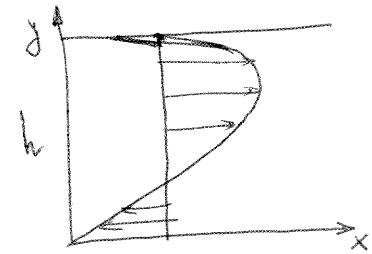
Problem I.

Write the velocity in the form $\vec{v} = (u(y), 0, 0)$, which automatically satisfies the continuity equation. With gravity acting in the plane of the figure, the N-S equations then become



$$(*) \quad \begin{cases} \mu \dfrac{d^2 u}{dy^2} = \dfrac{\partial p}{\partial x} - \rho g \cos\theta \\ 0 = \dfrac{\partial p}{\partial y} - \rho g \sin\theta \\ 0 = \dfrac{\partial p}{\partial z} \end{cases}$$

where θ is the angle between the x-axis and the direction of gravity. Since the viscous terms are functions of at most y, and since from the second and third equation p must be linear in y and independent of z, $\frac{\partial p}{\partial x}$ must be constant.

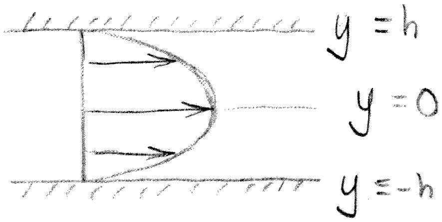
Integration of $(*)$ gives

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} - \rho g \cos\theta\right)\left(y^2 + c_1 y + c_2\right)$$

c_1, c_2 – are constants of integration.

Solving for $u|_{y=0} = u|_{y=h} = 0$ gives

$$\boxed{u(y) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} - \rho g \cos\theta\right)\left(y^2 - hy\right)}$$



newtonian fluid $\Rightarrow$

$$\tau_{ij} = -\frac{2}{3}\mu \partial_k v_k \delta_{ij} + 2\mu\, \partial_{(i} v_{j)}$$

(a) $\partial_k v_k = \partial_x u + \partial_y v + \partial_z w = 0 + 0 + 0 = 0 \Rightarrow$ incompressible

(b) (a) $\Rightarrow \tau_{ij} = 2\mu\, \partial_{(i} v_{j)}$. Because $v, w = 0$ and $u = f(y)$

The only non-zero component is $\partial_2 v_1 = -u_0 2y/h^2$

$$\Rightarrow \partial_{(i} v_{j)} = \begin{bmatrix} 0 & -u_0 y/h^2 & 0 \\ -u_0 y/h^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \tau_{ij} = \begin{bmatrix} 0 & -\frac{2\mu u_0 y}{h^2} & 0 \\ -\frac{2\mu u_0 y}{h^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left(= \frac{1}{2}(\partial_i v_j + \partial_j v_i) \right)$$

So, $\tau_{12} = \tau_{21} = -2\mu u_0 y/h^2$; else $\tau_{ij} = 0$

(c) $\tau_w = \tau_{xy}\big|_{y=-h} = -2\mu u_0(-h)/h^2 = \boxed{2\mu u_0/h = \tau_w}$

(d) $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$; $w_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(0 - u_0 \frac{2y}{h^2}\right) = \boxed{-uy_0/h^2}$$

(e) $\rho \partial_0 v_i + \rho v_j \partial_j v_i = -\partial_i P + \partial_j \tau_{ij}$ (no body force present)

$i=1$ $\rho \cancel{\partial_0 u} + \rho(\cancel{u\partial_1 u} + \cancel{v\partial_2 u} + \cancel{w\partial_3 u}) = -\partial_1 P + \cancel{\partial_1 \tau_{11}} + \partial_2 \tau_{21} + \cancel{\partial_3 \tau_{31}}$

$i=2$ $\cancel{\rho \partial_0 v} + \cancel{\rho(u\partial_1 v + v\partial_2 v + w\partial_3 v)} = -\partial_2 P + \cancel{\partial_1 \tau_{12}} + \cancel{\partial_2 \tau_{22}} + \cancel{\partial_3 \tau_{32}}$

$i=3$ $\rho \cancel{\partial_0 w} + \rho(\cancel{u\partial_1 w} + \cancel{v\partial_2 w} + \cancel{w\partial_3 w}) = -\partial_3 P + \cancel{\partial_1 \tau_{13}} + \cancel{\partial_2 \tau_{23}} + \cancel{\partial_3 \tau_{33}}$

So, $\partial_1 P = \partial_2 \tau_{21}$; $\partial_2 P = 0$; $\partial_3 P = 0$

X-momentum $\Rightarrow \partial_1 P = -2\mu u_0/h^2 \Rightarrow P = -\frac{2\mu u_0}{h^2} x + F(y,z)$

But from y & z momentum $F(y,z)$ must be a constant, say P_0

$$\Rightarrow \boxed{P = -\frac{2\mu u_0}{h^2} x + P_0}$$

Continuity: $\cancel{\frac{\partial u}{\partial x}}_{0} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}_{0} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$

$\Rightarrow \quad v(y) = C$

$v(0) = 0$ (impermeable wall) $\Rightarrow \quad C = 0$

$\Rightarrow \quad v = 0$

X-Momentum:

$$\rho \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0} + \rho \underbrace{\cancel{u \frac{\partial u}{\partial x}}}_{0} + \rho \underbrace{\cancel{v}}_{0} \frac{\partial u}{\partial y} + \rho w \underbrace{\cancel{\frac{\partial u}{\partial z}}}_{0} = \underbrace{\cancel{-\frac{\partial p}{\partial x}}}_{0} + \mu \left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{0} \right)$$

$\Rightarrow \quad 0 = \mu\, \partial^2 u / \partial y^2 \Rightarrow d^2 u / dy^2 = 0$

because $u \neq f(x, z)$

So, $u = C_1 y + C_2$

$u(0) = 0$ (no slip) $\Rightarrow C_2 = 0$

$u(h) = U \qquad \Rightarrow C_1 = U/h$

$\Rightarrow \boxed{u = (U/h)\, y}$

Energy Equation:

setting $\partial/\partial t$, $\partial/\partial x$ & $\partial/\partial z$ terms $= 0$

assume μ, k constant

$$0 = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \Rightarrow \frac{d^2 T}{dy^2} = -\frac{\mu}{k} \frac{U^2}{h^2} = \alpha$$

$$\Rightarrow \quad T = \frac{\alpha}{2} y^2 + C_3 y + C_4$$

$$T(0) = T_1 \quad \Rightarrow \quad C_4 = T_1$$

$$T(h) = T_2 \quad \Rightarrow \quad T_2 = -\frac{\mu}{k} \frac{v^2}{h^2} \frac{h^2}{2} + C_3 h + T_1$$

$$\Rightarrow \quad C_3 = \frac{T_2 - T_1 + (\mu v^2 / 2k)}{h}$$

So,
$$T = -\left(\frac{\mu v^2}{2kh^2}\right) y^2 + \left(\frac{T_2 - T_1 + \mu v^2/2k}{h}\right) y + T_1$$

Now
$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0$$

$$\Rightarrow \quad \frac{T_2 - T_1 + \mu v^2/2k}{h} = 0$$

$$\Rightarrow \quad v = \left[\frac{2k}{\mu}(T_1 - T_2)\right]^{1/2}$$