

SOLUTIONS TO EXERCISES

by

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HW 28. Find the derivatives of the following functions.

a. $y = 1/x^3$

$$y' = (x^{-3})' = -3x^{-4} = -\frac{3}{x^4}.$$

b. $y = 8 - x^3 + x^{-5}$

$$y' = -3x^2 - 5x^{-6}.$$

c. $y = \frac{1}{2x+1}$

$$y' = ((2x+1)^{-1})' = -(2x+1)^{-2}(2x+1)' = -(2x+1)^{-2}(2) = -\frac{2}{(2x+1)^2}.$$

d. $y = (x+1)^3$

$$y' = 3(x+1)^2.$$

e. $y = \frac{2x+3}{3x+4}$

$$\begin{aligned} y' &= \frac{(2x+3)'(3x+4) - (2x+3)(3x+4)'}{(3x+4)^2} \\ &= \frac{2(3x+4) - (2x+3)3}{(3x+4)^2} \\ &= \frac{6x+8-6x-9}{(3x+4)^2} \\ &= -\frac{1}{(3x+4)^2}. \end{aligned}$$

f. $y = (x^{100} + 1)(3x^2 + x + 1)$

$$\begin{aligned} y' &= (x^{100} + 1)'(3x^2 + x + 1) + (x^{100} + 1)(3x^2 + x + 1)' \\ &= 100x^{99}(3x^2 + x + 1) + (x^{100} + 1)(6x + 1). \end{aligned}$$

g. $y = x \sin x + 5 \cos x$

$$y' = (x \sin x)' + (5 \cos x)' = \sin x + x \cos x - 5 \sin x = x \cos x - 4 \sin x.$$

h. $y = x^2 \cos x$

$$y' = (x^2)' \cos x + x^2 (\cos x)' = 2x \cos x - x^2 \sin x.$$

i. $y = \sin^2 x$

$$y = (\sin x \sin x)' = (\sin x)' \sin x + \sin x (\sin x)' = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x$$

or

$$y' = 2 \sin x (\sin x)' = 2 \sin x \cos x.$$

j. $y = \tan x = \sin x / \cos x$

$$y' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

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k. $y = \sec x = 1/\cos x$

$$\begin{aligned}y' &= ((\cos x)^{-1})' = -(\cos x)^{-2}(\cos x)' \\&= (\cos x)^{-2} \sin x = \frac{\sin x}{(\cos x)^2} \\&= \frac{\sin x}{\cos x} \frac{1}{\cos x} \\&= \tan x \sec x\end{aligned}$$

or

$$\begin{aligned}y' &= \frac{1' \cos x - 1 \cdot (\cos x)'}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} \\&= \frac{\sin x}{\cos x} \frac{1}{\cos x} \\&= \tan x \sec x.\end{aligned}$$

l. $y = \csc x = 1/\sin x$

$$\begin{aligned}y' &= \frac{1' \sin x - 1 \cdot (\sin x)'}{(\sin x)^2} = -\frac{\cos x}{(\sin x)^2} \\&= -\frac{\cos x}{\sin x} \frac{1}{\sin x} \\&= \cot x \csc x.\end{aligned}$$

m. $y = \cot x = -\cos x/\sin x$

n. $y = \pi$

$$y' = 0.$$

o. $y = \pi x^2 + \sqrt{2}$

$$y' = (\pi)'x^2 + \pi(x^2)' + (\sqrt{2})' = 0 + 2\pi x + 0 = 3\pi x.$$

p. $y = \sqrt{x}$

$$y' = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

q. $y = 3x\sqrt{x}$

$$y' = (3x \cdot x^{1/2})' = (3x^{3/2})' = 3 \cdot \frac{3}{2}x^{1/2} = \frac{9}{2}\sqrt{x}.$$

r. $y = x^{-1/2}$

$$y' = -\frac{1}{2}x^{-3/2} = \frac{1}{2x^{3/2}} = \frac{1}{2x\sqrt{x}}.$$

s. $y = \frac{5\pi}{2x}$

$$y' = \frac{5\pi}{2} \left(\frac{1}{x}\right)' = \frac{5\pi}{2} (x^{-1})' = -\frac{5\pi}{2} x^{-2} = -\frac{5\pi}{2x^2}.$$

$$t. y = \frac{x^2 - 1}{x^2 + x + 1}$$

$$\begin{aligned} y' &= \frac{(x^2 - 1)'(x^2 + x + 1) - (x^2 - 1)(x^2 + x + 1)'}{(x^2 + x + 1)^2} \\ &= \frac{2x(x^2 + x + 1) - (x^2 - 1)(2x + 1)}{(x^2 + x + 1)^2} \\ &= \frac{2x^3 + 2x^2 + 2x - (2x^3 + x^2 - 2x - 1)}{(x^2 + x + 1)^2} \\ &= \frac{x^2 + 4x + 1}{(x^2 + x + 1)^2}. \end{aligned}$$

Formulas.

$$\tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}, \sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x}$$

10. $(\tan x)' = \sec^2 x$
11. $(\cot x)' = -\csc^2 x$
12. $(\sec x)' = \sec x \tan x$
13. $(\csc x)' = -\csc x \cot x$

Chain Rule

$$14. (f \circ g)'(x) = f'(g(x))g'(x).$$

HW31. Find the derivatives of the following functions:

$$a) f(x) = (2x - 7)^3$$

$$f'(x) = 3(2x - 7)^2(2x)' = 3(2x - 7)^2(2) = 6(2x - 7)^2.$$

$$b) f(x) = 3(4 - 9x)^4$$

$$f'(x) = 3((4 - 9x)^4)' = 3 \cdot 4(4 - 9x)^3(4 - 9x)' = 3 \cdot 4(4 - 9x)^3(-9) = -108(4 - 9x)^3.$$

$$c) f(t) = \sqrt{1 - t}$$

$$f'(t) = ((1 - t)^{1/2})' = \frac{1}{2}(1 - t)^{-1/2} = \frac{1}{2(1 - t)^{1/2}} = \frac{1}{2\sqrt{1 - t}}.$$

$$d) f(t) = (9x^2 + 4)^{1/3}$$

$$f'(t) = \frac{1}{3}(9x^2 + 4)^{-2/3}(9x^2 + 4)' = \frac{1}{3}(9x^2 + 4)^{-2/3}(9) = 3(9x^2 + 4)^{-2/3} = \frac{3}{(9x^2 + 4)^{2/3}}.$$

$$e) f(t) = \frac{1}{1 - t}$$

$$f'(t) = ((1 - t)^{-1})' = -(1 - t)^{-2}(1 - t)' = -(1 - t)^{-2}(-1) = (1 - t)^{-2} = \frac{1}{(1 - t)^2}.$$

$$f) f(t) = \left(\frac{1}{1 - t}\right)^3$$

$$f'(t) = ((1 - t)^{-3})' = -3(1 - t)^{-4}(1 - t)' = -3(1 - t)^{-4}(-1) = 3(1 - t)^{-4} = \frac{3}{(1 - t)^4}.$$

$$\text{g) } f(t) = \frac{1}{1-t^2}$$

$$f'(t) = ((1-t^2)^{-1})' = -(1-t^2)^{-2}(1-t^2)' = -(1-t^2)^{-2}(-2t) = (1-t^2)^{-2}(2t) = \frac{2t}{(1-t^2)^2}.$$

$$\text{h) } f(t) = x(x^2 + 2x - 1)^9$$

$$\begin{aligned} f'(t) &= (x)'(x^2 + 2x - 1)^9 + x((x^2 + 2x - 1)^9)' \\ &= (x^2 + 2x - 1)^9 + 9x(x^2 + 2x - 1)^8(x^2 + 2x - 1)' \\ &= (x^2 + 2x - 1)^9 + 9x(x^2 + 2x - 1)^8(2x + 2) \\ &= (x^2 + 2x - 1)^8[(x^2 + 2x - 1) + 9x(2x + 2)] \\ &= (x^2 + 2x - 1)^8(x^2 + 2x - 1 + 18x^2 + 18x) \\ &= (x^2 + 2x - 1)^8(19x^2 + 20x - 1). \end{aligned}$$

$$\text{i) } f(t) = \left(\frac{1+t}{1-t}\right)^3$$

$$\begin{aligned} f'(t) &= 3\left(\frac{1+t}{1-t}\right)^2 \left(\frac{1+t}{1-t}\right)' \\ &= 3\left(\frac{1+t}{1-t}\right)^2 \left(\frac{(1+t)'(1-t) - (1+t)(1-t)'}{(1-t)^2}\right)^3 \\ &= 3\left(\frac{1+t}{1-t}\right)^2 \left(\frac{(1-t) - (1+t)(-1)}{(1-t)^2}\right)^3 \\ &= 3\left(\frac{1+t}{1-t}\right)^2 \left(\frac{(1-t) + (1+t)}{(1-t)^2}\right)^3 \\ &= 3\left(\frac{1+t}{1-t}\right)^2 \left(\frac{2}{(1-t)^2}\right)^3 \\ &= 6\frac{(1+t)^2}{(1-t)^4}. \end{aligned}$$

$$\text{j) } f(t) = \tan^2(5x).$$

$$\begin{aligned} f'(t) &= 2 \tan(5x)(\tan(5x))' \\ &= 2 \tan(5x)(\sec^2(5x))(5x)' \\ &= 2 \tan(5x)(\sec^2(5x))5 \\ &= 10 \tan(5x) \sec^2(5x). \end{aligned}$$

HW #32. Find the equation of the tangent to the graph of the following functions:

(a) $y = x^4 - 3x^2 + 7$ at the point $(-1, 5)$,

Step I.

$$y' = 4x^3 - 6x$$

Step II.

If $x = -1$ then the slope $m = 4(-1)^3 - 6(-1) = -4 + 6 = 2$.

Step III.

$$y - 5 = 2(x - (-1)) = 2(x + 1).$$

(b) $y = (x - 1)^5$ at the point $(0, -5)$,

Step I.

$$y' = 5(x - 1)^4$$

Step II.

If $x = 0$ then the slope $m = 5(0 - 1)^4 = -5(1) = 5$.

Step III.

$y - (-5) = 5(x - 0)$ so $y + 5 = 5x$ or

$$y = 5x - 5.$$

(c) $y = \frac{1}{x^2}$ at the point $(2, 1/4)$,

Step I.

$$y' = -\frac{2}{x^3}$$

Step II.

If $x = 2$ then the slope

$$m = -\frac{2}{2^3} = -\frac{2}{8} = -\frac{1}{4}.$$

Step III.

$$y - \frac{1}{4} = -\frac{1}{4}(x - 2)$$

or

$$y - 1 = -(x - 2)$$

or

$$y = 3 - x.$$

(d) $y = \frac{x+1}{x-1}$ at the point $(0, -1)$,

Step I.

$$y' = \frac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1) - (x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

Step II.

If $x = 0$ then the slope

$$m = -\frac{2}{(0-1)^2} = -\frac{2}{1} = -2.$$

Step III.

$$y - (-1) = -2(x - 0)$$

or

$$y + 1 = -2x$$

or

$$y = -2x - 1.$$

(e) $y = \cos x$ at $(\pi/2, 0)$.

Step I.

$$y' = -\sin x$$

Step II.

If $x = \pi/2$ then the slope $m = -\sin(\pi/2) = -1$.

Step III.

$$y - 0 = -(x - \frac{\pi}{2})$$

or

$$y = \frac{\pi}{2} - x.$$

HW 33. (a) Given that $1 - xy = x - y$. Find dy/dx and the equation of the tangent at the point $(1, 1)$.

Step I.

$$\frac{d}{dx}(1) - (\frac{d}{dx}(x))y - x\frac{d}{dx}(y) = \frac{d}{dx}(x) - \frac{d}{dx}(y)$$

so

$$-y - x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

or

$$(1 - x)\frac{dy}{dx} = 1 + y$$

or

$$\frac{dy}{dx} = \frac{1 + y}{1 - x}.$$

Step II. Substitute $x = 1$ and $y = 1$ into Step I, we get:

$$m = \frac{dy}{dx} = \frac{1 + 1}{1 - 1}$$

which is undefined.

(b) Given that $\sqrt{x} + \sqrt{y} = 4$. Find dy/dx and the equation of the tangent at the point $(9, 1)$.

Step I.

$$\frac{d}{dx}(x^{1/2}) + (\frac{d}{dx}(y^{1/2})) = 0$$

so

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

or

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Step II. Substitute $x = 9$ and $y = 1$ into Step I, we get:

$$m = \frac{dy}{dx} = -\frac{\sqrt{1}}{\sqrt{9}} = -\frac{1}{3}.$$

Step III. The equation of the tangent line is:

$$y - 1 = -\frac{1}{3}(x - 9) = 3 - \frac{1}{3}x$$

or

$$y = -\frac{1}{3}(x - 9) = 4 - \frac{1}{3}x.$$

HW 34. Verify the Intermediate Value Theorem on the indicated interval and the given value.

(a) $f(x) = x^2 + x - 1$, $[0, 5]$, $f(c) = 11$.

Check the end points:

$$f(0) = -1, f(5) = 29$$

since $-1 < 11 < 29$ there is a point c between 0 and 5 such that $f(c) = 11$.

(b) $f(x) = \frac{x^2 + x}{x - 1}$, $[\frac{5}{2}, 4]$, $f(c) = 0$.

Check the end points:

$$f(5/2) = \frac{35}{6}, f(4) = 20/3$$

but 0 is not between $35/6$ and $20/3$ so this is impossible to decide.

HW 35. At 8:00 am on Saturday a man begins running up the side of a mountain to the top of the mountain. On Sunday morning at 8:00 am he runs back down (at the same speed as on the way up) the mountain. At some point on his way down he realized that he passed the same place at the same time on Saturday. Prove that he is correct by using the Intermediate Value Theorem.

Solution. Let $h_{\text{sat}}(t)$ be the height function of the runner on Saturday and $h_{\text{sun}}(t)$ be the height function on Sunday. Let $f(t) = h_{\text{sat}}(t) - h_{\text{sun}}(t)$. A root of the function f is the time when the runner passed the same place on both days. Suppose that it took the runner N hours to run to the top on Saturday and back down on Sunday. Let H be the height of the mountain. The domain of the function is the closed interval $[0, N]$ (so we set the clock to be 0 at 8 am on both days. We have the following information:

On Saturday: at $t = 0$, $h_{\text{sat}}(0) = 0$ (at 8 am the runner is at the bottom of the mountain) and $h_{\text{sat}}(N) = H$ (N hours after 8 am the runner is at the top of the mountain).

On Sunday: at $t = 0$, $h_{\text{sun}}(0) = H$ (at 8 am the runner is at the top of the mountain) and $h_{\text{sun}}(N) = 0$ (N hours after 8 am the runner is at the bottom of the mountain).

The preceding information means that

$$f(0) = h_{\text{sat}}(0) - h_{\text{sun}}(0) = 0 - H = -H < 0$$

and

$$f(N) = h_{\text{sat}}(N) - h_{\text{sun}}(N) = H - 0 = H > 0.$$

Since the function f is continuous on the closed interval $[0, N]$, we conclude by the IMVT (or the Elevator Theorem) that, at some time c between 0 and N , we have $f(c) = 0$. This means that at c hours after 8 am on both days the runner passed through the same point.

HW 36. Let $y = (1 + x^2)^{-1}$ and that $dx/dt = 2$. Find dy/dt when (a) $x = -2$, (b) $x = 0$ and (c) $x = 2$.

Solution. Follows the following steps:

Step I. Compute:

$$\frac{dy}{dt} = \frac{d}{dt}(1 + x^2)^{-1}.$$

Use power rule to compute the RHS.

Step II. (a) if $x = -2$ then $y = (1 + (-2)^2)^{-1} = 1/5$. Substitute $dx/dt = 2$, $x = -2$ and $y = 1/5$ into the equation that you got from Step I. \square

HW 37. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $dx/dt = 2$ cm/sec.

Solution. Follows the following steps:

Step I. First find the distance function between the origin (i.e., $(0,0)$) to a point (x,y) . This is easy, we get from the Pythagorean Theorem that

$$F(x, y) = \sqrt{x^2 + y^2}.$$

This function has two variables but we can eliminate the variable y by using $y = x^2 + 1$. Substituting this into F we get the distance as a function of x alone:

$$s = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}.$$

Step II. Differentiate with respect to t :

$$\frac{ds}{dt} = \frac{d}{dt} \sqrt{x^4 + 3x^2 + 1}.$$

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HW 38. A spherical balloon is inflated with gas at the rate of 800 cubic centimeter per minute. How fast is the radius of the balloon is increasing at the instant the radius is (a) 30 cm (b) 60 cm?

Solution. Follows the following steps:

Step I. First write down the volume of a sphere of radius r :

$$V = \frac{4}{3}\pi r^3.$$

Step II. Differentiate with respect to t :

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3}\pi r^3 \right).$$

Step III. Substitute $dV/dt = 800$ and $x = 30$ in the equation that you got from Step II.

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