Preliminary Exam II – Solutions

1. Find $\dfrac{dy}{dx}$ by implicit differentiation given that

$$\cos(\pi y) - 3\sin(\pi x) = 1.$$

Solution: Differentiation yields

$$-\pi\sin(\pi y)\frac{dy}{dx} - 3\pi\cos(\pi x) = 0$$

hence

$$\pi\sin(\pi y)\frac{dy}{dx} = \frac{-3\cos(\pi x)}{\sin(\pi y)}.$$

2. An object is moving along the circle $x^2 + y^2 = 16$. If $\dfrac{dy}{dt} = 2$ when it reaches the point with coordinates $x = 3$ and $y = 1$, find $\dfrac{dx}{dt}$ at that point.

Solution: Differentiating with respect to the variable t yields:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Substituting the given information yields:

$$3\frac{dx}{dt} + 2 = 0.$$

Thus $\dfrac{dx}{dt} = -\dfrac{2}{3}$.

3. Find the absolute extrema of the function

$$f(x) = x^3 - \frac{3}{2}x^2 + 5$$

on the closed interval $[-1, 2]$.

Solution: Differentiation yields

$$f'(x) = 3x^2 - 3x = 3x(x-1)$$

hence the critical numbers are $x = 0, 1$. The candidates are $-2, 0, 1, 2$ and 4

$$\begin{aligned} f(-1) &= (-1)^3 - \frac{3}{2}(-1)^2 + 5 = -1 - \frac{3}{2} + 5 = \frac{5}{2} \\ f(0) &= 5 \\ f(1) &= 1 - \frac{3}{2} + 5 = \frac{9}{2} \\ f(2) &= 2^3 - \frac{3}{2}2^2 + 5 = 8 - 6 + 5 = 7. \end{aligned}$$

Thus (a). $\begin{array}{rc} \max = & 7 \\ \min = & 5/2 \end{array}$

4. Find all of the critical numbers of the function

$$f(x) = \frac{x^2 + 1}{x}.$$

Solution: Differentiation yields

$$f'(x) = \frac{x^2 - 1}{x^2}.$$

The critical numbers are obtained by setting the numerator equals zero as well as setting the denominator equals. Thus the critical numbers are $x = 0, 1, -1$.

5. Compute $\frac{d^2y}{dx^2}$ for the curve $x^2 - y^2 = 25$.

First we compute dy/dx:

$$2x - 2y\frac{dy}{dx} = 0$$

hence

$$\frac{dy}{dx} = \frac{x}{y}.$$

Thus

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2} = \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = \frac{1}{y} - \frac{x^2}{y^3}.$$

We can express the answer in terms of the variable alone by using the original equation $x^2 = 25 + y^2$:

$$\frac{d^2y}{dx^2} = \frac{1}{y} - \frac{x^2}{y^3} = \frac{1}{y} - \frac{y^2 + 25}{y^3} = -\frac{25}{y^3}.$$

6. The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. Air is escaping from a spherical balloon, and its volume is decreasing at a constant rate of 3 cm^3/sec. At what rate is the radius of the balloon changing when $r = 10$ cm?

Differentiating with respect to t yields

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Thus

$$3 = 4\pi(10)^2 3\frac{dr}{dt}$$

or

$$\frac{dr}{dt} = \frac{-3}{400\pi} \text{ cm/sec.}$$

7. Let

$$f(x) = \frac{x^2}{4} + \sin(x)?$$

Then the first two derivatives are

$$f'(x) = \frac{x}{2} + \cos(x), \quad f''(x) = \frac{1}{2} - \sin(x)$$

Since

$$f''(\pi/6) = 0, f''(0) = \frac{1}{2}, f''(\frac{\pi}{2}) = -\frac{1}{2}$$

we conclude that the point $(\pi/6, f(\pi/6))$ is a point of inflection of f.

8. Compute

$$\lim_{x\to\infty} \frac{1 + x^2 + 2x^{15}}{1 + x^{14} + 3x^{15}} = \lim_{x\to\infty} \frac{2x^{15}}{+3x^{15}} = \frac{2}{3}.$$

9. Suppose that the second derivative of the function $f(x)$ exists for all points in the interval $(0, 2)$, and that $f''(1) > 0$. Which of the following must be true?

I. If $f'(1) = 0$, then f has a local minimum at $x = 1$.

II. f' is an increasing function on the interval $(0, 2)$.

III. If $f''(-1) < 0$, the graph of f has a point of inflection between $x = -1$ and $x = 1$.

Solution: By the second derivative test (I) is true.

10. The total costs per month to a certain company for making and storing x widgets is given by the function

$$C(x) = 4x + \frac{16,000}{x^2} \quad \text{dollars}$$

for $x \geq 1$. How many widgets should they manufacture to minimize the costs?

Solution: The domain of the function $C(x)$ is the interval (x, ∞). Since

$$C'(x) = 4 - \frac{32,000}{x^3}$$

there is only one critical number given by the solution $x^3 = 8,000$ or $x = 20$. Since $C'(10) = 4-32 < 0$ and $C'(40) = 4-2 > 0$. The function has an absolute minimum at $x = 20$ by the first derivative test. Alternatively,

$$C'(x) = \frac{95,000}{x^4}$$

is positive hence $x = 20$ is an absolute minimum (because there is only one critical point).

11. Suppose that the derivative of f exists for all numbers x and is continuous. Assume that the only critical number of f is $x = 0$, and that $f'(1) = 1$. Which of the following **must** be true?

I. If $f'(-1) = -1$, then f has an absolute minimum at the point $x = 0$.
II. f is increasing on the interval $(0, \infty)$.
III. The graph of f has an inflection point at $(0, f(0))$.
IV. The graph of f is concave up on $(0, \infty)$.

Both (I) and (II) are true.

12. An object is dropped from a height of 1000 feet. The height of the object t seconds later is given by the function $h(t) = -16t^2+1000$ feet. Verify the mean value theorem by finding a time t between 0 and 3 seconds at which the instantaneous velocity of the object is equal to its average velocity during the first 3 seconds.

Solution: $h(0) = 1,000, h(3) = 856$ hence the average velocity is

$$\frac{856 - 1000}{3} = \frac{-144}{3} = -48.$$

The instantaneous velocity is

$$v(t) = -32t.$$

MVT says that the instantaneous velocity is equal to the average velocity for some t between 0 and 3:

$$-32t = -48$$

or $t = 3/2$.

13. a. Give a formula for the surface area of a cube whose edges have length x.
b. All edges of a cube are expanding at a constant rate of 7 cm per second. How fast is the surface area changing when each edge is 12 cm in length?

Solution: Let be the surface area then $V = x^3$ and $A = 6x^2$. Differentiation yields:

$$\frac{dA}{dt} = 12x\frac{dx}{dt}.$$

Substituting the given information we get:

$$\frac{dA}{dt} = 12(12)(7) = 1008 \text{ cm}^2/\text{sec}.$$

14. Find the slope of the tangent line to the curve

$$x^2 + xy + y^2 = 3$$

at the point $x = 1$, $y = 1$.

We get

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = -1$$

at $x = 1, y = 1$.

15. Consider the function

$$f(x) = x^4 - 2x^3 + 1$$

a. Find the critical numbers of f.
b. Find the intervals on which f is increasing or decreasing.
c. Find the points at which relative extrema of f occur.

Solution:
(a) $f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$ so the critical numbers are $x = 0, 3/2$.
(b) $f'(-1) < 0, f'(1) < 0, f'(2) > 0$ so f is decreasing on $(-\infty, 0)$ and $(0, 3/2)$ and is increasing on the interval $(3/2, \infty)$.
(c) f has a relative minimum at $x = 3/2$.

16. Consider the function

$$g(x) = x^2 + \frac{1}{x}$$

a. $g'(x) = 2x - \frac{1}{x^2}$ and $g''(x) = 2(1 + \frac{1}{x^3})$. Thus $g''(0)$ does not exist and

$$0 = g''(x) = 2(1 + \frac{1}{x^3})$$

or $x^3 + 1 = 0$ or $x = -1$.

b. Find the intervals on which g is concave up or concave down.

$$g''(-2) = 2(1 - \frac{1}{8}) > 0$$
$$g''(-\frac{1}{2}) = 2(1 - 8) < 0$$
$$g''(1) = 2(1 + 1) > 0$$

Concave up : $(-\infty, -1)$ and $(0, \infty)$
Concave down : $(-1, 0)$.