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Conservation Equations of Gasdynamics in Curvilinear Coordinate Systems*

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This paper presents a new method of writing the conservation equations of gasdynamics in curvilinear coordinates which eliminates undifferentiated terms. One can thus readily apply difference schemes derived for Cartesian coordinates which conserve mass, momentum, and energy in the total flow field. The method is derived for orthogonal coordinates, and then extended to cover the most general class of coordinate transformations, using general tensor analysis. Several special features of the equations are discussed.

1. INTRODUCTION

An analytic solution of a physical problem usually depends on the equations being cast in some specialized form. On the other hand, a numerical solution can be based on more general forms of the equations, derived from fundamental principles. In applying a numerical technique, however, we often tend to accept as a starting point equations whose form is dictated by the requirements of an analytic solution. We may, thus, overlook other forms of the equations, ideally suited to that numerical technique, and extending its range of applicability. The present paper offers an illustration of this point.

For time-dependent flows, or in supersonic regions of steady-state flows, the equations of gasdynamics are hyperbolic with respect to some independent variable. They then admit weak solutions, i.e., those containing shock waves and contact surfaces across which certain dependent variables are discontinuous. There exist well developed techniques for numerically integrating the equations, in which the discontinuities are explicitly calculated [1, 2]. For three-dimensional flows past complex bodies, the system of shock waves and contact surfaces can be so involved as to make these techniques impractical. Problems in which shock waves suddenly arise are also difficult to handle with these techniques. For these reasons, alternate

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techniques have been developed which automatically integrate "through" the shock waves [3], with the resultant shock transition being spread over several mesh intervals. There is general agreement that the most accurate way to accomplish this is to difference equations which have been cast in conservation-law form [4]. These are differential forms of the basic integral relations which express the conservation of mass, momentum, and energy. Since the latter are valid in the presence of discontinuitites, appropriate methods of differencing the differential equations will result in the correct jump conditions being satisfied for the smeared out discontinuities. Recently, a hybrid method has been developed, which treats certain shock waves as sharp discontinuities by means of appropriate coordinate transformations, and automatically "captures" others by employing the conservation-law form [5].

In Cartesian coordinates, the gasdynamic equations can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} = 0, \tag{1}$$

where **U**, **E**, **F**, and **G** are five-component vectors in function space. As an example, for an ideal gas they are given by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ \rho uw \\ (e + p)u \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ \rho vw \\ (e + p)v \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ p + \rho w^2 \\ (e + p)w \end{pmatrix}, \tag{2}$$

where u, v, and w are velocity components in the x, y, and z directions, and p, ρ, and e are the pressure, density, and total energy per unit volume. The system of equations is completed by specifying an equation of state,

$$p = p(\epsilon, \rho), \tag{3}$$

where the specific internal energy ϵ is related to e through the defining relation,

$$e/\rho = \epsilon + (u^2 + v^2 + w^2)/2. \tag{4}$$

Equations (1) have two features. All terms are derivatives of the unknown vector components with respect to the independent variables. This shall be called a *strong* conservation-law form of the equations. The virtue of this form is that it is possible to difference the equations by a variety of stable schemes, each of which can be chosen so as to conserve mass, momentum, and energy for the total flow

region. The other feature possessed by (1) is that for an ideal gas, the vector components depend only on the physical unknowns. The resulting difference relations are straightforward.

In order to satisfy conditions on arbitrarily shaped boundaries, it is often convenient to employ general curvilinear coordinates x_1, x_2, and x_3. They are related to Cartesian coordinates by the transformations

$$\begin{aligned} x &= x(x_1, x_2, x_3), \\ y &= y(x_1, x_2, x_3), \\ z &= z(x_1, x_2, x_3). \end{aligned} \tag{5}$$

The transformed gasdynamic equations can always be written as

$$\frac{\partial \mathbf{U}'}{\partial t} + \frac{\partial \mathbf{E}'}{\partial x_1} + \frac{\partial \mathbf{F}'}{\partial x_2} + \frac{\partial \mathbf{G}'}{\partial x_3} + \mathbf{H} = 0, \tag{6}$$

where the form of the vectors $\mathbf{U}'$, $\mathbf{E}'$, $\mathbf{F}'$, and $\mathbf{G}'$ depends on the representation of the velocity components, as well as the coordinate transformation. In general, the vectors now exhibit an explicit dependence on the independent variables. The difference relations can therefore be generalized. For example, in a multistep procedure, the independent variables and the physical unknowns need not be evaluated at the same discrete points in any single term.

A more significant result of the transformation is the presence of the undifferentiated term $\mathbf{H}$. This term is analogous to the fictitious body force terms which result from the use of a non-inertial reference frame. Equations (6) will be said to be in *weak* conservation-law form. The presence of the $\mathbf{H}$ term prevents the achievement of overall conservation of mass, momentum, and energy. It would thus lead to inferior shock capturing capability in regions of very large shock curvature. The method of differencing the $\mathbf{H}$ term can be uncertain. For example, in difference schemes employing the splitting of operators in coordinate directions [6, 7], there is no obvious way to handle the undifferentiated term arising from the ignorable coordinate in axisymmetric flow.

The purpose of this paper is to present a new method of writing the conservation equations of gasdynamics which preserves the strong form for curvilinear coordinates, with minimum added complication. For simplicity, the method is first derived for orthogonal coordinates. It is then extended to general nonorthogonal and time-dependent coordinate transformations. Since this employs the tools of general tensor analysis, unfamiliar to most readers, the details of the extended derivation are relegated to an appendix. For illustrative purposes, an ideal gas will be assumed.

The results of this paper are formally applicable to steady-state flows in which

the velocity component in the marching direction is supersonic. There are important differences, relative to the time-dependent case, involving the decoding of the conservative variables. If the marching variable is not "straight", it will also affect the decoding procedure. These points are discussed at the end of the paper.

2. Strong Conservation Law for Orthogonal Curvilinear Coordinates

If the new coordinates defined by (5) are orthogonal, then the infinitesimal distance ds between two points whose coordinates differ by dx_1, dx_2, and dx_3 is given by

$$ds^2 = \sum_{j=1}^{3} h_j^2 \, dx_j^2, \tag{7}$$

where the scale factors h_j can be obtained from (5) as

$$h_j^2 = (\partial x/\partial x_j)^2 + (\partial y/\partial x_j)^2 + (\partial z/\partial x_j)^2. \tag{8}$$

The unit base vector $\mathbf{e}_j$ in the x_j direction is expressed in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ by

$$\mathbf{e}_j = \alpha_{j1}\mathbf{i} + \alpha_{j2}\mathbf{j} + \alpha_{j3}\mathbf{k}, \tag{9}$$

where the direction cosines α_{jk} are given by

$$\alpha_{j1} = \frac{1}{h_j}\frac{\partial x}{\partial x_j}, \qquad \alpha_{j2} = \frac{1}{h_j}\frac{\partial y}{\partial x_j}, \qquad \alpha_{j3} = \frac{1}{h_j}\frac{\partial z}{\partial x_j}. \tag{10}$$

From the orthogonality of the two coordinate systems, it follows that

$$\mathbf{i} = \sum_{j=1}^{3} \alpha_{j1}\mathbf{e}_j, \qquad \mathbf{j} = \sum_{j=1}^{3} \alpha_{j2}\mathbf{e}_j, \qquad \mathbf{k} = \sum_{j=1}^{3} \alpha_{j3}\mathbf{e}_j. \tag{11}$$

A vector $\mathbf{b}$ can be expressed in component form in curvilinear and Cartesian coordinates as

$$\mathbf{b} = \sum_{j=1}^{3} b_j \mathbf{e}_j \tag{12a}$$

or

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}. \tag{12b}$$

There are two types of conservation laws in physics. One is the *scalar conservation law* of the form

$$(\partial\alpha/\partial t) + \operatorname{div}\mathbf{b} = 0, \tag{13}$$

involving the scalar α and the vector $\mathbf{b}$. The continuity and energy equations are of this form. In orthogonal coordinates, we have

$$\operatorname{div} \mathbf{b} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 b_1) + \frac{\partial}{\partial x_2}(h_1 h_3 b_2) + \frac{\partial}{\partial x_3}(h_1 h_2 b_3)\right]. \tag{14}$$

Since h_1, h_2, and h_3 are independent of time, we obtain immediately the strong conservation-law form

$$\frac{\partial}{\partial t}(h_1 h_2 h_3 \alpha) + \frac{\partial}{\partial x_1}(h_2 h_3 b_1) + \frac{\partial}{\partial x_2}(h_1 h_3 b_2) + \frac{\partial}{\partial x_3}(h_1 h_2 b_3) = 0. \tag{15}$$

If it is desirable to use Cartesian components of $\mathbf{b}$ as unknown variables, we obtain, with the aid of (11) and (12), the alternate form

$$\begin{aligned}\frac{\partial}{\partial t}(h_1 h_2 h_3 \alpha) &+ \frac{\partial}{\partial x_1}[h_2 h_3(\alpha_{11} b_x + \alpha_{12} b_y + \alpha_{13} b_z)] \\ &+ \frac{\partial}{\partial x_2}[h_1 h_3(\alpha_{21} b_x + \alpha_{22} b_y + \alpha_{23} b_z)] + \frac{\partial}{\partial x_3}[h_1 h_2(\alpha_{31} b_x + \alpha_{32} b_y + \alpha_{33} b_z)] = 0.\end{aligned} \tag{16}$$

The momentum equation is an example of the *vector conservation law*,

$$(\partial \mathbf{a}/\partial t) + \operatorname{div} \mathbf{T} = 0. \tag{17}$$

Here $\mathbf{a}$ is a vector, and $\mathbf{T}$ is a symmetric tensor which can be written in component form as

$$\mathbf{T} = \sum_{k=1}^{3}\sum_{j=1}^{3} T_{jk}\mathbf{e}_j\mathbf{e}_k\,. \tag{18}$$

As a vector equation, the strong conservation-law form of (17) is

$$\frac{\partial}{\partial t}(h_1 h_2 h_3 \mathbf{a}) + \frac{\partial}{\partial x_1}(h_2 h_3 \mathbf{T}\cdot\mathbf{e}_1) + \frac{\partial}{\partial x_2}(h_1 h_3 \mathbf{T}\cdot\mathbf{e}_2) + \frac{\partial}{\partial x_3}(h_1 h_2 \mathbf{T}\cdot\mathbf{e}_3) = 0. \tag{19}$$

Introducing vector and tensor components using (12a) and (18), we can write (19) as

$$\begin{aligned}\frac{\partial}{\partial t}\left(h_1 h_2 h_3 \sum_{j=1}^{3} a_j \mathbf{e}_j\right) &+ \frac{\partial}{\partial x_1}\left(h_2 h_3 \sum_{j=1}^{3} T_{j1}\mathbf{e}_j\right) + \frac{\partial}{\partial x_2}\left(h_1 h_3 \sum_{j=1}^{3} T_{j2}\mathbf{e}_j\right) \\ &+ \frac{\partial}{\partial x_3}\left(h_1 h_2 \sum_{j=1}^{3} T_{j3}\mathbf{e}_j\right) = 0.\end{aligned} \tag{20}$$

The base vectors $\mathbf{e}_j$ are functions of position for curvilinear coordinates. Thus, the separation of (20) into three scalar equations introduces undifferentiated terms arising from the quantities $\partial \mathbf{e}_j/\partial x_k$.

One way to obtain scalar conservation equations in the strong form is to expand the base vectors $\mathbf{e}_j$ with respect to the Cartesian unit vectors **i**, **j**, and **k**. Since the latter are constant vectors, the scalar decomposition introduces no additional terms. Substituting (9) into (20), we obtain the strong conservation-law form

$$\frac{\partial}{\partial t}\left(h_1h_2h_3\sum_{j=1}^{3}\alpha_{jk}a_j\right)+\frac{\partial}{\partial x_1}\left(h_2h_3\sum_{j=1}^{3}\alpha_{jk}T_{j1}\right)+\frac{\partial}{\partial x_2}\left(h_1h_3\sum_{j=1}^{3}\alpha_{jk}T_{j2}\right)$$

$$+\frac{\partial}{\partial x_3}\left(h_1h_2\sum_{j=1}^{3}\alpha_{jk}T_{j3}\right)=0, \qquad (k=1 \text{ to } 3). \tag{21}$$

This result (for general coordinates) was derived by Anderson, Preiser, and Rubin [8] in a different manner. The direction cosines α_{jk} are in general nonzero functions of position. Only for coordinate systems containing cylindrical surfaces can one coordinate be aligned with a Cartesian coordinate, resulting in the vanishing of two of the α_{jk} . Other practical coordinate systems involving straight coordinates, such as polar or boundary-layer coordinates, do not offer any simplification. Equations (21) are thus fairly complicated, since they are of a hybrid nature, being partially Cartesian and partially curvilinear in character.

The presence of all three unknowns a_j in the time-derivative term requires a solution of a third order linear algebraic system to accomplish the decoding at each time step in the integration. This can be avoided by using Cartesian components of vectors and tensors, the latter being defined by the expansion

$$\mathbf{T}=T_{xx}\mathbf{ii}+T_{xy}\mathbf{ij}+T_{xz}\mathbf{ik}+T_{yx}\mathbf{ji}+T_{yy}\mathbf{jj}+T_{yz}\mathbf{jk}$$
$$+T_{zx}\mathbf{ki}+T_{zy}\mathbf{kj}+T_{zz}\mathbf{kk}. \tag{22}$$

Substituting (11), (12b), and (22) into (19), we obtain an equation whose **i** component is

$$\frac{\partial}{\partial t}(h_1h_2h_3a_x)+\frac{\partial}{\partial x_1}[h_2h_3(\alpha_{11}T_{xx}+\alpha_{12}T_{xy}+\alpha_{13}T_{xz})]$$
$$+\frac{\partial}{\partial x_2}[h_1h_3(\alpha_{21}T_{xx}+\alpha_{22}T_{xy}+\alpha_{23}T_{xz})]$$
$$+\frac{\partial}{\partial x_3}[h_1h_2(\alpha_{31}T_{xx}+\alpha_{32}T_{xy}+\alpha_{33}T_{xz})]=0. \tag{23}$$

The equations for the **j** and **k** components are analogously written. This form of

the strong conservation law was first proposed by MacCormack and Paullay [7], and implemented in a blunt-body flow calculation by Rizzi and Inouye [9]. Actually, these authors used an integral rather than a differential formulation of the problem, and employed Cartesian coordinates to calculate the geometric factors equivalent to the h_i and α_{ij} of (23). While the time derivative term is now simpler than the one in (21), the spatial derivative terms in the scalar equations (16) are more involved than the corresponding terms in (15).

Both (21) and (23) make an appeal to Cartesian coordinates to achieve the strong conservation-law form. While the use of curvilinear coordinates has simplified the handling of boundary conditions, the introduction of Cartesian directions in order to effect the scalar decomposition results in unduly complicated terms in the interior of the flow region. Since the directions of the base vectors are variable for curvilinear coordinates, the Cartesian decomposition, and the resultant complication, appear to be unavoidable. This conclusion results from looking at (20) as an *analytic* equation. But, by examining the *numerical* procedure used to solve that equation, a new natural method of scalar decomposition can be found, which is in general much simpler than either (21) or (23).

The numerical solution of Eqs. (15) and (20) is accomplished by using a time-continuous method (also known as the method of lines). The unknown variables are given only at discrete points fixed in space, and considered as functions of time at these points. (Alternatively, the points may be representative points inside discrete cells. The unknown variables would then be suitable averages over these cells. This interpretation is more consistent with a strong conservation-law formulation, as will be discussed later.) If the spatial derivatives are approximated by suitable finite differences involving the unknown point variables, the partial differential equations are then reduced to a system of ordinary differential equations in time. This latter system can be integrated numerically by any desired scheme.

Let the subscript P denote a quantity evaluated at one of the discrete points. The partial derivative with respect to time in Eq. (20) can be written as

$$\frac{\partial}{\partial t}\left(h_1 h_2 h_3 \sum_{j=1}^{3} a_j \mathbf{e}_j\right) = h_{1P} h_{2P} h_{3P} \sum_{j=1}^{3} \frac{da_{jP}}{dt}\, \mathbf{e}_{jP} \,. \tag{24}$$

In the finite difference approximations of the spatial derivatives appearing in the same equation, the base vectors $\mathbf{e}_j$ will be evaluated at points in space other than the discrete point P. (Note that whereas the dependent variables T_{ij} are defined only at the discrete points (or cell centers) of the numerical mesh, functions of the independent variables (such as h_j or $\mathbf{e}_j$) are continuously defined. It is thus possible to evaluate the h_j and $\mathbf{e}_j$ at points other than those of the discrete mesh, e.g., points on cell boundaries.) A *local* scalar decomposition can be achieved if the $\mathbf{e}_j$ at points other than P are expanded with respect to $\mathbf{e}_{jP}$. In evaluating the direction cosines

of the expansions, one notes that for each spatial derivative, only one coordinate is varied in the difference expression, the other coordinates being held fixed at their values at point P.

Let us introduce the notation that a subscript i appearing to the *left* of a quantity denotes variation with x_i only, the other coordinates being held fixed at their values at point P. Thus, for example,

$$ {}_1\mathbf{e}_2 = \mathbf{e}_2(x_1 , x_{2P} , x_{3P}). \tag{25} $$

The expansion for the base vector in the i direction appearing in the x_j derivative term can thus be written as

$$ {}_j\mathbf{e}_i = \sum_{k=1}^{3} {}_j\gamma_{ik}\mathbf{e}_{kP} . \tag{26} $$

(The notation ${}_j\gamma_{ik}$ reminds us that the role played by these coefficients is somewhat analogous to that of the Christoffel symbols of the second kind, Γ^k_{ji} .) The direction cosines ${}_j\gamma_{ik}$ will always be numbers very close to zero or one. They are easily calculated in terms of the α_{jk} defined by (10). Using Eq. (9), we obtain

$$ {}_j\gamma_{ik} = \mathbf{e}_{kP} \cdot {}_j\mathbf{e}_i = \sum_{m=1}^{3} {}_j\alpha_{im}\alpha_{kmP} . \tag{27} $$

The scalar decomposition of Eq. (20) in strong conservation form thus becomes

$$ \frac{\partial}{\partial t}(h_1h_2h_3a_k) + \frac{\partial}{\partial x_1}\left(h_2h_3 \sum_{j=1}^{3} {}_1\gamma_{jk}T_{j1}\right) + \frac{\partial}{\partial x_2}\left(h_1h_3 \sum_{j=1}^{3} {}_2\gamma_{jk}T_{j2}\right) $$

$$ + \frac{\partial}{\partial x_3}\left(h_1h_2 \sum_{j=1}^{3} {}_3\gamma_{jk}T_{j3}\right) = 0, \qquad (k = 1 \text{ to } 3). \tag{28} $$

The new form of the strong conservation law (28) is formally similar to Eq. (21) or (23). In general, (28) will be much simpler than (21) or (23). For an ideal gas,

$$ a_i = \rho u_i , \qquad T_{ij} = \rho u_i u_j + p\delta_{ij} , \tag{29} $$

where u_i is the velocity component in the x_i direction. Results obtained from Eq. (28) for the components of the momentum equation in cylindrical and spherical polar coordinates are found in Appendix A. These can be compared to the component equations obtained from (21) published in Ref. [8]. For completeness, the continuity and energy equations (obtained from (15) by setting $\alpha = \rho$, $b_i = \rho u_i$, and $\alpha = e$, $b_i = (e + p)\, u_i$, respectively) are also shown in Appendix A. A comparison reveals far fewer terms in the new momentum equations. This results

principally from the presence of the straight radial coordinate. By definition, if x_j is a straight coordinate, then

$$ {}_j\gamma_{ik} = \delta_{ik} \,. \tag{30} $$

Thus the x_j derivative is one term rather than being the sum of three terms. This simplification does not occur for the radial coordinate in formulations (21) or (23).

An additional simplification can be seen by considering the finite-difference representations of the equations. In a two-step difference method of the type first proposed by MacCormack [10], each spatial derivative is always approximated by a difference of two quantities, one of which is evaluated at the point P held constant in the time derivative. It is then advantageous to evaluate the ${}_j\gamma_{ik}$ at the same points as the T_{ij} (rather than at cell boundaries). Since

$$ ({}_j\gamma_{ik})_P = \delta_{ik} \,, \tag{31} $$

it follows that

$$ \left(\sum_{i=1}^{3} {}_j\gamma_{ik} T_{ij} \right)_P = T_{kjP} \,. \tag{32} $$

Thus, while the difference approximation of a spatial derivative in (21) or (23) will in general consist of six terms, it is reduced to at most four terms if (28) is used. A further simplification is possible for angular coordinates, using equal mesh spacing. The functions $\cos(\theta - \theta_P)$, $\sin(\theta - \theta_P)$, or $\cos(\phi - \phi_P)$, $\sin(\phi - \phi_P)$ (see Appendix A) evaluated at the point other than P are then all constants in absolute value, irrespective of the location of the point. This is not true in (21) or (23), where the corresponding trigonometric functions must be reevaluated at each point P along their respective coordinates.

3. Strong Conservation Law for General Curvilinear Coordinates

For some practical applications, formulation (28) must be generalized. Non-orthogonal curvilinear coordinates may be appropriate for certain geometries. For problems involving a time-varying free surface (such as a shock wave), a time-dependent coordinate transformation is found useful. It is also sometimes convenient to employ base vectors defined by one coordinate system, while the independent variables are given by a second coordinate system. In this case the vector and tensor components can be defined with respect to either of the coordinate systems.

The generalizations require the use of general tensor analysis. In order to include time-dependent coordinate transformations, it is convenient to utilize a new higher-

dimensional formulation of the conservation equations recently developed by the author [11]. The details of the derivation of the strong conservation law for the most general class of coordinate transformations will be found in Appendix B. For most cases it is sufficient to consider a more restricted class of transformations. Let x_1, x_2, and x_3 be an orthogonal coordinate system in which x_1 is straight, i.e.,

$$\partial \mathbf{e}_i/\partial x_1 = 0 \qquad (i = 1 \text{ to } 3). \tag{33}$$

Introduce a new coordinate system through the transformation

$$\begin{aligned} x_1' &= x_1'(x_1, x_2, x_3, t) \\ x_2' &= x_2'(x_2) \\ x_3' &= x_3'(x_3) \\ t' &= t. \end{aligned} \tag{34}$$

The determinant of the inverse transformation $| \partial x_i/\partial x_j' |$ is given by

$$\Delta = \left(\frac{\partial x_1'}{\partial x_1}\frac{dx_2'}{dx_2}\frac{dx_3'}{dx_3}\right)^{-1}. \tag{35}$$

In terms of the velocity components and base vectors defined with respect to the original coordinate system, the results of Appendix B can be specialized to yield the following strong conservation equations for an ideal gas:

$$\frac{\partial}{\partial t'}(h_1h_2h_3\,\Delta\rho) + \frac{\partial}{\partial x_1'}\left[h_1h_2h_3\,\Delta\rho\left(\frac{\partial x_1'}{\partial t} + \sum_{j=1}^{3}\frac{\partial x_1'}{\partial x_j}\frac{u_j}{h_j}\right)\right] + \frac{\partial}{\partial x_2'}\left(h_1h_3\,\Delta\rho\frac{dx_2'}{dx_2}u_2\right) + \frac{\partial}{\partial x_3'}\left(h_1h_2\,\Delta\rho\frac{dx_3'}{dx_3'}u_3\right) = 0, \tag{36a}$$

$$\frac{\partial}{\partial t'}(h_1h_2h_3\,\Delta\rho u_k) + \frac{\partial}{\partial x_1'}\left\{h_1h_2h_3\Delta\left[\frac{\partial x_1'}{\partial t}\rho u_k + \sum_{j=1}^{3}\frac{\partial x_1'}{\partial x_j}\frac{1}{h_j}(\rho u_k u_j + p\delta_{kj})\right]\right\} + \frac{\partial}{\partial x_2'}\left[h_1h_3\Delta\frac{dx_2'}{dx_2}\sum_{j=1}^{3}{}_2\gamma_{jk}(\rho u_j u_2 + p\delta_{j2})\right] + \frac{\partial}{\partial x_3'}\left[h_1h_2\Delta\frac{dx_3'}{dx_3}\sum_{j=1}^{3}{}_3\gamma_{jk}(\rho u_j u_3 + p\delta_{j3})\right] = 0, \qquad (k = 1 \text{ to } 3) \tag{36b}$$

$$\frac{\partial}{\partial t'}(h_1h_2h_3\,\Delta e) + \frac{\partial}{\partial x_1'}\left\{h_1h_2h_3\Delta\left[e\frac{\partial x_1'}{\partial t} + (e+p)\sum_{j=1}^{3}\frac{\partial x_1'}{\partial x_j}\frac{u_j}{h_j}\right]\right\} + \frac{\partial}{\partial x_2'}\left[h_1h_3\,\Delta(e+p)\frac{dx_2'}{dx_2}u_2\right] + \frac{\partial}{\partial x_3'}\left[h_1h_2\,\Delta(e+p)\frac{dx_3'}{dx_3}u_3\right] = 0. \tag{36c}$$

The coefficients ${}_2\gamma_{jk}$ and ${}_3\gamma_{jk}$ are determined by the original coordinate system using (27), but are considered as functions of x_2' and x_3', respectively.

A comparison of (36) with (15) and (28) shows that the new equations involve more terms arising from the time dependent transformation (34). There is a more fundamental difference, which can affect certain numerical procedures. While transformation (34) and condition (33) have the effect of making the base vector $\mathbf{e}_i$ independent of t' (as well as x_1'), the functions h_1, h_2, h_3, Δ, $\partial x_1'/\partial t$, and $\partial x_1'/\partial x_j$ at a fixed point P' in the computational space x_1', x_2', x_3' will in general be functions of t' through their dependence on x_1. Thus, in any multistep finite-difference technique in which quantities are evaluated at some "intermediate" moment in the time interval $\Delta t'$, the precise moment must be specified for the evaluation of these time-dependent functions. This problem does not exist for (15) and (28) (or more general equations where the spatial coordinates and time are uncoupled), since only the physical unknowns depend on time, and the precise moment at which the "intermediate" values of these unknowns are calculated is irrelevant.

If the original spatial coordinate which is coupled to the time is not straight, then time derivative term of each component of the momentum equation involves the coupling of all three velocity components (see Appendix B). This necessitates the solution of a third order linear algebraic system to obtain the velocity components, in contrast to their direct calculation from (36b). The new time coordinate is still basically distinct from the new spatial coordinates in the sense that the time derivatives involve mass, momentum, and energy densities, while the spatial derivatives involve fluxes, stresses, and work terms. This distinction is crucial in contrasting the decoding process in time-dependent and steady-state flow calculations.

Decoding can be illustrated by considering the system of Eqs. (1) to (4) for an ideal gas in Cartesian coordinates. In a time-dependent calculation, once a new vector $\mathbf{U}$ is determined, the first component gives ρ directly. The velocity components then follow from the next three components of $\mathbf{U}$, and the last component combined with Eq. (4) yields ϵ. The equation of state (3) then gives the pressure p, which must be obtained by some subroutine in the case of a real gas. This provides all the data necessary to determine the vectors $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$. For a steady-state flow in which z is the marching variable (w is supersonic), the vector $\mathbf{G}$ is determined at each integration step. The first three components readily give u and v. The first, fourth, and fifth components, combined with (3) and (4), then define an equivalent one-dimensional flow problem with two solutions, one subsonic and one supersonic. (In the present application the supersonic solution is required, while in the shock wave problem the supersonic solution is given and the subsonic solution is desired.) For a perfect gas, the one-dimensional problem can be reduced to the solution of a quadratic equation. For a real gas, an iterative solution is necessary, with repeated calls on the equation of state subroutine. If the steady flow is iso-

energetic (e.g., uniform upstream flow), the fifth components of **E**, **F**, and **G** can be replaced by the condition that the total enthalpy is constant. Equation (3) would then be written in terms of the specific enthalpy instead of ϵ. The decoding procedure would not be changed by this reduction in the number of equations to be integrated.

The above discussion indicates that for a real gas, the decoding for steady-state flows is iterative and time consuming compared to the direct procedure for time-dependent flows. While based on equations written in Cartesian coordinates, the discussion is essentially valid for any coordinate system in which the marching coordinate t' depends only on t. It is readily seen from Appendix B, Eq. (B.23), that the decoding remains direct under these conditions. On the other hand, for the most general transformation (B.9) in which the new marching coordinate depends also on the original spatial coordinates, the marching derivatives contain the same terms as the new spatial derivatives. The decoding for a real gas now involves the iteration associated with steady-state flows.

4. Concluding Remarks

In this paper a natural method of writing the strong conservation-law form of the equation of gasdynamics in arbitrary coordinates has been presented. Specific finite-difference relations have been deliberately avoided, and the equation cast in a form suitable for any desired numerical algorithm. While an ideal gas has been used for illustrative purposes, the extension to a viscous, heat-conducting gas are readily made. The primary motivation behind a strong conservation form is the ability to maintain exact numerical conservation of quantities that are physically conserved. In that regard, there are several points that merit further comment.

There exists an anomalous situation in which a strong conservation-law form cannot be achieved. It is the degenerate case of a curved, ignorable coordinate, exemplified by axisymmetric flow. Considering orthogonal coordinates for simplicity, let x_3 be the ignorable coordinate, so that $\partial/\partial x_3 = 0$ for all *physical* quantities. Thus, the last term in the scalar equation (15) vanishes. In the vector equation, assume also that

$$a_3 = T_{13} = T_{23} = 0. \tag{37}$$

This condition is satisfied for axisymmetric flow with no swirl velocity, i.e., $u_3 = 0$. Since x_3 is curved, the coefficient ${}_3\gamma_{3k}$ is a function of x_3. One must therefore examine the limit,

$$\lim_{x_3 \to x_{3P}} (\partial/\partial x_3)({}_3\gamma_{jk}),$$

to determine if the last term in (28) vanishes. It is readily shown for axisymmetric flow that the $k = 3$ equation vanishes identically. In the other two equations, the last term does not vanish, but

$$\lim_{x_3 \to x_{3P}} \frac{\partial}{\partial x_3} \left(h_1 h_2 \sum_{j=1}^{3} {}_3\gamma_{j1} T_{j3} \right) = -h_2 \frac{\partial h_3}{\partial x_1} T_{33}, \tag{38}$$

and

$$\lim_{x_3 \to x_{3P}} \frac{\partial}{\partial x_3} \left(h_1 h_2 \sum_{j=1}^{3} {}_3\gamma_{j2} T_{j3} \right) = -h_1 \frac{\partial h_3}{\partial x_1} T_{33}. \tag{39}$$

For an ideal gas, $T_{33} = p$. In cylindrical coordinates, where $x_3 = \theta$, all θ derivatives vanish except in

$$\text{Eq. (A.2a):} \quad \lim_{\theta \to \theta_P} (\partial/\partial\theta)(\quad) = -p. \tag{40}$$

In spherical polar coordinates, where $x_3 = \phi$, all ϕ derivatives vanish except in

$$\text{Eq. (A.5a):} \quad \lim_{\phi \to \phi_P} \frac{\partial}{\partial \phi}(\quad) = -rp \sin\theta, \tag{41}$$

$$\text{Eq. (A.5b):} \quad \lim_{\phi \to \phi_P} \frac{\partial}{\partial \phi}(\quad) = -rp \cos\theta. \tag{42}$$

The undifferentiated terms act as "sources" of momentum and appear to violate the conservation of momentum. The explanation is readily seen by applying the conservation law to a *finite* volume element. In cylindrical coordinates, forces on the two lateral faces have small additive components in the radial direction determined by the center of the volume element. These components do not vanish as $\Delta\theta \to 0$, but in fact result in the term given by (40). In summing the change in radial momentum over all the volume elements of the wedge-shaped sector of angular width $\Delta\theta$, the forces in the radial direction stemming from the lateral surfaces of all the elements will add. In the limit $\Delta\theta \to 0$ they therefore behave as "sources" of radial momentum, and a strong conservation-law form consequently does not exist for the radial momentum equation. A similar argument in spherical polar coordinates leads to an explanation of (41) and (42).

Having established the reason for the undifferentiated terms in axisymmetric flow, one must still determine how to treat them in a multistep finite-difference scheme. The appropriate representation in the two-step MacCormack scheme can be determined unambigously by taking the limit $\Delta x_3 \to 0$ in the three-dimensional algorithm. If there is further splitting of terms in coordinate directions, it is not obvious how to apportion the term arising from the ignorable coordinate. One reasonable solution is to apportion in a way to minimize unwanted oscillations in

regions of uniform flow caused by the splitting. As an illustration, consider the radial momentum equation in spherical polar coordinates (A.5a). For uniform pressure, differencing the radial term alone gives a force contribution

$$2rp \sin\theta[1 + O(\Delta r)],$$

while the polar angle term contribution is

$$-rp \sin\theta[1 + O(\Delta\theta)].$$

Therefore, to eliminate spurious first-order oscillations in regions of uniform flow, (41) should be apportioned as

$$\begin{matrix} & & \text{radial} & & \text{polar} \\ -rp \sin\theta & = & (-2rp \sin\theta) & + & (rp \sin\theta). \end{matrix} \tag{43}$$

It is clear that to achieve *exact* conservation of physically conserved quantities in the total flow field, the difference equations should be considered as finite-difference approximations to integral relations for finite volume elements, or cells. Appropriate differencing will assure exact cancellation over interior cell-faces. The quantities in the time derivative must also be defined as averages over cell volumes, rather than point values at the center of the cell, in order to achieve exact conservation. At the same time, an unambiguous point (such as the cell center) must be associated with each cell to define the base vector with respect to which the variable base vectors are expanded. The use of cell-averaged instead of point quantities has important implications for the handling of boundaries, the decoding process, and the analysis of a particular difference scheme. These will be discussed in a future publication.

APPENDIX A: Strong Conservation Equations for an Ideal Gas

1. *Cylindrical Coordinates*

$$\begin{matrix} x = r\cos\theta & y = r\sin\theta & z = z \\ x_1 = r & x_2 = \theta & x_3 = z \\ h_1 = 1 & h_2 = r & h_3 = 1 \\ u_1 = u_r & u_2 = u_\theta & u_3 = u_z \end{matrix}$$

$${}_1\gamma_{ij} = {}_3\gamma_{ij} = \delta_{ij}$$

$$\begin{matrix} {}_2\gamma_{11} = \cos(\theta - \theta_P) & {}_2\gamma_{12} = \sin(\theta - \theta_P) & {}_2\gamma_{13} = 0 \\ {}_2\gamma_{21} = -\sin(\theta - \theta_P) & {}_2\gamma_{22} = \cos(\theta - \theta_P) & {}_2\gamma_{23} = 0 \\ {}_2\gamma_{31} = 0 & {}_2\gamma_{32} = 0 & {}_2\gamma_{33} = 1 \end{matrix}$$

Continuity

$$\frac{\partial}{\partial t}(r\rho) + \frac{\partial}{\partial r}(r\rho u_r) + \frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(r\rho u_z) = 0. \tag{A.1}$$

Momentum

$$\frac{\partial}{\partial t}(r\rho u_r) + \frac{\partial}{\partial r}[r(\rho u_r^2 + p)] + \frac{\partial}{\partial \theta}[\cos(\theta - \theta_P)\,\rho u_r u_\theta - \sin(\theta - \theta_P)(\rho u^2\theta + p)] + \frac{\partial}{\partial z}(r\rho u_\theta u_z) = 0. \tag{A.2a}$$

$$\frac{\partial}{\partial t}(r\rho u_\theta) + \frac{\partial}{\partial r}(r\rho u_r u_\theta) + \frac{\partial}{\partial \theta}[\sin(\theta - \theta_P)\,\rho u_r u_\theta + \cos(\theta - \theta_P)(\rho u_\theta^2 + p)] + \frac{\partial}{\partial z}(r\rho u_\theta u_z) = 0. \tag{A.2b}$$

$$\frac{\partial}{\partial t}(r\rho u_z) + \frac{\partial}{\partial r}(r\rho u_r u_z) + \frac{\partial}{\partial \theta}(\rho u_\theta u_z) + \frac{\partial}{\partial z}[r(\rho u_z^2 + p)] = 0. \tag{A.2c}$$

Energy

$$\frac{\partial}{\partial t}(re) + \frac{\partial}{\partial r}[r(e + p)\,u_r] + \frac{\partial}{\partial \theta}[(e + p)\,u_\theta] + \frac{\partial}{\partial z}[r(e + p)\,u_z] = 0. \tag{A.3}$$

2. *Spherical Polar Coordinates*

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

$$x_1 = r \quad x_2 = \theta \quad x_3 = \phi$$

$$h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$$

$$u_1 = u_r \quad u_2 = u_\theta \quad u_3 = u_\phi$$

$${}_1\gamma_{ij} = \delta_{ij}$$

$${}_2\gamma_{11} = \cos(\theta - \theta_P) \quad {}_2\gamma_{12} = \sin(\theta - \theta_P) \quad {}_2\gamma_{13} = 0$$

$${}_2\gamma_{21} = -\sin(\theta - \theta_P) \quad {}_2\gamma_{22} = \cos(\theta - \theta_P) \quad {}_2\gamma_{23} = 0$$

$${}_2\gamma_{31} = 0 \quad {}_2\gamma_{32} = 0 \quad {}_2\gamma_{33} = 1$$

$${}_3\gamma_{11} = \cos^2\theta + \cos(\phi - \phi_P)\sin^2\theta \quad {}_3\gamma_{12} = \sin\theta\cos\theta[\cos(\phi - \phi_P) - 1]$$

$${}_3\gamma_{21} = \sin\theta\cos\theta[\cos(\phi - \phi_P) - 1] \quad {}_3\gamma_{22} = \sin^2\theta + \cos(\phi - \phi_P)\cos^2\theta$$

$${}_3\gamma_{31} = -\sin\theta\sin(\phi - \phi_P) \quad {}_3\gamma_{32} = -\cos\theta\sin(\phi - \phi_P)$$

$${}_3\gamma_{13} = \sin\theta\sin(\phi - \phi_P)$$

$${}_3\gamma_{23} = \cos\theta\sin(\phi - \phi_P)$$

$${}_3\gamma_{33} = \cos(\phi - \phi_P)$$

Continuity

$$\frac{\partial}{\partial t}(r^2\rho \sin\theta) + \frac{\partial}{\partial r}(r^2\rho u_r \sin\theta) + \frac{\partial}{\partial \theta}(r\rho u_\theta \sin\theta) + \frac{\partial}{\partial \phi}(r\rho u_\phi) = 0. \quad \text{(A.4)}$$

Momentum

$$\frac{\partial}{\partial t}(r^2\rho u_r \sin\theta) + \frac{\partial}{\partial r}[r^2 \sin\theta(\rho u_r^2 + p)] + \frac{\partial}{\partial \theta}[r \sin\theta\{\rho u_r u_\theta \cos(\theta - \theta_P) - (\rho u_\theta^2 + p)\sin(\theta - \theta_P)\}] + \frac{\partial}{\partial \phi}[r\{[\cos^2\theta + \cos(\phi - \phi_P)\sin^2\theta]\rho u_r u_\phi + \sin\theta\cos\theta[\cos(\phi - \phi_P) - 1]\rho u_\theta u_\phi - \sin\theta\sin(\phi - \phi_P)(\rho u_\phi^2 + p)\}] = 0. \quad \text{(A.5a)}$$

$$\frac{\partial}{\partial t}(r^2\rho u_\theta \sin\theta) + \frac{\partial}{\partial r}(r^2\rho u_r u_\theta \sin\theta) + \frac{\partial}{\partial \theta}[r \sin\theta\{\rho u_r u_\theta \sin(\theta - \theta_P) + (\rho u_\theta^2 + p)\cos(\theta - \theta_P)\}] + \frac{\partial}{\partial \phi}[r\{\sin\theta\cos\theta[\cos(\phi - \phi_P) - 1]\rho u_r u_z + [\cos^2\theta\cos(\phi - \phi_P) + \sin^2\theta]\rho u_\theta u_\phi - \cos\theta\sin(\phi - \phi_P)(\rho u_\phi^2 + p)\}] = 0. \quad \text{(A.5b)}$$

$$\frac{\partial}{\partial t}(r^2\rho u_\phi \sin\theta) + \frac{\partial}{\partial r}(r^2\rho u_r u_\phi \sin\theta) + \frac{\partial}{\partial \theta}(r\rho u_\theta u_\phi \sin\theta) + \frac{\partial}{\partial \phi}[r \sin\theta\sin(\phi - \phi_P)\rho u_r u_\phi + \cos\theta\sin(\phi - \phi_P)\rho u_\theta u_\phi + \cos(\phi - \phi_P)(\rho u_\phi^2 + p)\}] = 0. \quad \text{(A.5c)}$$

Energy

$$\frac{\partial}{\partial t}(r^2 e \sin\theta) + \frac{\partial}{\partial r}[r^2 \sin\theta(e + p)u_r] + \frac{\partial}{\partial \theta}[r \sin\theta(e + p)u_\theta] + \frac{\partial}{\partial \phi}[r(e + p)u_\phi] = 0. \quad \text{(A.6)}$$

APPENDIX B: Generalized Derivation of Strong Conservation Law

An arbitrary curvilinear coordinate system x^1, x^2, x^3, has as its natural basis the covariant base vectors

$$\mathbf{g}_i \equiv \partial \mathbf{r}/\partial x^i, \quad \text{(B.1)}$$

where $\mathbf{r}$ is the position vector in three-dimensional space. The covariant components of the metric tensor are defined by

$$g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j . \tag{B.2}$$

The determinant of g_{ij} is denoted by g. The conservation equations for an ideal gas can be written as

$$\frac{\partial}{\partial t}(g^{1/2}\rho) + \frac{\partial}{\partial x^i}(g^{1/2}\rho u^i) = 0, \tag{B.3a}$$

$$\frac{\partial}{\partial t}(g^{1/2}\rho u^j \mathbf{g}_j) + \frac{\partial}{\partial x^i}[g^{1/2}(\rho u^j u^i + p g^{ji})\,\mathbf{g}_j] = 0, \tag{B.3b}$$

$$\frac{\partial}{\partial t}(g^{1/2}e) + \frac{\partial}{\partial x^i}[g^{1/2}(e + p)\,u^i] = 0, \tag{B.3c}$$

where u^i is the contravariant component of the velocity vector, and the contravariant components of the metric tensor g^{ij} are related to the covariant components by

$$g^{ij}g_{jk} = \delta_k{}^i. \tag{B.4}$$

Summation over repeated indexes is assumed, Latin indexes being summed from 1 to 3. Equations (B.3) can be written as a single vector equation in a five-dimensional space. Let $x^4 \equiv t$, and define x^5 to be a fictitious energy coordinate having the property that all real, physical variables are independent of x^5. The corresponding covariant base vectors $\mathbf{g}_4$ and $\mathbf{g}_5$ are both assumed to be constant vectors of unit length, orthogonal to the physical base vectors and to each other. The covariant components of the five-dimensional metric tensor thus take the matrix form

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 \\ g_{31} & g_{32} & g_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{B.5}$$

We use Greek indexes for quantities defined over the five-dimensional space. Unless otherwise stated, Greek indexes are summed from 1 to 5.

For an ideal gas, the contravariant components of the five-dimensional flow tensor $\mathbf{T}$ are defined by

$$T^{\alpha\beta} \equiv \rho u^\alpha u^\beta + p \begin{bmatrix} g^{11} & g^{12} & g^{13} & 0 & u^1 \\ g^{21} & g^{22} & g^{23} & 0 & u^2 \\ g^{31} & g^{32} & g^{33} & 0 & u^3 \\ 0 & 0 & 0 & 0 & 0 \\ u^1 & u^2 & u^3 & 0 & 0 \end{bmatrix}, \tag{B.6}$$

where

$$u^4 \equiv 1, \qquad u^5 \equiv e/\rho. \tag{B.7}$$

Equations (B.3) are then equivalent to the condition that the flow tensor is divergence free, i.e.,

$$\operatorname{div}_5 \mathbf{T} = (\partial/\partial x^\alpha)[\, g^{1/2} T^{\beta\alpha} \mathbf{g}_\beta] = 0. \tag{B.8}$$

(Recall, from the definition of x^5, that $\partial/\partial x^5 \equiv 0$.) Further details and extension of this result may be found in Ref. [11].

The most general coordinate transformation involves the three physical coordinates and time. It can be written (along with the inverse transformation) as

$$\begin{aligned} x^{\alpha'} &= x^{\alpha'}(x^1, x^2, x^3, x^4) & x^\alpha &= x^\alpha(x^{1'}, x^{2'}, x^{3'}, x^{4'}) \\ x^{5'} &= x^5 & x^5 &= x^{5'} \end{aligned} \tag{B.9}$$

$$(\alpha, \alpha' = 1 \text{ to } 4).$$

The determinant of the inverse transformation $| \partial x^\alpha/\partial x^{\beta'} |$ is denoted by Δ. The transformed Eq. (B.8) is

$$(\partial/\partial x^{\alpha'})[(\, g')^{1/2} T^{\beta'\alpha'} \mathbf{g}_{\beta'}] = 0. \tag{B.10}$$

The unprimed coordinate system will in general be simpler than the primed one. Thus, a scalar decomposition along the unprimed directions will lead to simpler equations. It will also be generally simpler to define tensor components with respect to the unprimed coordinate system. Introducing the transformation law for contravariant components of vectors, one obtains

$$(\partial/\partial x^{\alpha'})[(\, g')^{1/2} (\partial x^{\alpha'}/\partial x^\nu)\, T^{\beta\nu} \mathbf{g}_\beta] = 0. \tag{B.11}$$

Since $\mathbf{g}_4$ and $\mathbf{g}_5$ are constant, it follows immediately that

$$(\partial/\partial x^{\alpha'})[(\, g')^{1/2} (\partial x^{\alpha'}/\partial x^\nu)\, T^{\beta\nu}] = 0 \qquad (\beta = 4, 5). \tag{B.12}$$

These two equations are the transformed continuity and energy equations. Let $x^{4'}$ be the new marching, or timelike coordinate. For an arbitrary coordinate system x^1, x^2, x^3, the general transformation (B.9) results in some $\mathbf{g}_i$ being a function of $x^{4'}$. Thus, at a point fixed in the computational space $x^{1'}$, $x^{2'}$, $x^{3'}$, the base vectors $\mathbf{g}_i$ are no longer constant. In order to achieve the natural scalar decomposition analogous to (28), we need an extension of the concepts used to derive that equation. Specifically, since the integration with respect to $x^{4'}$ is also accomplished by finite differences, all unknown variables are defined at discrete points fixed in the four-dimensional space $x^{1'}$, $x^{2'}$, $x^{3'}$, $x^{4'}$. These points define constant base vectors $\mathbf{g}_i$ which can be used at each integration step as a basis for the expansion of the variable $\mathbf{g}_i$.

Let the subscript P' denote a quantity evaluated at one of the discrete points in the four-dimensional space $x^{1'}$, $x^{2'}$, $x^{3'}$, $x^{4'}$. A subscript α' appearing to the *left* of a quantity again denotes variation with $x^{\alpha'}$ only, the other coordinates being held fixed at their values at point P'. The expansion for the base vector in the j direction appearing in the $x^{\alpha'}$ derivative term is

$$ {}_{\alpha'}\mathbf{g}_j = {}_{\alpha'}\gamma_j{}^k \mathbf{g}_{kP'} \,. \tag{B.13} $$

The scalar decomposition of the transformed momentum equation in strong conservation form becomes

$$ (\partial/\partial x^{\alpha'})[(g')^{1/2}\, {}_{\alpha'}\gamma_j{}^k (\partial x^{\alpha'}/\partial x^{\beta})\, T^{j\beta}] = 0 \qquad (k = 1 \text{ to } 3). \tag{B.14} $$

In terms of tensor components defined with respect to the primed coordinate system (B.12) and (B.14) become

$$ (\partial/\partial x^{\alpha'})[(g')^{1/2} (\partial x^{\beta}/\partial x^{\nu'})\, T^{\nu'\alpha'}] = 0 \qquad (\beta = 4, 5), \tag{B.15} $$

and

$$ (\partial/\partial x^{\alpha'})[(g')^{1/2}\, {}_{\alpha'}\gamma_j{}^k (\partial x^{j}/\partial x^{\beta})\, T^{\beta'\alpha'}] = 0 \qquad (k = 1 \text{ to } 3). \tag{B.16} $$

It will be seen that (16) and (23) are special cases of (B.12) and (B.14), while (15) and (21) are special cases of (B.15) and (B.16).

In actual calculations, it is customary to use physical components of vectors and tensors. The physical contravariant component of the velocity vector is related to the (mathematical) contravariant component by

$$ u^{(i)} \equiv (g_{ii})^{1/2}\, u^i \qquad \text{(unsummed)}. \tag{B.17} $$

The corresponding physical (or unit) covariant base vector is

$$ \mathbf{g}_{(i)} \equiv \mathbf{g}_i/(g_{ii})^{1/2} \qquad \text{(unsummed)}. \tag{B.18} $$

Expansion (B.13) in terms of the physical base vectors can be written as

$$ {}_{\alpha'}\mathbf{g}_{(j)} = {}_{\alpha'}\gamma^{(k)}_{(j)} \mathbf{g}_{(k)P'} \,. \tag{B.19} $$

If the curvilinear coordinate system x^1, x^2, x^3 is related to the Cartesian coordinate system y^1, y^2, y^3 by

$$ y^i = y^i(x^1, x^2, x^3) \qquad x^i = x^i(y^1, y^2, y^3), \tag{B.20} $$

and we let

$$ \alpha_j{}^i \equiv \partial y^i/\partial x^j \qquad \text{and} \qquad \beta_j{}^i \equiv \partial x^i/\partial y^j \tag{B.21} $$

one can easily show that

$$ {}_{\nu'}\gamma^{(k)}_{(j)} = (g_{kkP'}/{}_{\nu'} g_{jj})^{1/2}\, {}_{\nu'}\alpha_j{}^m \beta^k_{mP'} \qquad (j, k \text{ unsummed}). \tag{B.22} $$

For an ideal gas, the strong conservation equations in terms of physical components defined by the unprimed coordinate system take the form

$$\frac{\partial}{\partial x^{4'}}\left\{(g')^{1/2}\rho\left[\frac{\partial x^{4'}}{\partial t}+\frac{\partial x^{4'}}{\partial x^{j}}\frac{u^{(i)}}{(g_{jj})^{1/2}}\right]\right\}$$
$$+\frac{\partial}{\partial x^{i'}}\left\{(g')^{1/2}\rho\left[\frac{\partial x^{i'}}{\partial t}+\frac{\partial x^{i'}}{\partial x^{j}}\frac{u^{(j)}}{(g_{jj})^{1/2}}\right]\right\}=0, \tag{B.23a}$$

$$\frac{\partial}{\partial x^{4'}}\left[(g')^{1/2}\,{}_{4'}\gamma^{(k)}_{(j)}\left\{\frac{\partial x^{4'}}{\partial t}\rho u^{(j)}+\frac{\partial x^{4'}}{\partial x^{m}}\left[\frac{\rho u^{(j)}u^{(m)}}{(g_{mm})^{1/2}}+p(g_{jj})^{1/2}g^{jm}\right]\right\}\right]$$
$$+\frac{\partial}{\partial x^{i'}}\left[(g')^{1/2}\,{}_{i'}\gamma^{(k)}_{(j)}\left\{\frac{\partial x^{i'}}{\partial t}\rho u^{(j)}+\frac{\partial x^{i'}}{\partial x^{m}}\left[\frac{\rho u^{(j)}u^{(m)}}{(g_{mm})^{1/2}}+p(g_{jj})^{1/2}g^{jm}\right]\right\}\right]=0,$$
$$(k=1\text{ to }3), \tag{B.23b}$$

$$\frac{\partial}{\partial x^{4'}}\left\{(g')^{1/2}\left[\frac{\partial x^{4'}}{\partial t}e+\frac{\partial x^{4'}}{\partial x^{j}}\frac{(e+p)\,u^{(j)}}{(g_{jj})^{1/2}}\right]\right\}$$
$$+\frac{\partial}{\partial x^{i'}}\left\{(g')^{1/2}\left[\frac{\partial x^{i'}}{\partial t}e+\frac{\partial x^{i'}}{\partial x^{j}}\frac{(e+p)\,u^{(j)}}{(g_{jj})^{1/2}}\right]\right\}=0. \tag{B.23c}$$

Similar equations can be written in terms of physical components defined by the primed coordinate system. The determinants of the metric tensor in the two coordinate systems are related by

$$(g')^{1/2}=\Delta g^{1/2}. \tag{B.24}$$

In comparing (B.23) with (15) and (28), one sees that the derivative with respect to the new marching coordinate $x^{4'}$ now consists of many terms, particularly in the momentum equation (B.23b). This results in additional labor in the decoding for the physical unknowns required at each time step. There are two special cases of the general transformation (B.9) which result in simplications of the derivative. One case is that for which $x^{4'}$ is a straight coordinate, for which

$${}_{4'}\gamma^{(k)}_{(j)}=\delta_{j}^{\,k}. \tag{B.25}$$

One can show that the most general transformation satisfying this condition is

$$\begin{aligned}
x^{1'}&=x^{1'}(x^1,x^2,x^3,x^4) & x^1&=x^1(x^{1'},x^{2'},x^{3'},x^{4'})\\
x^{2'}&=x^{2'}(x^2,x^3) & x^2&=x^2(x^{2'},x^{3'})\\
x^{3'}&=x^{3'}(x^2,x^3) & x^3&=x^3(x^{2'},x^{3'})\\
x^{4'}&=x^{4'}(x^1,x^2,x^3,x^4) & x^4&=x^4(x^{1'},x^{2'},x^{3'},x^{4'})\\
x^{5'}&=x^5 & x^5&=x^{5'},
\end{aligned} \tag{B.26}$$

where x^1 must be a straight coordinate. It follows from (B.26) that $x^{1'}$ is also straight, which therefore simplifies the $x^{1'}$ derivative. The other case is that for which $x^{4'} = x^4$. This eliminates all terms involving $\partial x^{4'}/\partial x^j$ appearing in the $x^{4'}$ derivative. Both of these cases are found in most practical coordinate transformations. If, additionally, x^2 and x^3 undergo only stretching transformations, so that $x^{2'}$ and $x^{3'}$ are uncoupled, it follows that

$$ {}_{i'}\gamma_{(j)}^{(k)} = {}_{i}\gamma_{(j)}^{(k)} \qquad (i = 2, 3). \tag{B.27} $$

Finally, if the original coordinate system is orthogonal, the metric tensor components can be expressed in terms of scale factors through the relations

$$ g^{ij} = \frac{\delta_{ij}}{h_i^2}, \qquad g_{ij} = h_i^2 \delta_{ij} \qquad \text{(unsummed)}. \tag{B.28} $$

In terms of a new notation defined by

$$ x_i \equiv x^i, \qquad x_i' \equiv x^{i'}, \qquad u_i \equiv u^{(i)}, \qquad \text{and} \qquad {}_i\gamma_{jk} \equiv {}_i\gamma_{(j)}^{(k)}, \tag{B.29} $$

the resulting set of strong conservation equations are given as Eqs. (36).

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