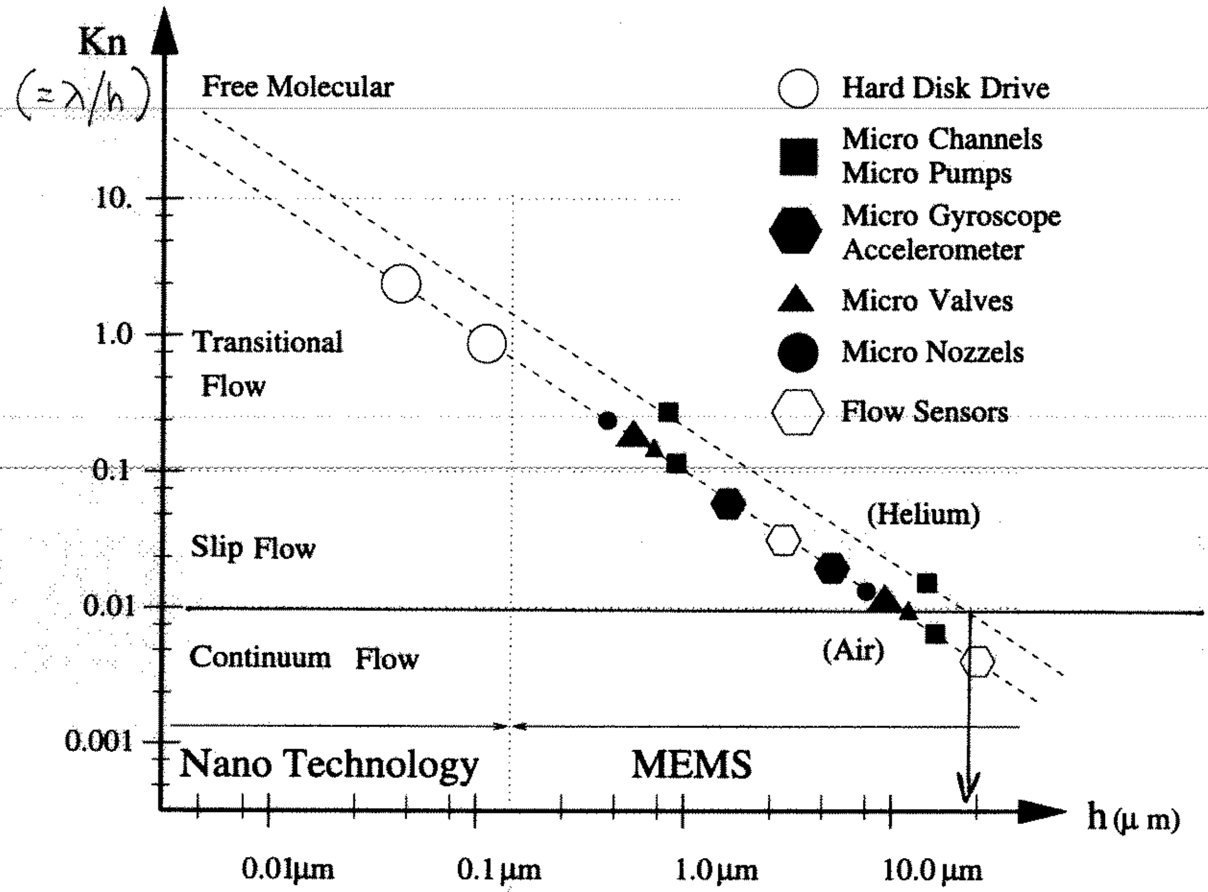


FIGURE 1.10. Typical MEMS and nano technology applications in standard atmospheric conditions span the entire Knudsen regime (Continuum, slip, transition and free-molecular flow). Here *h* denotes a characteristic length scale for the micro flow.

Ref: MicroFlows by Karniadakis & Beskok

The Mean Free Path

- The concept of a continuum is based upon the mean free path, λ.
- λ is the <u>average</u> distance that a molecule will travel in a gas between collisions

- From the kinetic theory of gases

$$\lambda = 1/(\sqrt{2}\pi n d^2)$$

n: molecules/volume $= p/k_B T$

k_B: Boltzman's constant $= 1.3806 \times 10^{-23}\ J/K$

d: molecular diameter $\cong 3.7 \times 10^{-10}\ m$

- For air at STP ($p = 1.01325 \times 10^5\ Pa$; $T = 288.15\ K$)

$$\lambda_{STP} = 6.6 \times 10^{-8}\ m \cong \text{one thousandth of the diameter of human hair}$$

- Also, $n_{STP} = \dfrac{(1.01325 \times 10^5)}{(1.3806 \times 10^{-23})(288.15)} = 2.547 \times 10^{25}\ \dfrac{\text{molecules}}{m^3}$

$\Rightarrow$ average distance between molecules $= 3.4 \times 10^{-9}\ m$

$\Rightarrow$ molecule travels ~ 21 molecular distances before a collision

- The Knudsen number, $Kn \equiv \lambda/L$ ← characteristic length

$Kn \lessapprox 0.01 \Rightarrow$ continuum

$Kn \gtrapprox 10 \Rightarrow$ free-molecular

$0.01 \lessapprox Kn \lessapprox 0.1 \Rightarrow$ slip flow

$0.1 \lessapprox Kn \lessapprox 10 \Rightarrow$ transition flow

Forms of Governing Fluid Equations

- Integral: $\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = -\int_{CS} \rho \bar{V} \cdot d\bar{A}$ Cons. of mass

useful in gross behavior of flow field and its effects on devices

- Differential: $-(\nabla \cdot \rho \bar{V}) = \frac{\partial \rho}{\partial t}$ cons. of mass

detailed, pt-by-pt knowledge of a flow field

—//—

- Conservative (aka divergence, reduced)

$$\frac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j) = \rho f_i - \partial_i p + \partial_j \tau_{ji}$$

first choice for numerical simulation
easy to discretize

- Non-conservative (word seldom used now)

$$\rho \frac{D v_i}{Dt} = \rho f_i - \partial_i p + \partial_j \tau_{ji}$$

Shorter; easier to physical interpretation
here (l.h.s.): time rate of change in momentum/volume of a fluid element as it moves through space

REFERENCE FRAMES

- There are 2 main reference frames used in fluid mechanics:

Eulerian	Lagrangian
↓	↓
fixed in space	moving with material substance (fluid element)

- Each provide different mathematical representations of the <u>same</u> physical field.

- One field may be steady ⇒ easier to analyze

  Ex: moving 'on' a wave makes the analysis 'steady'

DERIVATIVES

- Let $\mathbf{c}(x, y, z, t)$ be the concentration of fish swimming around in a river

- The Partial Time Derivative $\partial/\partial t$

  Stand on a bridge and observe $\mathbf{c}$ just below as a function of time. The time rate of change of $\mathbf{c}$ at a fixed location is

$$\left(\frac{\partial \mathbf{c}}{\partial t}\right)\Bigg|_{x,y,z}$$

- The Total Time Derivative, d/dt

  Now speed around the river in a motor boat. The time rate of change of $\mathbf{c}$ is

$$\frac{dc}{dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} + \frac{dx}{dt}\left(\frac{\partial c}{\partial x}\right)_{y,z,t} + \frac{dy}{dt}\left(\frac{\partial c}{\partial y}\right)_{x,z,t} + \frac{dz}{dt}\left(\frac{\partial c}{\partial z}\right)_{x,y,t}$$

velocity components of the boat

- The Substantial Time Derivative, D/Dt

  Now just float along with the current
  $\Rightarrow$ velocity of observer = velocity of the river ($\bar{v}$)
  The time rate of change of $\mathbf{c}$ is

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x\frac{\partial c}{\partial x} + v_y\frac{\partial c}{\partial y} + v_z\frac{\partial c}{\partial z} = \frac{\partial c}{\partial t} + (\bar{v}\cdot\nabla c)$$

material derivative

local derivative

convective derivative

Governing Equations (incompressible, Newtonian, constant properties)

$$\nabla \cdot \bar{u} = 0$$ cons. of mass (1 eqn)

$$\rho \frac{D\bar{u}}{Dt} = -\nabla p + \rho \bar{g} + \mu \nabla^2 \bar{u}$$ cons. of momentum (3 eqns)

$\Big\}$ N-S Eqns

$$\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla T = \kappa \nabla^2 T$$ cons. of energy (1 eqn)

physical properties & 'constants': $\rho, \mu, \kappa, \bar{g}$

unknowns: $p, T, \bar{u}$ (5 total) — 5 eqns

Scalar (Latin: scala = ladder): magnitude

Ex: mass, length, volume, p, T

Vector (Latin: vectus = carried): magnitude & uni-direction

Ex: force, velocity, acceleration

Tensor (Latin: tendere = to stretch): magnitude & multi-direction

Merriam-Webster Definition: "a generalized vector with more than 3 components, each of which is a function of the coordinates of an arbitrary point in space of an appropriate number of dimensions"

Ex: Stress Tensor

THE STRESS TENSOR

p.30 Kundu & Cohen

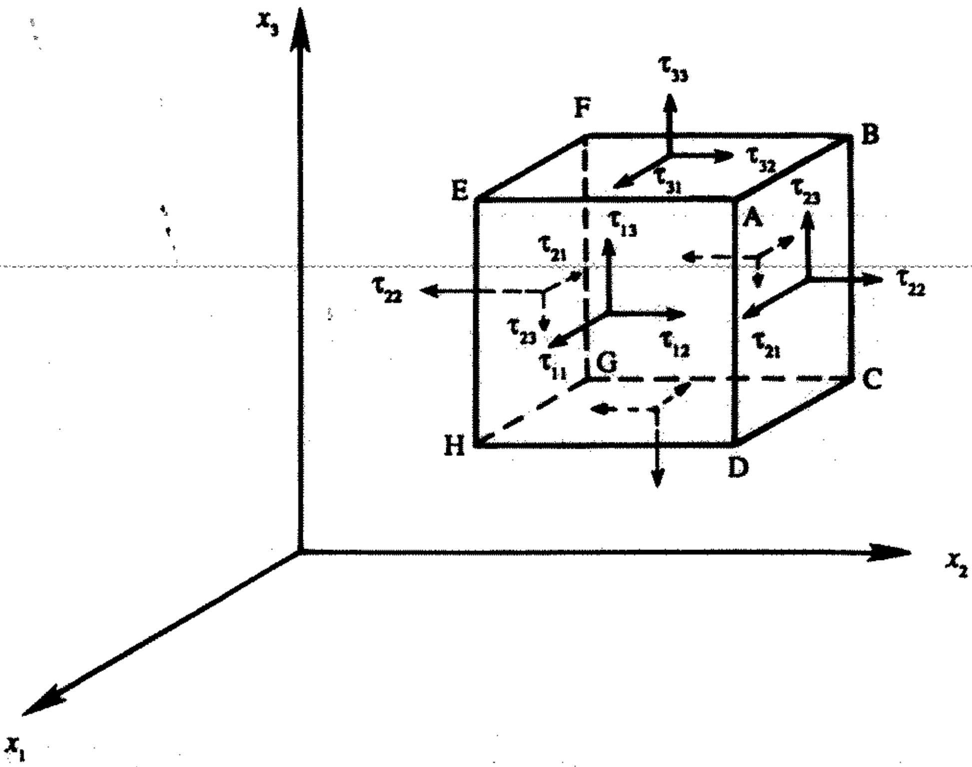


Figure 2.4 Stress field at a point. Positive normal and shear stresses are shown. For clarity, the stresses on faces FBCG and CDHG are not labeled.

the state of stress at a point can be completely specified by the nine components τ_{ij} (where $i, j = 1, 2, 3$), which can be written as the matrix

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}.$$

τ_{ij} :

i denotes face perpendicular to i-th direction

j denotes direction of component

DIVERGENCE OPERATOR (brings order down by 1)

$$\nabla^T \cdot \bar{u} = \underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}_{\text{Scalar}} = \text{div}\, \bar{u}$$

operator 'nabla' — Vector

$$= \partial_i u_i$$

$$= \nabla \cdot u \quad \text{(assumes transpose is understood)}$$

- div of a vector produces a scalar
- div of a 2nd-order tensor produces a row vector

$$\partial_i T_{ij} = \partial_1 T_{1j} + \partial_2 T_{2j} + \partial_3 T_{3j} = \text{div}\, \mathbf{T} = \nabla^T \cdot \bar{\bar{T}}$$

- It can be shown that when $\bar{u}$ denotes velocity

$$\nabla \cdot \bar{u} = \frac{1}{\delta\forall} \frac{D(\delta\forall)}{Dt} \qquad \delta\forall \text{ fluid element volume}$$

which is the time rate of change of a moving fluid element volume per unit volume

GRADIENT OPERATOR (brings order up by 1)

- Of a scalar field, p

$$\nabla p = \underbrace{\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}}_{\text{vector}} = \partial_i p$$

Gibbs notation → ∇p ; del p ; operator, scalar ; Einstein notation → $\partial_i p$

* conservative force, F, can be written as $F = \nabla \phi$ ← potential (scalar)

- Of a vector field, $\bar{u}$

$$\nabla \bar{u}^T = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} (u \ v \ w) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}$$

operator, vector

$$= \underbrace{\partial_j u_i}_{\text{2nd-order tensor}} dx_j$$

tensor
9 components for 3-D

written as ∇u because it is assumed that the transpose operation is understood (vectors must conform to multiply $a_{ij} b_{jk} = c_{ik}$)

- Of a tensor $\underbrace{\partial_k T_{ij}}_{\text{3rd-order tensor}} dx_k$ (27 components!)

Curl Operator (keeps same order)

$$\nabla \times \bar{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \bar{S}$$

$$= \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{S_1} \hat{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{S_2} \hat{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{S_3} \hat{k}$$

- Curl of a vector produces a vector

- What if $\nabla \times \bar{u} = 0$?

Then $S_1 = 0$, $S_2 = 0$, $S_3 = 0$

$$\Rightarrow \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \quad \& \quad \frac{\partial u}{\partial z} = \frac{\partial z}{\partial x} \quad \& \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad \text{A}$$

- When $\bar{u}$ denotes velocity $\nabla \times \bar{u}$ denotes fluid vorticity, $\bar{\omega}$. That is,

$$\nabla \times \bar{u} = \bar{\omega}$$

Thus, conditions A hold when $\bar{\omega} = 0$, which is called irrotational flow.

- Note: $\frac{1}{2}\omega$ is the average velocity of two small fluid lines that, at that instant, are mutually perpendicular

Thus, vorticity 'represents' the local rotation or 'spin' of a fluid element.

- Curl $F = 0$ if F is a conservative force because

$$\nabla \times F = \nabla \times (\nabla \phi) = 0$$

LAPLACIAN OPERATOR (keeps same order)

$$\nabla^2 u = \text{div grad } \bar{u} = \partial_i \partial_i u_j$$

- occurs in waves, heat conduction problems
  – use Fourier series solution techniques

- can have Laplacian of a scalar, vector or tensor

INCOMPRESSIBLE, CONSTANT PROPERTY EQUATIONS

Conservation of Mass:

$$\partial_i v_i = 0 \qquad \text{OR} \qquad \nabla \cdot \bar{v} = 0$$

Conservation of Momentum:

$$\rho \partial_0 v_i + \rho v_j \partial_j v_i = \rho f_i - \partial_i p + \mu \partial_j \partial_j v_i$$

OR

$$\rho \frac{D\bar{v}}{Dt} = \rho \bar{f} - \nabla p + \mu \nabla^2 \bar{v}$$

Conservation of Energy:

$$\rho c \partial_0 T + \rho c v_j \partial_j T = k \partial_i \partial_i T + 2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}$$

OR

$$\rho c \frac{DT}{Dt} = k \nabla^2 T + \Phi$$

EINSTEIN NOTATION SUMMARY (see Powers)

free index: appears only once in a term

Ex: $\frac{\partial}{\partial x_i} = \partial_i$ or u_i

dummy index: appears only twice in a term

Ex: $\frac{\partial}{\partial x_j} v_i \, dx_j$ repeated $\Rightarrow$ summation

$$\frac{\partial}{\partial x} v_i \, dx + \frac{\partial}{\partial y} v_i \, dy + \frac{\partial}{\partial z} v_i \, dz$$

Kronecker Delta Function:

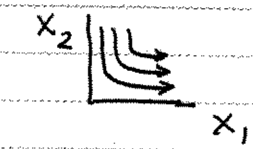
$$\delta_{ij} \equiv \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow X_{ij} \delta_{ij} = X_{11} + X_{22} + X_{33}$$

Cross Vector Product

$$\epsilon_{ijk} \equiv \begin{cases} 1 & ijk = 123, 231, 312 \curvearrowright \text{cw} \\ 0 & \text{any 2 terms the same} \\ -1 & ijk = 321, 213, 132 \curvearrowleft \text{ccw} \end{cases}$$

Using index notation, compute the vorticity components of stagnation-point flow where

$$u_1 = c x_1 \qquad u_2 = -c x_2 \qquad u_3 = 0$$



Note: $\bar{\omega} = \nabla \times \bar{u} \qquad \omega_i = \epsilon_{ijk} \partial_j u_k$

$$\Rightarrow \omega_1 = \epsilon_{1jk} \partial_j u_k = \epsilon_{123} \partial_2 u_3 + \epsilon_{132} \partial_3 u_2$$

$$= \partial_2 u_3 - \partial_3 u_2 = 0 - 0 = 0$$

$$\omega_2 = \epsilon_{2jk} \partial_j u_k = \epsilon_{231} \partial_3 u_1 + \epsilon_{213} \partial_1 u_3$$

$$= \partial_3 u_1 - \partial_1 u_3 = 0$$

$$\omega_3 = \epsilon_{3jk} \partial_j u_k = \epsilon_{312} \partial_1 u_2 + \epsilon_{321} \partial_2 u_1$$

$$= \partial_1 u_2 - \partial_2 u_1 = 0$$

$$\Rightarrow \bar{\omega} = 0 = \nabla \times \bar{u} \Rightarrow \text{irrotational flow}$$

A tensor can be decomposed into symmetric and anti-symmetric tensors

$$\underset{\text{symmetric}}{\tau_{ij} = \tau_{(ij)}} + \underset{\text{anti-symmetric}}{\tau_{[ij]}}$$

where

$$\tau_{(ij)} = \frac{1}{2}\left(\tau_{ij} + \tau_{ji}\right) \qquad (\text{sym} \Rightarrow \tau_{ij} = \tau_{ji})$$

$$\tau_{[ij]} = \frac{1}{2}\left(\tau_{ij} - \tau_{ji}\right) \qquad (\text{anti-sym} \Rightarrow \tau_{ij} = -\tau_{ji})$$

The product of a symmetric and an anti-symmetric tensor is 0. Example $\underset{\text{anti-sym}}{\epsilon_{ijk}}\, \underset{\text{sym}}{\tau_{(jk)}} = 0$

Whereas $\underset{\text{anti-sym}}{\epsilon_{ijk}}\, \underset{\text{anti-sym}}{\tau_{[jk]}} \neq 0 \equiv d_i$ (the dual vector of a tensor)

Given the 2-D velocity field with components

$$u_1 = c(x_1^2 + x_2^2)\,; \quad u_2 = c(x_1^2 - x_2^2)\,; \quad u_3 = 0$$

c: conversion factor = 1/(m·s) in SI

Find the components of the stress tensor $\partial_i u_j$ at the point (10, 5, 0) and of the symmetric $\partial_{(i}u_{j)}$ and anti-symmetric $\partial_{[i}u_{j]}$ tensors. Show $\partial_i u_j = \partial_{(i}u_{j)} + \partial_{[i}u_{j]}$

#

$$\partial_i u_j = \begin{bmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 \\ \partial_3 u_1 & \partial_3 u_2 & \partial_3 u_3 \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_1 & 0 \\ 2x_2 & -2x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 20 & 0 \\ 10 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $\partial_{[i}u_{j]} = \frac{1}{2}\left[\partial_i u_j - \partial_j u_i\right]$ (associated with rotation)

$$= \begin{bmatrix} 0 & \frac{20-10}{2} & 0 \\ \frac{10-20}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 \\ -5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also, $\partial_{(i}u_{j)} = \frac{1}{2}\left[\partial_i u_j + \partial_j u_i\right]$ (associated with shear)

$$= \begin{bmatrix} 20 & \frac{20+10}{2} & 0 \\ \frac{10+20}{2} & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 15 & 0 \\ 15 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\partial_{[i}u_{j]} + \partial_{(i}u_{j)} =$

$$\begin{bmatrix} 0 & 5 & 0 \\ -5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 20 & 15 & 0 \\ 15 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 20 & 0 \\ 10 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \partial_i u_j$$

EXAMPLE: Inner Product of a Tensor with Itself

- Recall the energy equation:

$$\rho c\, \partial_0 T + \rho c\, v_j \partial_j T = k\, \partial_i \partial_i T + 2\mu\, \partial_{(i v_j)} \partial_{(i v_j)}$$

OR

$$\rho c \frac{DT}{Dt} = k \nabla^2 T + \underset{\text{dissipation function}}{\Phi}$$

$$\underset{\text{scalar}}{\Phi} = \underset{\text{scalar}}{2\mu}\, \underbrace{\partial_{(i v_j)} \partial_{(i v_j)}}_{\substack{\text{inner product} \\ \text{of a tensor} \\ \text{with itself} \\ = \text{scalar}}} \; ; \quad \underset{\text{tensor}}{\partial_{(i v_j)}} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

- Fully expand $\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$. (drop $\frac{1}{2}$ factor for now)

$$\begin{array}{c|ccc|}
 & j=1 & j=2 & j=3 \\
\hline
i=1 & 2\frac{\partial u}{\partial x} & \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\
i=2 & \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\frac{\partial v}{\partial y} & \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\
i=3 & \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\frac{\partial w}{\partial z} \\
\hline
\end{array}$$

- So, the inner product of $\partial_{(i v_j)}$ with itself, which is the sum of the squares of each of the nine terms, becomes

$$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2$$

$$+\frac{1}{4}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2$$

$$+\frac{1}{4}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2$$

which reduces to (noting its symmetry)

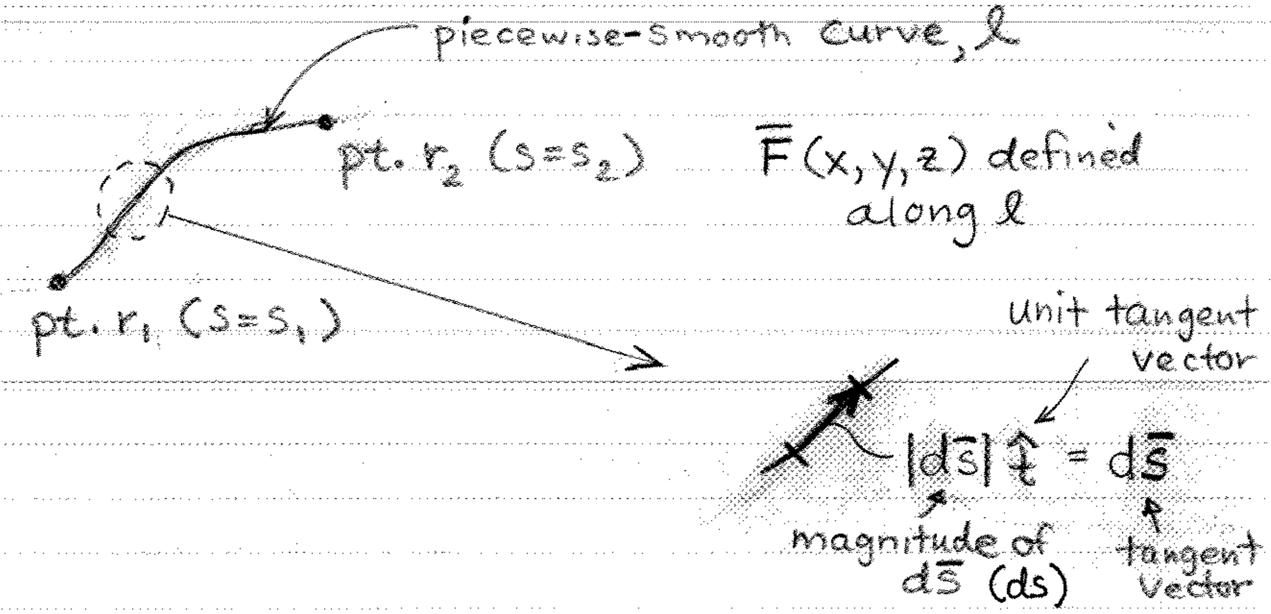
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2$$

$$+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2$$

- Example reduction: For fully developed ($\partial/\partial x = 0$), two-dimensional ($\frac{\partial}{\partial z} = 0$ and $w = 0$), incompressible ($\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ and $v = 0$ at wall $\Rightarrow v = 0$), Φ reduces to

$$\Phi = \mu\left(\frac{\partial u}{\partial y}\right)^2$$

LINE INTEGRAL



- The line integral, I, is defined as

$$I \equiv \int_{s_1}^{s_2} \bar{F} \cdot \hat{t}\, ds = \int_{s_1}^{s_2} \bar{F} \cdot d\bar{s}$$

- If $\bar{F}$ is the force vector, then the work done while moving along a piecewise-smooth curve from s_1 to s_2 is: $W = \int_{s_1}^{s_2} \bar{F} \cdot d\bar{s}$

- If the curve becomes a <u>closed</u> contour, C, then the integral is expressed as $\oint_C \bar{F} \cdot d\bar{s}$

- If the vector $\overline{F}$ is now the velocity of a fluid, $\overline{u}$, the line integral of the tangential component of the velocity, $\overline{u} \cdot d\overline{s}$ $(= |\overline{u}|\,|d\overline{s}| \cos\theta)$, around a <u>closed</u> contour, C, is defined as the circulation, Γ:

$$\Gamma \equiv \oint_C \overline{u} \cdot d\overline{s}$$

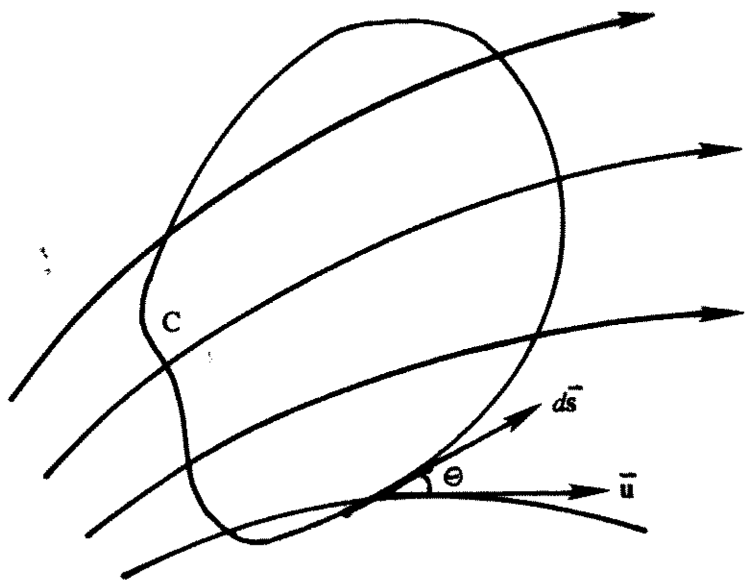
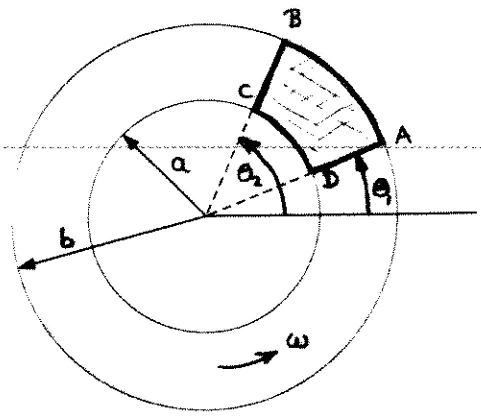


Figure 3.11 Circulation around contour C. p.60 Kundu & Cohen

- Circulation is a scalar.

AME 538 Mid-Semester Exam: Fall 2004

Problem 1: Determine (derive an expression in terms of known variables) the circulation around the path ABCD that is shown in Figure 1. The streamlines for this flow are circles; the velocity is given by $u_\theta = \omega r$, where ω is the angular velocity of the rotating fluid.



Solution:

$$\Gamma = \oint_{ABCD} \vec{V} \cdot d\vec{s} \qquad \left(\begin{array}{ll} \text{for } AB, CD & \vec{V} \cdot d\vec{s} = v_\theta \, r \, d\theta \\ \quad\;\; BC, DA & \qquad\;\; = v_r \, dr \end{array} \right)$$

$$= \int_{AB} v_\theta \, b \, d\theta + \int_{BC} v_r \, dr + \int_{CD} v_\theta \, a \, d\theta + \int_{DA} v_r \, dr \quad (1)$$

Since $v_r = 0$ and $v_\theta = \omega r$, Eq. (1) becomes

$$\Gamma = \int_{\theta_1}^{\theta_2} \omega b^2 \, d\theta + 0 + \int_{\theta_2}^{\theta_1} \omega a^2 \, d\theta + 0$$

$$= \omega b^2 (\theta_2 - \theta_1) + \omega a^2 (\theta_1 - \theta_2)$$

or

$$\Gamma = \omega (\theta_2 - \theta_1)(b^2 - a^2) = \underline{\underline{\omega \Delta\theta \, (b^2 - a^2)}}$$

STOKES THEOREM

- relates line and surface integrals
- $$\oint_C \overline{F} \cdot d\overline{s} = \int_A (\nabla \times \overline{F}) \cdot d\overline{A}$$

where the positive direction of $d\overline{s}$ is such that the surface is on its left-hand side.

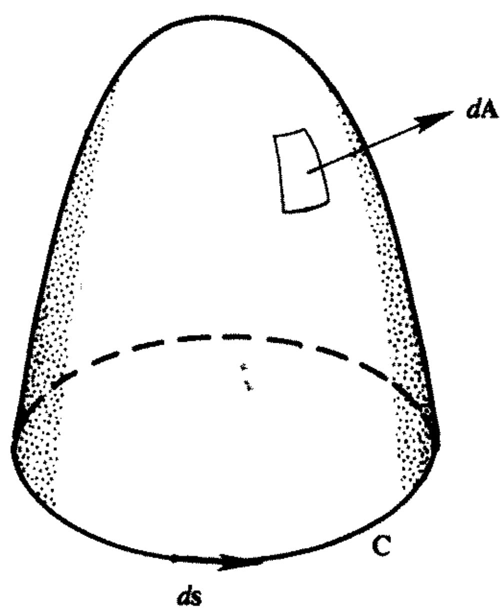


Figure 2.11 Illustration of Stokes' theorem. p. 45 Kundu & Cohen

- The line integral is called the "circulation of $\overline{F}$ about C," Γ.

- Now reduce the surface area to an infinitesimal surface (a point!)

$$\lim_{A\to 0} \frac{1}{A} \int_A (\nabla \times \bar{F}) \cdot d\bar{A} = \bar{n} \cdot (\nabla \times \bar{F})$$

unit vector normal to the local tangent of A

$$\lim_{A\to 0} \frac{1}{A} \int_C \bar{F} \cdot d\bar{s} = \lim_{A\to 0} \frac{\Gamma}{A}$$

$\Rightarrow$ $\nabla \times \bar{F}$ <u>at a point</u> equals Γ/A (circulation per unit area).

- Application of Stoke's Theorem gives (when $\overline{F}$ is $\overline{u}$)

$$\Gamma \equiv \oint_C \overline{u} \cdot d\overline{s} = \underbrace{\int_A (\nabla \times \overline{u}) \cdot d\overline{A}}_{\text{'flux' of } \nabla \times \overline{u} \text{ through surface } A \text{ bounded by } C}$$

- Because $\overline{\omega}$ (vorticity) $\equiv \nabla \times \overline{u}$

$$\Gamma = \underbrace{\int_A \overline{\omega} \cdot d\overline{A}}_{\text{'flux' of vorticity}}$$

$\Rightarrow$ vorticity at a point equals the circulation per unit area

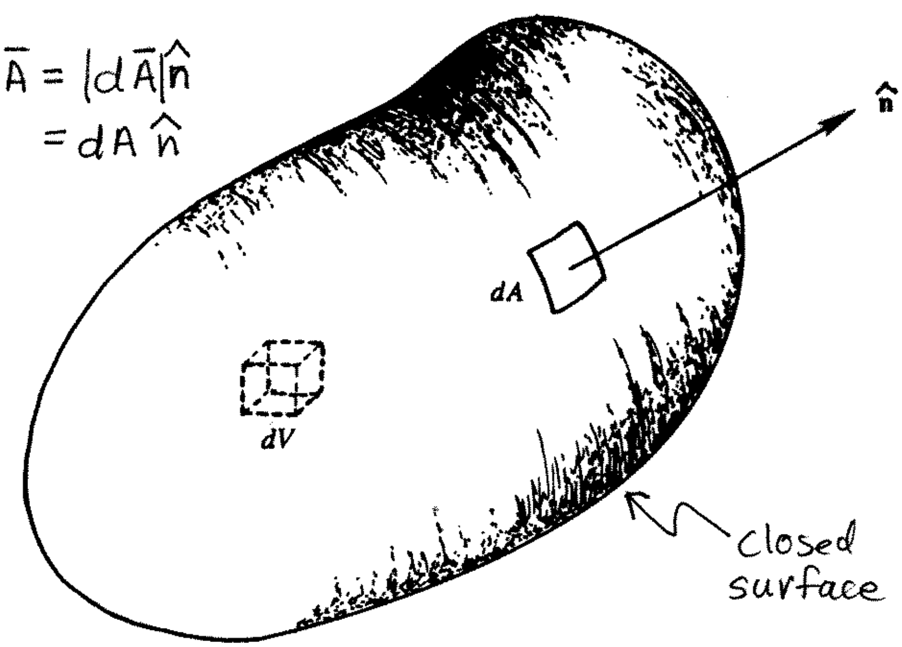
- Mathematically, the 'flux', $\Phi(t)$, at time t is defined as the integral of the component of the vector function $\overline{F}(\overline{r},t)$ normal to the surface $S(t)$

$$\Phi(t) = \int_{S(t)} \overline{F}(\overline{r},t) \cdot d\overline{s}$$

- For engineering, the flux of G is G per unit area per unit time (sometimes, just G per unit area)

GAUSS' THEOREM (aka Divergence Theorem)

- relates surface and volume integrals
- $\int_A \overline{F} \cdot d\overline{A} = \int_{\forall} (\nabla \cdot \overline{F}) d\forall$ (a scalar result)
- When $\overline{F}$ is a vector, this is called the divergence theorem.



$d\overline{A} = |d\overline{A}|\hat{n} = dA\,\hat{n}$

Figure 2.10 Illustration of Gauss' theorem. p.42 Kundu & Cohen

- In index notation

$$\int_A dA_i F_i = \int_{\forall} \frac{\partial F_i}{\partial x_i} d\forall$$

GRADIENT THEOREM

- relates surface and volume integrals
- For a scalar p

$$\int_A p\, d\bar{A} = \int_{\forall} \nabla p\, d\forall \qquad \text{(a vector result)}$$

(the surface A encloses the volume $\forall$)

- $$\int_A p\, d\bar{A} \cong \sum_A p\, \hat{n}\, dA$$

$$= \underbrace{\Delta y \Delta z}_{\substack{\text{area} \\ \text{normal to } \Delta x}} p\Big|_{\Delta x} + \underbrace{\Delta z \Delta x}_{\substack{\text{area} \\ \text{normal to } \Delta y}} p\Big|_{\Delta y} + \underbrace{\Delta x \Delta y}_{\substack{\text{area} \\ \text{normal to } \Delta z}} p\Big|_{\Delta z}$$

$$= \Delta y \Delta z \sum_x \left(\frac{\partial p}{\partial x}\right) \Delta x + \Delta z \Delta x \sum_y \left(\frac{\partial p}{\partial y}\right) \Delta y$$

$$+ \Delta x \Delta y \sum_z \left(\frac{\partial p}{\partial z}\right) \Delta z$$

$$= \sum_{\forall} \left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z}\right) \Delta x \Delta y \Delta z$$

$$\cong \int_{\forall} \nabla p\, d\forall$$

LEIBNITZ THEOREM AND 'VOLUMES'

- Consider a volume whose surfaces move wrt the local fluid velocity
- Let $F(\bar{x},t)$ be a tensor of <u>any</u> rank
- Leibnitz's theorem relates the time derivatives of the integral quantities of F that occur within and through surfaces

$$\frac{d}{dt}\int_V F\,dV = \int_V \frac{\partial F}{\partial t}\,dV + \int_A \overline{dA}\cdot\bar{u}_A F$$

where V denotes the moving volume
A the bounding surface of V
$\bar{u}_A$ the velocity of A

- Conceptually, this is easier to understand for 1-D

$$\frac{d}{dt}\int_{x=a(t)}^{b(t)} F(\bar{x},t)\,dx = \int_a^b \frac{\partial F}{\partial t}\,dx + \frac{db}{dt}F(b,t) - \frac{da}{dt}F(a,t)$$

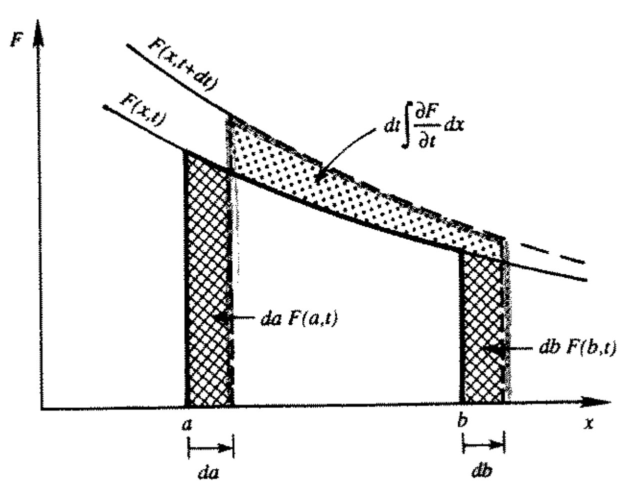


Figure 4.1 Graphical illustration of Leibniz's theorem.

- When the volume is 'fixed', $\overline{u}_A = 0$

$$\Rightarrow \frac{d}{dt}\int_{\forall} F(\overline{x}, t)\, d\forall = \int_{\forall} \frac{\partial F}{\partial t}\, d\forall$$

(because $\forall \neq f(t)$)

- When the volume is 'material'

$$\frac{D}{Dt}\int_{\mathcal{V}} F(\overline{x}, t)\, d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial F}{\partial t}\, d\mathcal{V} + \int_{A} \overline{dA} \cdot \overline{u}\, F$$

notations for material volume

Here, the surface moves at the fluid velocity, $\overline{u}$.

This is called Reynolds transport theorem.