}^j$  of  $Q^j$ . By the assumption (ii), we have  $\tilde{Q}^j \subset B$  for each  $j$ . Hence we obtain

$$A \subset \cup_j \tilde{Q}^j \subset B.$$

We relabel  $\{\tilde{Q}^j\}$  so that they are non-overlapping. Therefore we get

$$|A| \leq \sum_i |A \cap \tilde{Q}^i| \leq \delta \sum_i |\tilde{Q}^i| \leq \delta |B|.$$

This finishes the proof.  $\square$

Now we begin to prove the Harnack inequality. First we prove a local maximum principle for subsolutions.

**Theorem 2.6.** *Suppose  $Lu \geq f$  in  $\Omega$ . Then for any ball  $B_{2R}(x_0) \subset \Omega$  and  $p > 0$ , there holds*

$$\sup_{B_R(x_0)} u \leq C \left\{ \left( \frac{1}{R^n} \int_{B_{2R}(x_0)} (u^+)^p \right)^{\frac{1}{p}} + R \|f^-\|_{L^n(B_{2R}(x_0))} \right\},$$

where  $C = C(n, \lambda, \Lambda, p)$  is a positive constant.

*Proof.* Without loss of generality, we can assume  $B_{2R}(x_0) = B_1(0)$ , the general case being recovered by means of coordinate transformation  $x \mapsto \frac{x - x_0}{2R}$ .

A cut-off function is needed since there is no control on  $\partial B_1$ . For  $\beta \geq 1$ , a cut-off function  $\eta$  is defined by

$$\eta(x) = (1 - |x|^2)^\beta.$$

By a differentiation, we obtain

$$D_i \eta(x) = -2\beta x_i (1 - |x|^2)^{\beta-1},$$

and

$$D_{ij} \eta(x) = -2\beta \delta_{ij} (1 - |x|^2)^{\beta-1} + 4\beta(\beta - 1) x_i x_j (1 - |x|^2)^{\beta-2}.$$

Hence we get

$$|D\eta(x)| = 2\beta|x|(1 - |x|^2)^{\beta-1} \leq 2\beta\eta^{1-\frac{1}{\beta}},$$

and

$$L\eta = 4\beta(\beta - 1) a_{ij} x_i x_j (1 - |x|^2)^{\beta-2} - 2\beta \sum a_{ii} (1 - |x|^2)^{\beta-1}.$$

Setting  $v = \eta u$ , we then have

$$Lv = \eta Lu + uL\eta + 2a_{ij} D_i \eta D_j u.$$

Consider the upper contact set  $\Gamma^+ = \Gamma_{v^+}$  of  $v$  in  $B_1$ . We clearly have  $v > 0$  in  $\Gamma^+$ , since  $v = 0$  on  $\partial B_1$ . Hence for any  $x \in \Gamma^+$

$$|Dv(x)| \leq \frac{v(x)}{1 - |x|} \leq \frac{2v(x)}{1 - |x|^2},$$

and then in  $\Gamma^+$

$$|Du(x)| = \frac{1}{\eta} |Dv - uD\eta| \leq \frac{1}{\eta} \left( \frac{2v}{1 - |x|^2} + u|D\eta| \right) \leq 2(1 + \beta)\eta^{-1-\frac{1}{\beta}} v.$$

Then in  $\Gamma^+$  we have

$$|a_{ij} D_i \eta D_j u| \leq \Lambda |Du| |D\eta| \leq 4\beta(1 + \beta)\Lambda \eta^{-\frac{2}{\beta}} v.$$

Note we also have  $u > 0$  in  $\Gamma^+$ . Hence, we get

$$\begin{aligned} -uL\eta &= 2\beta u \sum a_{ii}(1-|x|^2)^{\beta-1} - 4\beta(\beta-1)ua_{ij}x_ix_j(1-|x|^2)^{\beta-1} \\ &\leq 2\beta u \sum a_{ii}(1-|x|^2)^{\beta-1} \\ &= 2\beta n\Lambda u\eta^{1-\frac{1}{\beta}} = 2n\Lambda\beta v\eta^{-\frac{1}{\beta}} \\ &\leq 2n\Lambda\beta v\eta^{-\frac{2}{\beta}}. \end{aligned}$$

Therefore on  $\Gamma^+$  we obtain the inequality

$$-a_{ij}D_{ij}v \leq c(n, \beta, \lambda, \Lambda)\eta^{-\frac{2}{\beta}}v + \eta f^- \leq c\eta^{-\frac{2}{\beta}}v + f^-.$$

Applying the Alexandroff maximum principle for  $\beta \geq 2$ , we obtain

$$\begin{aligned} \sup_{B_1} v &\leq C \left\{ \|\eta^{-\frac{2}{\beta}}v^+\|_{L^n(B_1)} + \|f^-\|_{L^n(B_1)} \right\} \\ &\leq C \left\{ \left( \sup_{B_1} v^+ \right)^{1-\frac{2}{\beta}} \|(u^+)^{\frac{2}{\beta}}\|_{L^n(B_1)} + \|f^-\|_{L^n(B_1)} \right\}. \end{aligned}$$

If we choose  $\beta > 2$ , then using Cauchy inequality we get

$$\begin{aligned} \sup_{B_1} v &\leq C \left\{ \|(u^+)^{\frac{2}{\beta}}\|_{L^n(B_1)}^{\frac{\beta}{2}} + \|f^-\|_{L^n(B_1)} \right\} \\ &= C \left\{ \|u^+\|_{L^{\frac{2n}{\beta}}(B_1)} + \|f^-\|_{L^n(B_1)} \right\}. \end{aligned}$$

For  $\beta > 2$ , we have  $\frac{2n}{\beta} \in (0, n)$ . Hence we have for any  $p \in (0, n)$

$$\sup_{B_1} v^+ \leq C \left\{ \|u^+\|_{L^p(B_1)} + \|f^-\|_{L^n(B_1)} \right\}.$$

We use Hölder inequality to get the result for  $p \geq n$ .  $\square$

By replacing  $u$  with  $-u$ , Theorem 2.6 extends automatically to supersolutions and solutions of the equation  $Lu = f$ .

**Corollary 2.7.** *Suppose  $Lu = f$  in  $\Omega$ . Then for any ball  $B_{2R}(x_0) \subset \Omega$  and  $p > 0$ , there holds*

$$\sup_{B_R(x_0)} |u| \leq C \left\{ \left( \frac{1}{R^n} \int_{B_{2R}(x_0)} |u|^p \right)^{\frac{1}{p}} + R\|f\|_{L^n(B_{2R}(x_0))} \right\},$$

where  $C$  is a positive constant depending only on  $n, \lambda, \Lambda$  and  $p$ .

Now we turn to supersolutions. For convenience we work in cubes instead of balls. The following result is the key ingredient. It claims that if a solution is small somewhere in  $Q_3$  then it is under control in a good portion of  $Q_1$ .

**Lemma 2.8.** *Suppose  $Lu \leq f$  in  $B_{2\sqrt{n}}$  for some  $f \in C(B_{2\sqrt{n}})$ . Then there exist constants  $\varepsilon_0 > 0$ ,  $\mu \in (0, 1)$  and  $M > 1$ , depending only on  $n, \lambda$  and  $\Lambda$ , such that if*

$$(1) \quad \begin{aligned} &u \geq 0 \text{ in } B_{2\sqrt{n}}, \\ &\inf_{Q_3} u \leq 1, \\ &\|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0, \end{aligned}$$

there holds

$$|\{u \leq M\} \cap Q_1| > \mu.$$

We will construct a function  $g$ , which is very concave outside  $Q_1$ , such that, if we correct  $u$  by  $g$ , the lower contact set of  $u + g$  occurs in  $Q_1$  and occupies a large portion of  $Q_1$ . In other words, we localize where the contact occurs by choosing suitable functions.

*Proof.* Note  $B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{2\sqrt{n}}$ . Define  $g$  in  $B_{2\sqrt{n}}$  by

$$g(x) = -M \left(1 - \frac{|x|^2}{4n}\right)^\beta,$$

for large  $\beta > 0$  to be determined and some  $M > 0$ . We choose  $M$ , according to  $\beta$ , such that

$$(2) \quad g = 0 \text{ on } \partial B_{2\sqrt{n}} \quad \text{and} \quad g \leq -2 \text{ in } Q_3.$$

Set  $w = u + g$  in  $B_{2\sqrt{n}}$ . We will show by choosing  $\beta$  large that

$$(3) \quad Lw \leq f \text{ in } B_{2\sqrt{n}} \setminus Q_1.$$

We need to calculate the Hessian matrix of  $g$ . Note

$$D_{ij}g(x) = \frac{M}{2n}\beta \left(1 - \frac{|x|^2}{4n}\right)^{\beta-1} \delta_{ij} - \frac{M}{(2n)^2}\beta(\beta-1) \left(1 - \frac{|x|^2}{4n}\right)^{\beta-2} x_i x_j.$$

This implies that

$$a_{ij}D_{ij}g = \frac{M}{2n}\beta \left(1 - \frac{|x|^2}{4n}\right)^{\beta-2} \left\{ \left(1 - \frac{|x|^2}{4n}\right) \sum a_{ii} - \frac{1}{2n}(\beta-1)a_{ij}x_i x_j \right\}.$$

Therefore for  $|x| \geq 1/4$  we have

$$Lg \leq \frac{M}{2n}\beta \left(1 - \frac{|x|^2}{4n}\right)^{\beta-2} \left\{ \left(1 - \frac{|x|^2}{4n}\right)a_{ii} - \frac{1}{2n}(\beta-1)\lambda|x|^2 \right\} \leq 0,$$

if we choose  $\beta$  large, depending only on  $n, \lambda$  and  $\Lambda$ . This finishes the proof of (3). In fact we obtain

$$Lw \leq f + \eta \text{ in } B_{2\sqrt{n}},$$

for some  $\eta \in C_0^\infty(Q_1)$  and  $0 \leq \eta \leq C(n, \lambda, \Lambda)$ .

Now we apply Theorem 2.1 to  $-w$  in  $B_{2\sqrt{n}}$ . Note  $\inf_{Q_3} w \leq -1$  and  $w \geq 0$  on  $\partial B_{2\sqrt{n}}$  by (1) and (2). We obtain

$$\begin{aligned} 1 &\leq C \left( \int_{B_{2\sqrt{n}} \cap \Gamma_w^-} (|f| + \eta)^n \right)^{\frac{1}{n}} \\ &\leq C \|f\|_{L^n(B_{2\sqrt{n}})} + C |\Gamma_w^- \cap Q_1|^{\frac{1}{n}}, \end{aligned}$$

where  $\Gamma_w^-$  is the lower contact set of  $w$ . Choosing  $\varepsilon_0$  small enough we get

$$\frac{1}{2} \leq C |\Gamma_w^- \cap Q_1|^{\frac{1}{n}} \leq C |\{u \leq M\} \cap Q_1|^{\frac{1}{n}},$$

since  $w(x) \leq 0$  and hence  $u(x) \leq -g(x) \leq M$  on  $\Gamma_w^-$ .  $\square$

Next we prove the power decay of distribution functions for supersolutions.

**Lemma 2.9.** *Suppose  $Lu \leq f$  in  $B_{2\sqrt{n}}$  for some  $f \in C(B_{2\sqrt{n}})$ . Then there exist positive constants  $\varepsilon_0$ ,  $\varepsilon$  and  $C$ , depending only on  $n, \lambda$  and  $\Lambda$ , such that if*

$$(1) \quad \begin{aligned} &u \geq 0 \text{ in } B_{2\sqrt{n}}, \\ &\inf_{Q_3} u \leq 1, \\ &\|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0, \end{aligned}$$

there holds

$$|\{u \geq t\} \cap Q_1| \leq Ct^{-\varepsilon} \text{ for } t > 0.$$

*Proof.* We will prove that under the assumption (1) there holds

$$(2) \quad |\{u > M^k\} \cap Q_1| \leq (1 - \mu)^k \text{ for } k = 1, 2, \dots,$$

where  $M$  and  $\mu$  are as in Lemma 2.8.

For  $k = 1$ , (2) is just Lemma 2.8. Suppose now (2) holds for  $k - 1$ . Set

$$A = \{u > M^k\} \cap Q_1, \quad B = \{u > M^{k-1}\} \cap Q_1.$$

We will use Lemma 2.5 to prove that

$$(3) \quad |A| \leq (1 - \mu)|B|.$$

Clearly  $A \subset B \subset Q_1$  and  $|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu$  by Lemma 2.8. We claim that if  $Q = Q_r(x_0)$  is a cube in  $Q_1$  such that

$$(4) \quad |A \cap Q| > (1 - \mu)|Q|,$$

then  $\tilde{Q} \cap Q_1 \subset B$  for  $\tilde{Q} = Q_{3r}(x_0)$ . We prove it by contradiction. Suppose not. We may take  $\tilde{x} \in \tilde{Q}$  such that  $u(\tilde{x}) \leq M^{k-1}$ . Consider the transformation

$$x = x_0 + ry \quad \text{for } y \in Q_1 \text{ and } x \in Q = Q_r(x_0),$$

and the function

$$\tilde{u}(y) = \frac{1}{M^{k-1}} u(x).$$

Then  $\tilde{u} \geq 0$  in  $B_{2\sqrt{n}}$  and  $\inf_{Q_3} \tilde{u} \leq 1$ . It is easy to check that  $L\tilde{u} \leq \tilde{f}$  in  $B_{2\sqrt{n}}$  with  $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0$ . In fact we have

$$\tilde{f}(y) = \frac{r^2}{M^{k-1}} f(x) \text{ for } y \in B_{2\sqrt{n}},$$

and hence

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{M^{k-1}} \|f\|_{L^n(B_{2\sqrt{n}})} \leq \|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0.$$

Therefore,  $\tilde{u}$  satisfies the assumption (1). We may apply Lemma 2.8 to  $\tilde{u}$  to get

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = r^{-n} |\{u(x) \leq M^k\} \cap Q|.$$

Hence  $|Q \cap A^C| > \mu|Q|$ , which contradicts (4). We are in a position to apply Lemma 2.5 to get (3).  $\square$

Now we rewrite the power decay of the distribution function for supersolutions.

**Corollary 2.10.** *Suppose  $Lu \leq f$  with  $u \geq 0$  in  $B_{2\sqrt{n}}$ . Then there exist positive constants  $\gamma$  and  $C$ , depending only on  $n$ ,  $\lambda$  and  $\Lambda$ , such that*

$$|\{x \in B_1; u(x) > t\}| \leq Ct^{-\gamma} \left( \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_{2\sqrt{n}})} \right)^\gamma \text{ for any } t > 0.$$

*Proof.* This follows from Lemma 2.9 easily. Consider

$$u_\delta = \frac{u}{\inf_{Q_3} u + \delta + \frac{1}{\varepsilon_0} \|f\|_{L^n(B_{2\sqrt{n}})}},$$

for  $\delta > 0$ . We apply Lemma 2.9 to  $u_\delta$  to get, after letting  $\delta \rightarrow 0$ ,

$$|\{x \in Q_1; u(x) > t\}| \leq Ct^{-\gamma} \left( \inf_{Q_3} u + \|f\|_{L^n(B_{2\sqrt{n}})} \right)^\gamma \text{ for any } t > 0.$$

This clearly implies Corollary 2.10.  $\square$

Now we prove the weak Harnack inequality for supersolutions.

**Corollary 2.11.** *Suppose  $Lu \leq f$  with  $u \geq 0$  in  $B_{2\sqrt{n}}$ . Then there exist positive constants  $p$  and  $C$ , depending only on  $n$ ,  $\lambda$  and  $\Lambda$ , such that*

$$\left( \int_{B_1} u^p \right)^{\frac{1}{p}} \leq C \left\{ \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_{2\sqrt{n}})} \right\}.$$

*Proof.* Set  $A(t) = \{x \in B_1; u(x) > t\}$  for any  $t > 0$ . First we have for any  $\xi > 0$ ,

$$\int_{B_1} u^p = p \int_0^\infty t^{p-1} |A(t)| dt = p \int_0^\xi t^{p-1} |A(t)| dt + p \int_\xi^\infty t^{p-1} |A(t)| dt = I + II.$$

For  $I$ , we have easily

$$I \leq p|B_1| \int_0^\xi t^{p-1} dt = |B_1| \xi^p.$$

For  $II$ , we get by Corollary 2.10

$$\begin{aligned} II &= p \int_{\xi}^{\infty} t^{p-1} |A(t)| dt \leq Cp \int_{\xi}^{\infty} t^{p-\gamma-1} dt \left( \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_{2\sqrt{n}})} \right)^{\gamma} \\ &= C\xi^{p-\gamma} \left( \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_{2\sqrt{n}})} \right)^{\gamma}, \end{aligned}$$

if we choose  $p < \gamma$ . Combining these two estimates, we obtain

$$\int_{B_1} u^p \leq C \left\{ \xi^p + \xi^{p-\gamma} \left( \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_{2\sqrt{n}})} \right)^{\gamma} \right\}.$$

Next we may choose

$$\xi = \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_{2\sqrt{n}})}.$$

This finishes the proof.  $\square$

Now we can state the Harnack inequality. It is an easy corollary to Theorem 2.6 and Corollary 2.10.

**Theorem 2.12.** *Suppose  $Lu = f$  in  $B_R$  with  $u \geq 0$  in  $B_R$  for some  $f \in C(B_R)$ . Then there holds*

$$\sup_{B_{\frac{R}{2}}} u \leq C \left\{ \inf_{B_{\frac{R}{2}}} u + R\|f\|_{L^n(B_R)} \right\},$$

where  $C$  is a positive constant depending only on  $n, \lambda$  and  $\Lambda$ .

*Proof.* Without loss of generality, we may assume that  $R = 1$ , the general case being recovered by means of the coordinate transformation  $x \mapsto x/R$ . The case  $R = 1$  is implied by Theorem 2.6 and Corollary 2.10, rescaled, together with a simple covering argument.  $\square$

The interior Hölder estimates of solutions is a direct consequence.

**Corollary 2.13.** *Suppose  $Lu = f$  in  $B_R$  for some  $f \in C(B_R)$ . Then there exists an  $\alpha \in (0, 1)$ , depending only on  $n, \lambda$  and  $\Lambda$ , such that there holds*

$$\text{osc}_{B_r} u \leq C \left( \frac{r}{R} \right)^{\alpha} \left\{ \text{osc}_{B_R} u + R\|f\|_{L^n(B_R)} \right\} \quad \text{for any } r \leq R,$$

where  $C = C(n, \lambda, \Lambda)$  is a positive constant.

*Proof.* Again we prove the case  $R = 1$ . Let  $M(r) = \max_{B_r} u$  and  $m(r) = \min_{B_r} u$  for  $r \in (0, 1)$ . Then  $M(r) < +\infty$  and  $m(r) > -\infty$ . It suffices to show that

$$\omega(r) \equiv M(r) - m(r) \leq Cr^{\alpha} \left\{ \omega(1) + \|f\|_{L^n(B_1)} \right\} \quad \text{for any } r < 1.$$

Apply Theorem 2.12 to  $M(r) - u \geq 0$  in  $B_r$  to get

$$\sup_{B_{\frac{r}{2}}} (M(r) - u) \leq C \left\{ \inf_{B_{\frac{r}{2}}} (M(r) - u) + r\|f\|_{L^n(B_r)} \right\},$$

i.e.,

$$(1) \quad M(r) - m\left(\frac{r}{2}\right) \leq C \left\{ (M(r) - M\left(\frac{r}{2}\right)) + r \|f\|_{L^n(B_r)} \right\}.$$

Similarly, apply Theorem 2.12 to  $u - m(r) \geq 0$  in  $B_r$  to get

$$(2) \quad M\left(\frac{r}{2}\right) - m(r) \leq C \left\{ (m\left(\frac{r}{2}\right) - m(r)) + r \|f\|_{L^n(B_r)} \right\}.$$

Then by adding (1) and (2) together we get

$$\omega(r) + \omega\left(\frac{r}{2}\right) \leq C \left\{ (\omega(r) - \omega\left(\frac{r}{2}\right)) + r \|f\|_{L^n(B_r)} \right\},$$

or

$$\omega\left(\frac{r}{2}\right) \leq \gamma \omega(r) + Cr \|f\|_{L^n(B_r)},$$

for some  $\gamma = \frac{C-1}{C+1} < 1$ .

Choosing  $\mu$  satisfying  $(1 - \mu) \log \gamma / \log \tau = \mu$  and applying Lemma 2.14 below with  $\sigma(r) = Cr \|f\|_{L^n(B_1)}$ , we obtain

$$\omega(r) \leq Cr^\alpha \left\{ \omega(1) + \|f\|_{L^n(B_1)} \right\} \quad \text{for any } r \in (0, 1].$$

This finishes the proof.  $\square$

**Lemma 2.14.** *Let  $\omega$  and  $\sigma$  be non-decreasing functions in an interval  $(0, R]$  satisfying for any  $r \leq R$*

$$\omega(\tau r) \leq \gamma \omega(r) + \sigma(r),$$

for some  $0 < \gamma, \tau < 1$ . Then for any  $\mu \in (0, 1)$  and  $r \leq R$ , there holds

$$\omega(r) \leq C \left\{ \left(\frac{r}{R}\right)^\alpha \omega(R) + \sigma(r^\mu R^{1-\mu}) \right\},$$

where  $C = C(\gamma, \tau)$  and  $\alpha = \alpha(\gamma, \tau, \mu)$  are positive constants. In fact  $\alpha = (1 - \mu) \log \gamma / \log \tau$ .

*Proof.* Fix some number  $r_1 \leq R$ . Then for any  $r \leq r_1$  we have

$$\omega(\tau r) \leq \gamma \omega(r) + \sigma(r_1),$$

since  $\sigma$  is nondecreasing. We now iterate this inequality to get for any positive integer  $k$

$$\omega(\tau^k r_1) \leq \gamma^k \omega(r_1) + \sigma(r_1) \sum_{i=0}^{k-1} \gamma^i \leq \gamma^k \omega(R) + \frac{\sigma(r_1)}{1 - \gamma}.$$

For any  $r \leq r_1$  we choose  $k$  in such a way that

$$\tau^k r_1 < r \leq \tau^{k-1} r_1.$$

Hence we have

$$\begin{aligned}\omega(r) &\leq \omega(\tau^{k-1}r_1) \leq \gamma^{k-1}\omega(R) + \frac{\sigma(r_1)}{1-\gamma} \\ &\leq \frac{1}{\gamma} \left(\frac{r}{r_1}\right)^{\frac{\log \gamma}{\log \tau}} \omega(R) + \frac{\sigma(r_1)}{1-\gamma}.\end{aligned}$$

Now let  $r_1 = r^\mu R^{1-\mu}$ . We obtain

$$\omega(r) \leq \frac{1}{\gamma} \left(\frac{r}{R}\right)^{(1-\mu)\frac{\log \gamma}{\log \tau}} \omega(R) + \frac{\sigma(r^\mu R^{1-\mu})}{1-\gamma}.$$

This finishes the proof.  $\square$

In the rest of the section, we give another application of Harnack inequality to the normal derivatives of solutions on boundary. We introduce the notions  $B_r^+$  and  $\Gamma_r$  as follows

$$\begin{aligned}B_r^+ &= \{(x', x_n) = x; |x| < r, x_n > 0\}, \\ \Gamma_r &= \{(x', 0); |x'| < r\}.\end{aligned}$$

**Theorem 2.15.** *Let  $u$  be a solution of*

$$(1) \quad Lu = f \text{ in } B_4^+, u(x', 0) = 0,$$

and assume

$$(2) \quad |u|, |\nabla u|, |f| \leq K \text{ in } \bar{B}_4^+,$$

for some  $K > 0$ . Then there are constants  $\alpha \in (0, 1)$  and  $C > 0$  depending only on  $n, \lambda$  and  $\Lambda$  such that

$$(3) \quad \left| \frac{\partial u}{\partial x_n} \right|_{C^\alpha(\Gamma_1)} \leq C \left( \sup_{B_4^+} |\nabla u| + \sup_{B_4^+} |f| \right).$$

*Proof.* Since  $u = 0$  on  $\Gamma_4$ , we have

$$\frac{\partial u}{\partial x_n}(x', 0) = \lim_{x_n \rightarrow 0} \frac{u(x', x_n) - 0}{x_n - 0} = \lim_{x_n \rightarrow 0} \frac{u(x)}{x_n};$$

thus we estimate

$$(4) \quad v(x) \equiv \frac{u(x)}{x_n}.$$

It is convenient to introduce some notations. For  $R \leq 1$  and some  $\delta > 0$  to be fixed (one can take  $\delta = \lambda/(9n\Lambda) < \frac{1}{2}$ ), let

$$\begin{aligned}Q(R) &= \{|x'| \leq R, 0 \leq x_n \leq \delta R\}, \\ Q^+(R) &= \{|x'| \leq R, \delta R/2 \leq x_n \leq \delta R\},\end{aligned}$$

and

$$m_R = \inf_{Q_R} v, \quad M_R = \sup_{Q_R} v.$$

Two preliminary results are needed.

**Lemma 2.16.** *If  $Lu \leq f$  in  $Q(R)$  with  $u \geq 0$  and  $u(x', 0) = 0$ , then*

$$(5) \quad \inf_{Q^+(R)} v \leq \frac{2}{\delta} \inf_{Q(\frac{R}{2})} v + \frac{R}{\lambda} \sup |f|.$$

*Proof.* Let

$$(6) \quad \gamma = \inf v(x) \text{ for } x_n = \delta R, |x'| \leq R,$$

and introduce the comparison function

$$(7) \quad \omega(x) = \gamma x_n \left( \delta - 2\delta \frac{|x'|^2}{R^2} + \frac{x_n}{R} \right) - \frac{x_n(\delta R - x_n)}{2\lambda} \sup |f|.$$

It is straightforward to verify that if  $0 < \delta \leq \frac{1}{2}$ , then in  $Q(R)$

(i)  $z(x', 0) = 0$ ;

(ii)  $z(x) \leq 0$  on the sides of  $Q(R) : \{|x'| = R, 0 \leq x_n \leq \delta R\}$ ;

(iii)  $z(x) \leq 2\gamma\delta^2 R \leq \gamma\delta R$  on the top of  $Q(R) : \{|x'| \leq R, x_n = \delta R\}$ ;

and

(iv)  $Lz \geq \sup |f| \geq f$  if  $\delta > 0$  is sufficiently small.

In particular, these all hold if  $\delta = \lambda/9n\Lambda$ . In fact, for part (iv), we find

$$\begin{aligned} z_{ij} &= -\frac{4\delta\gamma}{R^2} x_n \delta_{ij}, & 1 \leq i, j \leq n-1, \\ z_{in} &= -\frac{4\delta\gamma}{R^2} x_i, & 1 \leq i \leq n-1, \\ z_{nn} &= \frac{2\gamma}{R} + \frac{1}{\lambda} \sup |f|, \end{aligned}$$

and hence

$$Lz = \frac{a_{nn}}{\lambda} \sup |f| + \frac{2\gamma}{R} a_{nn} - \frac{4\delta\gamma}{R^2} \left( \sum_{i=1}^{n-1} a_{ii} x_n + \sum_{i=1}^{n-1} a_{in} x_i \right).$$

So (iv) follows easily.

Since  $u \geq 0$  in  $Q(R)$  and by (6)  $u = x_n v \geq \gamma\delta R$  on the top of  $Q(R)$ , then  $L(u - z) \leq 0$  in  $Q(R)$  with  $u \geq z$  on  $\partial Q(R)$ . Hence, by the maximum principle,  $u \geq z$  in  $Q(R)$ , or

$$v(x) \geq \frac{z(x)}{x_n} \text{ for } x \in Q(R).$$

In particular, using the explicit formula (7), we obtain

$$\inf_{Q(\frac{R}{2})} v \geq \frac{\delta}{2} \left( \gamma - \frac{R}{\lambda} \sup |f| \right).$$

Since  $\gamma \geq \inf v(x)$  for  $x$  in  $Q^+(R)$ , this gives (5). □

The second lemma we need is a Harnack-type inequality.

**Lemma 2.17.** *Let  $Lu = f$  in  $B_4^+$  and  $u \geq 0$  in  $Q(2R)$ , with  $R \leq 1$ . There is a constant  $c > 0$ , depending only on  $n, \lambda$  and  $\Lambda$ , so that*

$$(8) \quad \sup_{Q^+(R)} \leq c \left( \inf_{Q^+(R)} v + R \sup |f| \right).$$

*Proof.* Observe that every point  $x$  in  $Q^+(R)$  there is a ball  $B_{\frac{\delta R}{2}}(x)$  in  $Q(2R)$ . Since  $u \geq 0$  in  $Q(2R)$ , we apply the scaled version of Harnack inequality with  $4r = \delta R/2$  to get

$$\sup_{B_r(x)} u \leq c \left( \inf_{B_r(x)} u + r^2 \sup |f| \right).$$

Since a finite number (independent of  $R$ ) of the balls  $B_{\frac{\delta R}{8}}(x)$ ,  $x \in Q^+(R)$ , cover  $Q^+(R)$ , we conclude that, with a new  $c$ ,

$$(9) \quad \sup_{Q^+(R)} u \leq c \left( \inf_{Q^+(R)} u + R^2 \sup |f| \right).$$

In  $Q^+(R)$  we have  $\delta R/2 \leq x_n \leq \delta R$ . Since  $u = x_n v$  this gives

$$\frac{1}{2} \delta R \sup_{Q^+(R)} v \leq \sup_{Q^+(R)} u, \quad \text{and} \quad \inf_{Q^+(R)} u \leq \delta R \inf_{Q^+(R)} v.$$

Combined with (9), this proves (8) with some possibly larger constant  $c$ .  $\square$

Now we continue the proof of Theorem 2.15. In what follows, we assume  $R \leq 1$  and denote by  $C$  a constant depending only on  $n, \lambda$  and  $\Lambda$ . By Lemma 2.17, with  $u$  replaced by  $u - m_{2R}x_n \geq 0$  in  $Q(2R)$ , we obtain

$$\sup_{Q^+(R)} (v - m_{2R}) \leq C \left( \inf_{Q^+(R)} (v - m_{2R}) + R \sup |f| \right).$$

Thus using Lemma 2.16, we get

$$\begin{aligned} \sup_{Q^+(R)} (v - m_{2R}) &\leq C \left( \inf_{Q(\frac{R}{2})} (v - m_{2R}) + R \sup |f| \right) \\ &= C \left( m_{\frac{R}{2}} - m_{2R} + R \sup |f| \right). \end{aligned}$$

Repeating these same inequalities with  $u$  replaced by  $M_{2R}x_n - u \geq 0$  in  $Q(2k)$ , we find that

$$\sup_{Q^+(R)} (M_{2R} - v) \leq C \left( M_{2R} - M_{\frac{R}{2}} + R \sup |f| \right).$$

Adding these two inequalities thus gives

$$M_{2R} - m_{2R} \leq C \left( (M_{2R} - m_{2R}) - \left( M_{\frac{R}{2}} - m_{\frac{R}{2}} \right) + R \sup |f| \right).$$

Let  $\omega(R) = M_R - m_R$  denote the oscillation of  $v$  in  $Q(R)$ . Then we obtain

$$\omega \left( \frac{R}{2} \right) \leq \theta \omega(2R) + R \sup |f| \quad \text{for any } R \leq 1,$$

for some  $0 < \theta < 1$ . Then by using Lemma 2.14 as in the proof of Corollary 2.13, we obtain for some  $\alpha \in (0, 1)$ ,

$$\omega(R) \leq CR^\alpha (\omega(1) + \sup |f|) \quad \text{for any } R \leq 1,$$

where  $C$  is a positive constant depending only on  $n, \lambda$  and  $\Lambda$ . Note that (2) implies that  $w(1)$  is bounded. Letting  $x_n \rightarrow 0$  we find that, in the set  $\Gamma_R$  with  $R \leq 1$ , the oscillation of  $\frac{\partial u}{\partial x_n} \leq cR^\alpha$ . This is the desired Hölder estimate (3).  $\square$

**2.2. Interior Hölder Estimates for Second Derivatives.** In this section, we derive interior Hölder estimates for second derivatives. We first need the following result on symmetric matrices.

**Lemma 2.18.** *Let  $\mathcal{S}(\lambda, \Lambda)$  denote the set of positive matrices in  $\mathbb{R}^{n \times n}$  with eigenvalues lying in the interval  $[\lambda, \Lambda]$ , where  $0 < \lambda < \Lambda$ . Then there exists a finite set of unit vectors  $\gamma_1, \dots, \gamma_N \in \mathbb{R}^n$  and positive numbers  $\lambda^* < \Lambda^*$ , depending only on  $n, \lambda$  and  $\Lambda$  such that any matrix  $A = (a_{ij}) \in \mathcal{S}(\lambda, \Lambda)$  can be written in the form*

$$(1) \quad A = \sum_{k=1}^N \beta_k \gamma_k \otimes \gamma_k, \quad \text{i.e.,} \quad a_{ij} = \sum_{k=1}^N \beta_k \gamma_{ki} \gamma_{kj},$$

where  $\beta_k \in [\lambda^*, \Lambda^*]$ ,  $k = 1, \dots, N$ . Furthermore, the directions  $\gamma_1, \dots, \gamma_N$  can be chosen to include the coordinate directions  $e_i, i = 1, \dots, n$ , together with the directions  $\frac{1}{\sqrt{2}}(e_i \pm e_j), i < j, i, j = 1, \dots, n$ .

*Proof.* Let  $\mathcal{S}_+$  denote the cone of positive matrices in  $\mathcal{S}$  and  $n' = n(n+1)/2 = \dim(\mathcal{S})$ . We may represent any  $A \in \mathcal{S}_+$  in the form

$$(2) \quad A = \sum_{k=1}^{n'} \gamma_k \otimes \gamma_k,$$

for some vectors  $\gamma_1, \dots, \gamma_{n'} \in \mathbb{R}^n$  with the property that the dyadic matrices  $\gamma_k \otimes \gamma_k = [\gamma_{ki} \gamma_{kj}]$  are linearly independent. To see this we observe that any two matrices in  $\mathcal{S}_+$  are similar, and hence in particular each  $A \in \mathcal{S}_+$  is similar to the matrix  $A_0$  whose diagonal and nondiagonal terms are  $n$  and  $1$ , respectively. With

$$A_0 = \sum_{i=1}^n e_i \otimes e_i + \sum_{i < j}^n (e_i + e_j) \otimes (e_i + e_j),$$

(2) follows by an appropriate base change. Consequently, the family of sets of the form

$$U(\gamma_1, \dots, \gamma_{n'}) = \left\{ \sum_{k=1}^{n'} \beta_k \gamma_k \otimes \gamma_k; \beta_k > 0, k = 1, \dots, n' \right\},$$

where  $\gamma_k \otimes \gamma_k$  are linearly independent, forms an open cover of  $\mathcal{S}(\lambda, \Lambda) \subset \mathcal{S}_+$ , and since  $\mathcal{S}(\lambda, \Lambda)$  is compact there exists a finite subcover. Accordingly, there exists a fixed set of unit vectors  $\gamma_1, \dots, \gamma_N$ , depending only on  $\lambda, \Lambda$  and  $n$  such that any  $A \in \mathcal{S}(\lambda, \Lambda)$  may be written

$$(3) \quad A = \sum_{k=1}^N \beta_k \gamma_k \otimes \gamma_k,$$

with  $0 \leq \beta_k \leq \Lambda$ , for any  $k = 1, \dots, N$ . Note in this step, any particular finite set of unit vectors may be included among  $\gamma_k$ .

To create a common positive lower bound, we proceed as follows. We apply the previous process to  $\mathcal{S}(\lambda/2, \Lambda)$  to get unit vectors  $\gamma_1, \dots, \gamma_N$ . For any  $A \in \mathcal{S}(\lambda, \Lambda)$ ,

consider

$$A - \lambda^* \sum_{k=1}^N \gamma_k \otimes \gamma_k \in \mathcal{S}\left(\frac{\lambda}{2}, \Lambda\right),$$

for sufficiently small  $\lambda^*$  ( $\lambda^* = \frac{\lambda}{2N}$  is sufficient). Now if we write  $A$  as in (3), we get  $\lambda^* \leq \beta_k \leq \Lambda + \lambda^*$ .  $\square$

Applying Lemma 2.18 to uniformly elliptic operators, we get a relation among *pure* second derivatives of solutions. Suppose  $L$  is a linear operator of the form  $L = a_{ij}\partial_{ij}$  with

$$\lambda I \leq (a_{ij}) \leq \Lambda I.$$

Then  $L$  can be written as

$$L = \sum_{k=1}^N \beta_k(x) \partial_{\gamma_k \gamma_k},$$

where functions  $\beta_1, \dots, \beta_N$  satisfy

$$\lambda^* \leq \beta_k(x) \leq \Lambda^*.$$

Now we derive interior Hölder estimates for second derivatives. Suppose  $F : \mathcal{S} \rightarrow \mathbb{R}$  is a  $C^2$  function defined in the space of  $n \times n$  symmetric matrices and  $u$  is a  $C^4$  function defined in an open set  $\Omega \subset \mathbb{R}^n$ . We consider the equation

$$F(D^2u) = f \text{ in } \Omega,$$

for some function  $f \in C^2(\Omega)$ . We assume

(i)  $F$  is uniformly elliptic with respect to  $u$ , i.e., there exist positive constants  $\lambda$  and  $\Lambda$  such that

$$\lambda|\xi|^2 \leq F_{ij}(D^2u)\xi_i\xi_j \leq \Lambda|\xi|^2, \text{ for all } \xi \in \mathbb{R}^n;$$

(ii)  $F$  is a concave function on the range of  $D^2u$ , i.e.,

$$F_{ij,kl}(D^2u)m_{ij}m_{kl} \leq 0,$$

for any  $M = (m_{ij}) \in \mathcal{S}$ .

**Theorem 2.19.** *Let  $u$  be a smooth solution in  $\Omega$  to the equation*

$$(1) \quad F(D^2u) = f \text{ in } \Omega,$$

and assume (i) and (ii) hold. Then there holds for any ball  $B_{R_0} \subset \Omega$  and  $R \leq R_0$ ,

$$\text{osc}_{B_R} D^2u \leq C \left( \frac{R}{R_0} \right)^\alpha \left\{ \text{osc}_{B_{R_0}} D^2u + R_0 |Df|_0 + R_0^2 |D^2f|_0 \right\},$$

where  $C > 0$  and  $\alpha \in (0, 1)$  are constants depending only on  $n$ ,  $\lambda$  and  $\Lambda$ .

*Proof.* Let  $\gamma$  be an arbitrary unit vector in  $\mathbb{R}^n$  and differentiate (1) twice in the direction  $\gamma$ . We thus obtain

$$F_{ij} D_{ij\gamma} u = D_\gamma f,$$

and

$$F_{ij} D_{ij\gamma\gamma} u + F_{ij,kl} D_{ij\gamma} u D_{kl\gamma} u = D_{\gamma\gamma} f.$$

Consequently, the function  $w = u_{\gamma\gamma}$  satisfies the differential inequality

$$(2) \quad F_{ij}D_{ij}w \geq D_{\gamma\gamma}f \text{ in } \Omega.$$

We shall apply the weak Harnack inequality. Let  $B_R, B_{2R}$  be concentric balls in  $\Omega$  of radius  $R, 2R$ , respectively, and set for  $s = 1, 2$ ,

$$M_s = \sup_{B_{sR}} w, \quad m_s = \inf_{B_{sR}} w.$$

Applying Corollary 2.11 to the function  $M_2 - w$ , we thus obtain

$$(3) \quad \left\{ \frac{1}{R^n} \int_{B_R} (M_2 - w)^p \right\}^{\frac{1}{p}} \leq C \{M_2 - M_1 + R^2|D^2f|_0\} \\ \leq C \{(M_2 - m_2) - (M_1 - m_1) + R^2|D^2f|_0\}.$$

To conclude a Hölder estimate for  $w$  from (3), we need a corresponding inequality for  $-w$ , which we obtain by considering (1) as a functional relationship between the second derivatives of  $u$ . To begin with, using the concavity of  $F$  again, we have for any  $x, y \in \Omega$

$$F_{ij}(D^2u(y))(D_{ij}u(x) - D_{ij}u(y)) \geq F(D^2u(x)) - F(D^2u(y)) = f(x) - f(y).$$

Applying Lemma 2.18 to the matrix  $(F_{ij})$ , we obtain

$$(4) \quad \sum_{k=1}^N \beta_k (w_k(y) - w_k(x)) \leq f(y) - f(x),$$

where  $w_k = D_{\gamma_k\gamma_k}u$  and  $\beta_k = \beta_k(y)$  satisfy

$$0 < \lambda^* \leq \beta_k \leq \Lambda^*,$$

the vectors  $\gamma_1, \dots, \gamma_N$  and numbers  $\lambda^*, \Lambda^*$  depending only on  $n, \lambda$  and  $\Lambda$ . Set

$$M_k(sR) = \sup_{B_{sR}} w_k, \quad m_k(sR) = \inf_{B_{sR}} w_k,$$

and

$$\omega_k(sR) = M_k(sR) - m_k(sR), \quad s = 1, 2; \quad k = 1, \dots, N.$$

Then each of the function  $w_k$  satisfy (3); so that

$$(5) \quad \left\{ \frac{1}{R^n} \int_{B_R} (M_k(2R) - w_k)^p \right\}^{\frac{1}{p}} \leq C \{\omega_k(2R) - \omega_k(R) + R^2|D^2f|_{0;\Omega}\}.$$

By a summation over  $k \neq l$  for some fixed  $l$ , we obtain

$$\left\{ \frac{1}{R^n} \int_{B_R} \left[ \sum_{k \neq l} (M_k(2R) - w_k) \right]^p \right\}^{\frac{1}{p}} \\ \leq C \left\{ \sum_{k \neq l} (\omega_k(2R) - \omega_k(R)) + R^2|D^2f| \right\} \\ \leq C \{\omega(2R) - \omega(R) + R^2|D^2f|\},$$

where, for  $s = 1, 2$ ,

$$\omega(sR) = \sum_{k=1}^N \text{osc}_{B_{sR}} w_k = \sum_{k=1}^N (M_k(sR) - m_k(sR)).$$

By (4), we have for  $x \in B_{2R}, y \in B_R$ ,

$$\beta_l[w_l(y) - w_l(x)] \leq f(y) - f(x) + \sum_{k \neq l} \beta_k(w_k(x) - w_k(y)),$$

so that

$$w_l(y) - m_l(2R) \leq \frac{1}{\lambda^*} \left\{ 3R|Df|_{0,\Omega} + \Lambda^* \sum_{k \neq l} (M_k(2R) - w_k) \right\}.$$

This implies

$$(6) \quad \left\{ \frac{1}{R^n} \int_{B_R} (w_l - m_l(2R))^p \right\}^{\frac{1}{p}} \leq C \{ \omega(2R) - \omega(R) + R|Df|_0 + R^2|D^2f|_0 \},$$

where  $C$  again depends only on  $n, \lambda, \Lambda$ . By taking  $k = l$  in (5) and adding this to (6), we get

$$M_l(2R) - m_l(2R) \leq C \{ \omega(2R) - \omega(R) + R|Df|_0 + R^2|D^2f|_0 \}.$$

Summing over  $l = 1, \dots, N$ , we obtain

$$\omega(2R) \leq C \{ \omega(2R) - \omega(R) + R|Df|_0 + R^2|D^2f|_0 \},$$

and hence

$$\omega(R) \leq \delta \omega(2R) + R|Df|_0 + R^2|D^2f|_0,$$

for  $\delta = 1 - \frac{1}{C} \in (0, 1)$ . This implies Hölder estimates for the functions  $w_k, k = 1, \dots, N$ . Hence, for any ball  $B_{R_0} \subset \Omega$  and  $R \leq R_0$ , we get

$$\omega(R) \leq C \left( \frac{R}{R_0} \right)^\alpha \{ \omega(R_0) + R_0|Df|_0 + R_0^2|D^2f|_0 \},$$

where  $C > 0$  and  $\alpha \in (0, 1)$  are positive constants depending only on  $n, \lambda$  and  $\Lambda$ . By using the last assertion of Lemma 2.18, we obtain the desired estimate for  $D^2u$ .  $\square$

**2.3. Global Hölder Estimates for Second Derivatives.** In this section, we derive the global Hölder estimates for second derivatives. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ . We consider the Dirichlet problem

$$\begin{aligned} F(D^2u) &= f(x) \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

for some smooth function  $f$  in  $\bar{\Omega}$  and  $\varphi$  on  $\partial\Omega$ .

As in the previous section, we suppose  $F : \mathcal{S} \rightarrow \mathbb{R}$  is a  $C^2$  function defined in the space of  $n \times n$  symmetric matrices and  $u$  is a  $C^4$  function defined in  $\Omega \subset \mathbb{R}^n$ . We assume

(i)  $F$  is uniformly elliptic with respect to  $u$ , i.e., there exists positive constants  $\lambda, \Lambda$  such that

$$\lambda|\xi|^2 \leq F_{ij}(D^2u)\xi_i\xi_j \leq \Lambda|\xi|^2, \text{ for any } \xi \in \mathbb{R}^n;$$

(ii)  $F$  is a concave function on the range of  $D^2u$ , i.e.,

$$F_{ij,kl}(D^2u)m_{ij}m_{kl} \leq 0,$$

for any  $M = (m_{ij}) \in \mathcal{S}$ .

**Theorem 2.20.** *Let  $u \in C^4(\bar{\Omega})$  be a solution of the Dirichlet problem*

$$(1) \quad \begin{aligned} F(D^2u) &= f(x) \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

and assume (i) and (ii) hold. If  $u$  satisfies in  $\bar{\Omega}$

$$(2) \quad |u|_2 \leq K,$$

then there holds in  $\bar{\Omega}$  for some positive  $\alpha < 1$ ,

$$(3) \quad |u|_{2+\alpha} \leq C,$$

where  $\alpha$  and  $C$  depend only on  $K, |f|_2, |\varphi|_3, F$  and  $\Omega$ .

*Proof.* We divide the proof of (3) into several steps. In the following, a universal constant  $C$  depends only on  $K, |f|_2, |\varphi|_3, F$  and  $\Omega$ .

Step 1. We estimate  $C^\beta$ -norm of  $(D^2u)|_{\partial\Omega}$ , for a universal  $\beta \in (0, 1)$ . We prove

$$(4) \quad \|D^2u(x_1) - D^2u(x_0)\| \leq C_1|x_1 - x_0|^\beta, \text{ for any } x_0, x_1 \in \partial\Omega.$$

For a fixed  $x_0 \in \partial\Omega$ , we flatten the boundary  $\partial\Omega$  near  $x_0$ ; that is, we consider smooth diffeomorphisms

$$x = \chi(y), y = \chi^{-1}(x) = \eta(x), x \in \Omega, y \in A = \eta(\Omega),$$

so that  $\eta(x_0) = 0$ , and

$$\eta(U \cap \Omega) = B_4^+ = \{y \in \mathbb{R}^n; |y| < 4, y_n > 0\},$$

$$\eta(U \cap \partial\Omega) = \Gamma_4 = \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; |y'| < 4, y_n = 0\},$$

where  $U$  is a neighborhood of  $x_0$  which contains a ball centered at  $x_0$  with a universal radius. We also have that  $\|\eta\|_{C^3(\bar{\Omega})}$  and  $\|\chi\|_{C^3(\bar{A})}$  are bounded by a universal constant.

We consider the function

$$v(y) \equiv u(\chi(y)) - \varphi(\chi(y)) = u(x) - \varphi(x), y \in \bar{A},$$

which vanishes on  $\partial A$  and satisfies the equation

$$(5) \quad \begin{aligned} F \left( \left[ \sum_{k,l} \eta_i^k(\chi(y)) \eta_j^l(\chi(y)) v_{kl} + \sum_k \eta_{ij}^k(\chi(y)) v_k + \varphi_{ij}(\chi(y)) \right]_{ij} \right) \\ - f(\chi(y)) = 0, \end{aligned}$$

where  $\eta = (\eta^1, \dots, \eta^n)$ . Hence

$$(6) \quad G(D^2v, Dv, y) = 0 \text{ in } A,$$

where

$$(7) \quad \begin{aligned} & G(M, p, y) \\ &= F \left( \left[ \sum_{k,l} \eta_i^k(\chi(y)) \eta_j^l(\chi(y)) m_{kl} + \sum_k \eta_{ij}^k(\chi(y)) p_k + \varphi_{ij}(\chi(y)) \right]_{ij} \right. \\ & \quad \left. - f(\chi(y)) \right). \end{aligned}$$

The estimate (4) will be proved if we show that

$$(8) \quad \|D^2v(y) - D^2v(0)\| \leq C|y|^\beta \quad \text{for any } y \in \Gamma_1,$$

where

$$\Gamma_1 = \{y = (y', y_n); |y'| < 1, y_n = 0\}.$$

Note that

$$\frac{\partial G}{\partial m_{nn}} = F_{ij} \eta_i^n \eta_j^n \geq C\lambda > 0;$$

the implicit function theorem implies that (6) may be written as

$$(9) \quad v_{nn} = H((v_{kl})_{1 \leq k \leq n-1, 1 \leq l \leq n}, Dv, y).$$

The ellipticity implies that (6) is exactly equivalent (not only locally, but globally) to (9). By the chain rule, we have

$$|DH(\tilde{M}, p, y)| \leq (C\lambda)^{-1} |DG| \leq C(1 + |p| + |\tilde{M}|),$$

where  $\tilde{M}$  denotes  $(M_{kl})_{1 \leq k \leq n-1, 1 \leq l \leq n}$ . By (2), we know that

$$\|Dv\|_{L^\infty(A)} + \|D^2v\|_{L^\infty(A)} \leq K.$$

Hence (9) implies

$$|v_{nn}(y) - v_{nn}(0)| \leq C \left( \sup_{1 \leq k \leq n-1, 1 \leq l \leq n} |v_{kl}(y) - v_{kl}(0)| + |y| \right).$$

This inequality, together with  $v_{kl}|_{\Gamma_1} \equiv 0$  if  $1 \leq k, l \leq n-1$  (recall that  $v|_{\Gamma_1} \equiv 0$ ), implies that, in order to prove (8), we only need to show

$$(10) \quad |v_{pn}(y) - v_{pn}(0)| \leq C|y|^\beta \quad \text{for any } y \in \Gamma_1, 1 \leq p \leq n-1.$$

To prove (10), we differentiate (5) with respect to  $y_p$  and get

$$F_{ij} \eta_i^k \eta_j^l v_{klp} + \sum_{k,l} F_{ij} v_{kl} \partial_p (\eta_i^k \eta_j^l) + F_{ij} \partial_p \left( \sum_k \eta_{ij}^k v_k + \varphi_{ij} \right) - \partial_p f = 0,$$

where  $\eta_i^k, \eta_{ij}^k$ , and  $f$  are evaluated at  $\chi(y)$ . This equation can be put into the form

$$\tilde{L}v_p = \tilde{f}(y) \quad \text{in } A,$$

where

$$\tilde{L} = \sum_{k,l=1}^n a_{kl}(y) \partial_{kl},$$

and

$$a_{kl}(y) = F_{ij}(\chi(y))\eta_i^k(\chi(y))\eta_j^l(\chi(y)),$$

and hence  $\tilde{L}$  is uniformly elliptic with ellipticity constants  $C^{-1}\lambda$  and  $C\Lambda$  (for a universal constant  $C \geq 1$ ). Since  $\|\chi\|_{C^3(A)}$ ,  $\|\eta\|_{C^3(\bar{\Omega})}$ ,  $\|\varphi\|_{C^3(\bar{\Omega})}$ ,  $\|Dv\|_{L^\infty(A)}$ ,  $\|D^2v\|_{L^\infty(A)}$  are bounded, we have that

$$\|\tilde{f}\|_{L^\infty(A)} \leq C,$$

for a universal  $C$ .

We therefore have for  $k = 1, \dots, n-1$ ,

$$\begin{aligned} \tilde{L}v_k &= \tilde{f}(y) \quad \text{in } B_4^+ \\ v_k &= 0 \quad \text{on } \Gamma_4^+, \end{aligned}$$

where  $\tilde{L}$  is uniformly elliptic and

$$\|v_k\|_{L^\infty(B_4^+)} + \|Dv_k\|_{L^\infty(B_4^+)} + \|\tilde{f}\|_{L^\infty(B_4^+)} \leq K_*.$$

By Theorem 2.15, the normal derivative  $\partial_n$  of  $v_k$  along  $\Gamma_1$  is  $C^\beta$ , i.e.,

$$\|\partial_n v_k\|_{C^\beta(\Gamma_1)} \leq C,$$

or for any  $y, y' \in \Gamma_1$

$$|\partial_{nk}v(y) - \partial_{nk}v(y')| \leq C|y - y'|^\beta,$$

for some universal constant  $\beta \in (0, 1)$  and  $C > 0$ . This obviously implies (10).

Note we did not use the concavity in this step.

Step 2. We shall prove

$$(11) \quad \|D^2u(x) - D^2u(x_0)\| \leq C_2|x - x_0|^\beta \quad \text{for any } x \in \bar{\Omega}, x_0 \in \partial\Omega.$$

We need the following Lemma.

**Lemma 2.21.** *Suppose  $F$  is uniformly elliptic. Then for any  $M_1, M_2 \in \mathcal{S}$*

$$\|M_2 - M_1\| \leq \frac{\Lambda + \lambda}{\lambda} \sup_{|e|=1} (e^t(M_2 - M_1)e)^+ + \frac{1}{\lambda} [F(M_1) - F(M_2)].$$

*Proof.* Using  $\|M_2 - M_1\| \leq \|(M_2 - M_1)^+\| + \|(M_2 - M_1)^-\|$ , we have

$$\begin{aligned} F(M_2) - F(M_1) &= \int_0^1 \frac{d}{dt} F(M_1 + t(M_2 - M_1)) dt \\ &= \int_0^1 F_{ij}(tM_2 + (1-t)M_1) dt \cdot (M_2 - M_1)_{ij} \\ &\leq \Lambda \|(M_2 - M_1)^+\| - \lambda \|(M_2 - M_1)^-\| \\ &\leq (\Lambda + \lambda) \|(M_2 - M_1)^+\| - \lambda \|M_2 - M_1\|. \end{aligned}$$

This finishes the proof.  $\square$

Now we begin to prove (11). For any unit vector  $\gamma \in \mathbb{R}^n$ , we have

$$Lu_{\gamma\gamma} \geq f_{\gamma\gamma} \geq -C,$$

where  $C = |D^2f|_{0,\Omega}$ . Note that the linear operator  $L = F_{ij}\partial_{ij}$  satisfies

$$\lambda I \leq (F_{ij}) \leq \Lambda I.$$

Fix an  $x_0 \in \partial\Omega$  and consider

$$w = u_{\gamma\gamma} + \frac{C}{2n\lambda}|x - x_0|^2.$$

Then we have

$$Lw = Lu_{\gamma\gamma} + \frac{C}{n\lambda} \sum F_{ii} \geq 0.$$

By (4), we have

$$w(x) - w(x_0) \leq C|x - x_0|^\beta \quad \text{for any } x \in \partial\Omega.$$

We claim

$$(12) \quad w(x) - w(x_0) \leq C|x - x_0|^{\frac{\beta}{2}} \quad \text{for any } x \in \Omega.$$

This implies

$$u_{\gamma\gamma}(x) - u_{\gamma\gamma}(x_0) \leq C_*|x - x_0|^{\frac{\beta}{2}} \quad \text{for any } x \in \partial\Omega.$$

Note this holds for arbitrary unit vector  $\gamma \in \mathbb{R}^n$ . By Lemma 2.21, we obtain

$$\begin{aligned} \lambda \|D^2u(x) - D^2u(x_0)\| &\leq \sup_{|\gamma|=1} (u_{\gamma\gamma}(x) - u_{\gamma\gamma}(x_0)) + f(x) - f(x_0) \\ &\leq C|x - x_0|^{\frac{\beta}{2}} + |x - x_0| \|Df\|_{0,\Omega}. \end{aligned}$$

This implies (11).

To prove (12), we transform the boundary  $\partial\Omega$  near  $x_0$  to a parabola. Take  $x_0 = 0 \in \partial\Omega$  and assume  $w(x_0) = w(0) = 0$ . We consider smooth diffeomorphisms

$$x = \chi(y), y = \chi^{-1}(y) = \eta(x), \quad x \in \Omega, \quad y \in A = \eta(\Omega),$$

such that  $\eta(x_0) = 0$ , and

$$\begin{aligned} \eta(U \cap \partial\Omega) &= \{y \in \mathbb{R}^n; |y|^2 = 2y_n, 0 \leq y_n < a\}, \\ \eta(U \cap \Omega) &= \{y \in \mathbb{R}^n; |y|^2 < 2y_n, 0 < y_n < a\} = P_a, \end{aligned}$$

where  $a$  is a constant in  $(0, 1)$  and  $U$  is a neighborhood of  $x_0$  which contains a ball centered at  $x_0$  with a universal radius. We also have that  $\|\eta\|_{C^2(\bar{A})}$  and  $\|\chi\|_{C^2(\bar{\Omega})}$  are bounded by a universal constant. Moreover, by writing  $\eta = (\eta^1, \dots, \eta^n)$ , we also require that

$$|D\eta^n(\chi(y))| \geq C \quad \text{for any } y \in P_a,$$

for a universal constant  $C > 0$ .

We consider the function

$$v(y) = w(\chi(y)) = w(x) \quad \text{for } y \in \bar{A}.$$

Since  $v(0) = 0$ , we define

$$K = \sup_{y \in \partial A} \frac{|v(y)|}{|y|^\beta}.$$

Note  $v$  satisfies the following equation

$$\tilde{L}v \equiv F_{ij}(\chi(y))\eta_i^k(\chi(y))\eta_j^l(\chi(y))v_{kl} + F_{ij}(\chi(y))\eta_{ij}^k(\chi(y))v_k = 0,$$

where  $\tilde{L}$  is uniformly elliptic.

For simplicity, we denote  $\psi = w|_{\partial\Omega}$ . We already proved  $\psi \in C^\beta(\partial\Omega)$  in Step 1. There hold

$$v(y) \leq K|y|^\beta = 2^{\frac{\beta}{2}}Ky_n^{\frac{\beta}{2}} \quad \text{for any } y \in \partial A \cap \partial P_a,$$

and

$$v(y) \leq |\psi|_0 = \frac{|\psi|_0}{a^{\frac{\beta}{2}}}y_n^{\frac{\beta}{2}} \quad \text{for any } y \in A \cap \partial P_a.$$

Hence on  $\partial P_a$ , we have

$$v(y) \leq C_*y_n^{\frac{\beta}{2}},$$

where

$$C_* = C_0|\psi|_{\beta, \partial\Omega},$$

for some universal constant  $C_0$ . A direct calculation shows

$$\begin{aligned} Ly_n^{\frac{\beta}{2}} &= \frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right) y_n^{\frac{\beta}{2}-2} (F_{ij}\eta_i^n\eta_j^n)(\chi(x)) + \frac{\beta}{2} y_n^{\frac{\beta}{2}-1} (F_{ij}\eta_{ij}^n)(\chi(y)) \\ &= \frac{\beta}{2} y_n^{\frac{\beta}{2}-2} \left( - \left( 1 - \frac{\beta}{2} \right) (F_{ij}\eta_i^n\eta_j^n)(\chi(x)) + y_n (F_{ij}\eta_{ij}^n)(\chi(y)) \right) \leq 0, \end{aligned}$$

if  $a$  is small. Hence we obtain

$$Lv \geq L \left( C_*y_n^{\frac{\beta}{2}} \right) \quad \text{in } P_a$$

$$v \leq C_*y_n^{\frac{\beta}{2}} \quad \text{on } \partial P_a.$$

By the maximum principle, we get for  $y \in P_a$

$$v(y) \leq C_*y^{\frac{\beta}{2}} \leq C_*|y|^{\frac{\beta}{2}}.$$

Transform back to  $x$  and recall the definition of  $C_*$ . We obtain

$$w(x) - w(x_0) \leq C_*|\psi|_{\beta, \Omega} |x - x_0|^{\frac{\beta}{2}},$$

for any  $x$  in  $B_r(x_0) \cap \bar{\Omega}$ , where  $r$  is universal. This is obviously true for any other  $x \in \bar{\Omega}$  by increasing  $C$ . Hence we finishes the proof of (12).

Step 3. Take  $\alpha$  to be the minimum of  $\alpha$  in Theorem 2.19 and  $\beta/2$  in Step 2, and we prove

$$(13) \quad \|D^2u(x) - D^2u(y)\| \leq C|x - y|^\alpha \quad \text{for any } x, y \in \bar{\Omega}.$$

We first recall Theorem 2.19, the interior estimates. For any  $B_R(x_0) \subset \Omega$ , there holds for any  $x, y \in B_{R/2}(x_0)$

$$(14) \quad R^\alpha \frac{|D^2u(x) - D^2u(y)|}{|x - y|^\alpha} \leq C (|D^2u|_{L^\infty(B_R)} + R|Df|_{L^\infty} + R^2|D^2f|_{L^\infty}).$$

For any  $x, y \in \Omega$ , set  $d_x = \text{dist}(x, \Omega)$  and  $d_y = \text{dist}(y, \Omega)$ . Suppose  $d_y \leq d_x$ . Take  $x_0, y_0 \in \partial\Omega$  such that  $|x - x_0| = d_x$  and  $|y - y_0| = d_y$ . Assume first that  $|x - y| \leq d_x/2$ . Then we have  $y \in \bar{B}_{d_x/2}(x) \subset B_{d_x}(x) \subset \Omega$ . Consider

$$w = u - u(x_0) - Du(x_0)(x - x_0) - \frac{1}{2}(x - x_0)^t Du^2(x_0)(x - x_0).$$

Then  $w$  satisfies

$$G(D^2w) = f(x) \quad \text{in } \Omega,$$

where

$$G(M) = F(M + D^2u(x_0)).$$

Obviously,  $G$  is elliptic and concave with the same ellipticity constant as  $F$ . We apply the interior estimate (scaled version) to  $w$  in  $B_{d_x}(x)$  and get by (14)

$$\begin{aligned} & d_x^\alpha \frac{|D^2u(x) - D^2u(y)|}{|x - y|^\alpha} \\ & \leq C \left( |D^2u - D^2u(x_0)|_{L^\infty(B_{d_x}(x))} + d_x|Df|_{L^\infty} + d_x^2|D^2f|_{L^\infty} \right). \end{aligned}$$

By (11), we obtain

$$|D^2u - D^2u(x_0)|_{L^\infty(B_{d_x}(x))} \leq C_2 d_x^\alpha.$$

Hence we obtain

$$|D^2u(x) - D^2u(y)| \leq C|x - y|^\alpha (C_2 + |Df|_{L^\infty} + |D^2f|_{L^\infty}) \equiv C|x - y|^\alpha.$$

Assume now that  $d_y \leq d_x \leq 2|x - y|$ . Then by (11) again we have

$$\begin{aligned} & |D^2u(x) - D^2u(y)| \\ & \leq |D^2u(x) - D^2u(x_0)| + |D^2u(x_0) - D^2u(y_0)| + |D^2u(y_0) - D^2u(y)| \\ & \leq C(d_x^\alpha + |x_0 - y_0|^\alpha + d_y^\alpha) \\ & \leq C|x - y|^\alpha, \end{aligned}$$

since  $|x_0 - y_0| \leq d_x + |x - y| + d_y \leq 5|x - y|$ . This finishes the proof of (13).  $\square$

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