Solutions to Old Exam IIIb

1. (d). First we have $\Delta x = (b-a)/n = (5-1)/4 = 1$. The points of the subdivision are $x_0 = 1, x_1 = x_0 + \Delta x = 2, x_2 = x_1 + \Delta x = 2+1 = 3, x_3 = x_2 + \Delta x = 3+1 = 4, x_4 = x_3 + \Delta x = 5$. The lower sum is

$$f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x = \big(f(1) + f(2) + f(3) + f(4)\big)\Delta x.$$

Since $f(x) = x^2$ and $\Delta x = 1$ we get

$$1^2 + 2^2 + 3^2 + 4^2.$$

2. (b). Since $y = x - 100$,

$$P = xy = x(x-100)$$

hence $P'(x) = 2x - 100$. Set $0 = P'(x) = 2x - 100$ we get $x = 50$ and $y = -50$ so $P = xy = -2500$.

3. (a).

$$\int \frac{x^3 + x^2 + 1}{\sqrt{x}} dx = \int (x^{5/2} + x^{3/2} + x^{-1/2}) dx = \frac{2}{7}x^{7/2} + \frac{2}{5}x^{5/2} + 2x^{1/2} + C.$$

Factoring out the common factor $2\sqrt{x}$ we get $7/2 = 3 + (1/2), 5/2 = 2 + (1/2)$

$$2\sqrt{x}(\frac{x^3}{7} + \frac{x^2}{5} + 1) + C.$$

4. (a).

$$\frac{d}{dx}\int_0^{x^2} \frac{t}{1+t+t^2} dt = \frac{x^2}{1+x^2+x^4}\frac{d}{dx}x^2 = \frac{2x^3}{1+x^2+x^4}.$$

5. (d). Since $f''(x) = 1$ we get via integration $f'(x) = x + C$. Set $2 = f'(0) = 0 + A$ we see that $A = 2$ thus $f'(x) = x + 2$. Integration yields

$$f(x) = \frac{x^2}{2} + 2x + B.$$

Set $f(0) = 3$ we get $3 = 0 + 0 + B$ so $B = 3$ and

$$f(x) = \frac{x^2}{2} + 2x + 3.$$

6. (a).

$$|x-1| = \begin{cases} x-1, & \text{for } x \geq 1, \\ 1-x, & \text{for } x \leq 1, \end{cases}$$

Thus

$$\int_{-1}^{3} |x-1| dx = \int_{-1}^{1} (1-x) dx + \int_{1}^{3} (x-1) dx = \big[x - \frac{x^2}{2}\big]_{-1}^{1} + \big[\frac{x^2}{2} - x\big]_1^3 = 4.$$

7. (b).

$$\int_0^{\pi/2} (\sin x - \cos x) dx = -\big[\cos x + \sin x\big]_0^{\pi/2} = -[0 + 1 - 1 - 0] = 0.$$

8. (b).

$$\int_1^4 (3 - \frac{1}{x^2})dx = \big[3x + \frac{1}{x}\big]_1^4 = [12 + \frac{1}{4} - 3 - 1] = 8\frac{1}{4}.$$

9. (c). The velocity function is

$$v(t) = -32t + v_0 = -32t + 100.$$

Thus $v(2) = -64 + 100 = 36$.

10. (e). The linear approximation formula is

$$f(x) \approx f(a) + f'(a)(x - a).$$

Take $f(x) = x^5, x = 1.98$ and $a = 2$ then $f'(x) = 5x^4$ hence

$$(1.98)^5 \approx 2^5 + 5(2)^4(1.98 - 2) = 32 - 80(.02) = 32 - 1.6 = 30.4$$

11. (d).

$$y = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}.$$

12. (d). The width of the partition is $\Delta x = (b-a)/n = 3/n$ and $x_0 = 2, x_1 = 2 + \Delta x = 2 + \frac{3}{n}, x_2 = 2 + 2\Delta x = 2 + 2\frac{3}{n}, x_i = 2 + i\Delta x = 2 + \frac{3i}{n}$, ... thus the Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} (2 + \frac{3i}{n})^4 \frac{3}{n}.$$

13. (a) $\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = 1/n$. The points of the partition are

$$\begin{cases} x_0 = 1, \\ x_1 = 1 + \Delta x = 1 + \frac{1}{n}, \\ x_2 = 1 + 2\Delta x = 1 + \frac{2}{n}, \\ x_3 = 1 + 3\Delta x = 1 + \frac{3}{n}, \\ x_i = 1 + i\Delta x = 1 + \frac{i}{n}. \end{cases}$$

The Riemann sum is given by

$$\begin{aligned} S &= \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(1 + \frac{i}{n})\Delta x = \sum_{i=1}^{n} (2(1 + \frac{i}{n}) + 1)\frac{1}{n} \\ &= \sum_{i=1}^{n} (\frac{3}{n} + \frac{2i}{n^2}) \\ &= \frac{3}{n}n + \frac{2}{n^2}\frac{n(n+1)}{2} \\ &= 3 + \frac{n(n+1)}{n^2}. \end{aligned}$$

Thus

$$\int_1^2 (2x+1)dx = \lim_{n\to\infty} 3 + \frac{n(n+1)}{n^2} = 3 + 1 = 4.$$

$$\int_1^2 (2x+1)dx = \left[x^2+x\right]_1^2 = 6-2 = 4.$$

14. Let x be the length of the side of each of the cut-out squares. Then

$$V = x(y-2x)^2, y^2 = 1200$$

so $V = x(\sqrt{1200}-2x)^2$. Differentiation yields $V'(x) = 1200 - 8\sqrt{1200}x + 12x^2$ so, setting $V'(x) = 0$ we get

$$(2x-\sqrt{1200})(6x-\sqrt{1200}) = 0$$

so $x = \sqrt{1200}/2$ or $x = \sqrt{1200}/6$. If we take $x = \sqrt{1200}/2$ then volume is zero (minimum). For $x = \sqrt{1200}/6, y-2x = 2\sqrt{1200}/3$ and

$$V = x(y-2x)^2 = \frac{\sqrt{1200}}{6}\frac{4(1200)}{9} = \frac{2400\sqrt{1200}}{27}.$$

16. $D = x^2+y^2, y = 2x+5$ so $D(x) = x^2+(2x+5)^2$ and

$$D'(x) = 2x + 4(2x+5) = 10x+20.$$

Set $D'(x) = 0$ we get $x = -2$ and $y = 1$. Distance $= \sqrt{4+1} = \sqrt{5}$.