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Asymptotic Stability and Disturbance Attenuation Properties for a Class of Networked Control Systems

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Abstract: In this paper, stability and disturbance attenuation issues for a class of Networked Control Systems (NCSs) under uncertain access delay and packet dropout effects are considered. Our aim is to find conditions on the delay and packet dropout rate, under which the system stability and $\mathcal{H}^\infty$ disturbance attenuation properties are preserved to a desired level. The basic idea in this paper is to formulate such Networked Control System as a discrete-time switched system. Then the NCSs' stability and performance problems can be reduced to corresponding problems for the switched systems, which have been studied for decades and for which a number of results are available in the literature. The techniques in this paper are based on recent progress in the discrete-time switched systems and piecewise Lyapunov functions.

Keywords: Networked Control Systems, Switched Systems, Piecewise Quadratic Lyapunov Function, Average Dwell Time

1 Introduction

By Networked Control Systems (NCSs), we mean feedback control systems where networks, typically digital band-limited serial communication channels, are used for the connections between spatially distributed system components like sensors and actuators to controllers, see Figure 1 for illustration. These channels may be shared by other feedback control loops. In traditional feedback control systems, these connections are established by point-to-point

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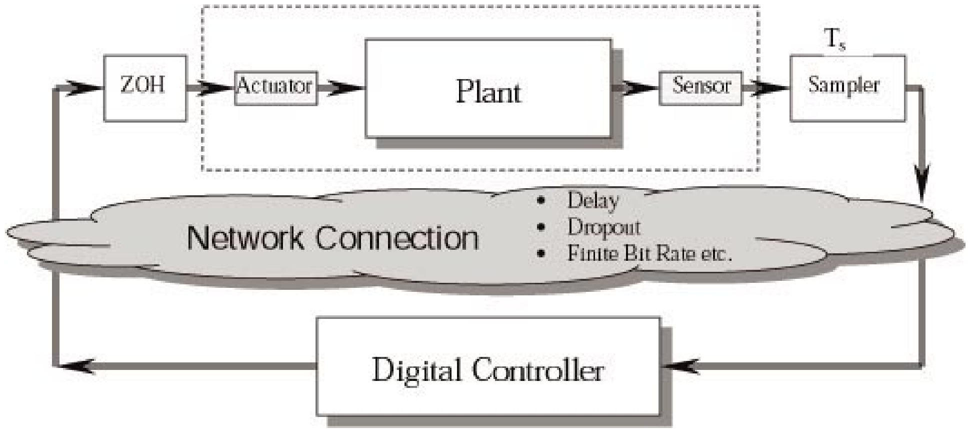

Figure 1: The block diagram for a typical Networked Control System.

cables. Compared with point-to-point cables, the introduction of serial communication networks has several advantages, such as high system testability and resource utilization, as well as low weight, space, power and wiring requirements [11, 25]. These advantages have made the networks connecting sensors/actuators to controllers increasingly popular in many applications, including traffic control, satellite clusters, mobile robotics, etc. However, the connection by digital serial communication channels also brings several new challenges for our control community. For example, the network link is band-limited, so one needs to quantize the signals and send digital bits. In addition, the network could be noisy and collisions between packets could occur, then one needs to concern the packet dropouts and delays etc. Recently, modeling, analysis and control of networked control systems with limited communication capability has emerged as a topic of significant interest to control community, see for example [22, 3, 6, 25, 11], and recent special issue [1].

Time delay typically has negative effects on the NCSs' stability and performance. There are several situations where time delay may arise. First, transmission delay is caused by the limited bit rate of the communication channels. Secondly, the channel in NCSs is usually shared by multiple sources of data, and the channel is usually multiplexed by a time-division method. Therefore, there are delays caused by a node waiting to send out a message through a busy channel, which is usually called accessing delay and serves as the main source of delays in NCSs. There are also some delays caused by processing and propagation, which are usually negligible for NCSs. Another interesting problem in NCSs is the packet dropout phenomenon. Because of the uncertainties and noise in the communication channel, there may exist unavoidable errors or losses in the transmitted packet even when an error control coding and/or Automatic Repeat reQuest (ARQ) mechanisms are employed. If this happens, the corrupted packet is dropped and the receiver (controller or actuator) uses the packet that it received most recently. In addition, packet dropout may occur when one packet, say sampled values from the sensor, reaches the destination later than its successors. In this situation, the old packet is dropped, and its successive packet is used instead. There is another important issue in NCSs, namely the quantization effect. With the finite bit-rate constraints, quantization has to be taken into consideration in NCSs. Therefore, quantization and limited bit rate issues have attracted many researchers' attention with the aim to identify the minimum bit rate required to stabilize a NCS, see for example [5, 3, 6, 21, 18]. In this paper, we will focus on packet exchange networks, in which the minimum unit of data transmission is the packet which typically is with the size of several hundred bits. Therefore, sending a single bit or several hundred of bits does not make significant difference in the network resource usage. Hence, we will omit the quantization effects here and focus our attention on the effects of network induced delay and packet dropout on NCSs' stability and performance.

The effects of network induced delay on the NCS's stability have been studied in the literature. In [2], the delay was assumed to be constant and then the NCS could be transformed into a time-invariant discrete-time system. Therefore, the NCS's stability could be checked by the Schurness of certain augmented state matrix. Since most network protocols introduce delays that can vary from packet to packet, the authors extended the results to non-constant delay case in [25]. They employed Lyapunov methods, in particular a common quadratic Lyapunov function, to study bounds on the maximum delay allowed by the NCSs. However, the choice of a common quadratic Lyapunov function could make the conclusion for maximum allowed delay conservative in some cases. The packet dropouts have also been studied, and there are two typical ways to model packet dropouts in the literature. The first approach assumes that the packet dropouts follow certain probability distributions, which is difficult to verify, and describes NCSs with packet dropouts via stochastic models, such as Markovian jump linear systems. The second approach is deterministic, and specifies the dropouts in the time average sense or in terms of bounds on maximum allowed consecutive dropouts. For example, [8] modeled a class of NCSs with package dropouts as asynchronous dynamical systems, and derived a sufficient condition on packet dropouts in the time-average sense for NCSs' stability based on common Lyapunov function approach. Notice that most of the results obtained so far are for the NCS's stability problem, and the delay and packet dropouts are usually dealt with separately.

In this paper, we will consider both network induced delay and packet dropouts in a unified switched system model. In addition, the disturbance attenuation issues for NCSs are investigated as well as stability problems. The strength of this approach comes from the solid theoretic results existing in the literature for stability, robust performance etc. for switched systems. By a switched system, we mean a hybrid dynamical system consisting of a finite number of subsystems described by differential or difference equations and a logical rule that orchestrates switching between these subsystems. Properties of this type of model have been studied for the past fifty years to consider engineering systems that contain relays and/or hysteresis. Recently, there has been increasing interest in the stability analysis and switching control design of switched systems (see, for example, the survey papers [13, 17, 4, 19], recent books [12, 20] and the references cited therein).

In this paper, we investigate the asymptotic stability and disturbance attenuation properties for a class of Networked Control Systems (NCSs) under uncertain access delay and packet dropout effects. Our aim is to find conditions concerning the delay and packet dropout rate, under which the system stability and $\mathcal{H}^\infty$ disturbance attenuation properties are preserved to a desired level. We first analyze the nature of the uncertain access delay and packet dropout effects on NCSs in Section 2. Then in Section 3, we model the NCS as a discrete-time switched system. Therefore the NCSs' asymptotic stability and robust performance problems can be

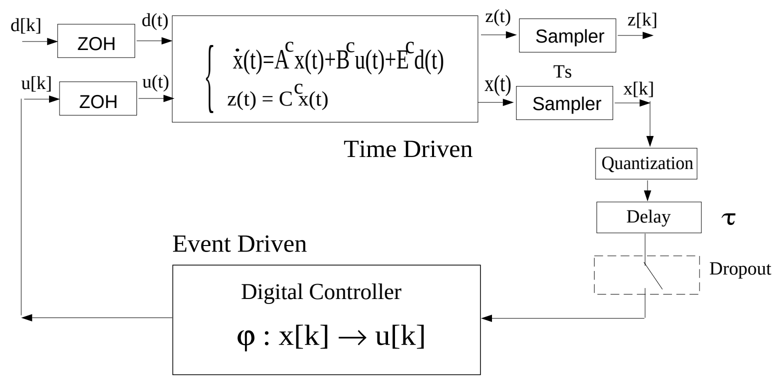

Figure 2: The Networked Control Systems' model.

boiled down to the stability analysis and disturbance attenuation problems of switched systems. In Section 4, the asymptotic stability for such NCSs with uncertain access delay and packet dropout effects is studied, and disturbance attenuation properties for such NCSs are studied in Section 5. The techniques employed in this paper are based on recent progress in the continuous-time and discrete-time switched systems [9, 24, 23], i.e., multiple Lyapunov functions and average dwell time methods in particular. Finally, concluding remarks are presented.

2 The Access Delay and Packet Dropout

For the network link layer, we assume that the delays caused by processing and propagation are ignored, and we only consider the access delay which serves as the main source of delays in NCSs. Dependent on the data traffic, the communication bus is either busy or idle (available). If the link is available, the communication between sender and receiver is assumed to be instantaneous. Errors may occur during the communication and destroy the packet, and this is considered as a packet dropout.

The model of the NCS used in this paper is shown in Figure 2. For simplicity, but without loss of generality, we may combine all the time delay and packet dropout effects into the sensor to controller path and assume that the controller and the actuator communicate ideally.

We assume that the plant can be modeled as a continuous-time linear time-invariant system described by

$$\begin{cases} \dot{x}(t) = A^c x(t) + B^c u(t) + E^c d(t) \\ z(t) = C^c x(t) \end{cases}, \qquad t \in \mathbb{R}^+ \tag{1}$$

where $\mathbb{R}^+$ stands for nonnegative real numbers, $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is control input, and $z(t) \in \mathbb{R}^p$ is the controlled output. The disturbance input $d(t)$ is contained in $\mathcal{D} \subset \mathbb{R}^r$. $A^c \in \mathbb{R}^{n\times n}$, $B^c \in \mathbb{R}^{n\times m}$ and $E^c \in \mathbb{R}^{n\times r}$ are constant matrices related to the system state, and $C^c \in \mathbb{R}^{p\times n}$ is the output matrix.

For the above NCS, it is assumed that the plant output node (sensor) is time driven. In other words, after each clock cycle (sampling time T_s), the output node attempts to send a

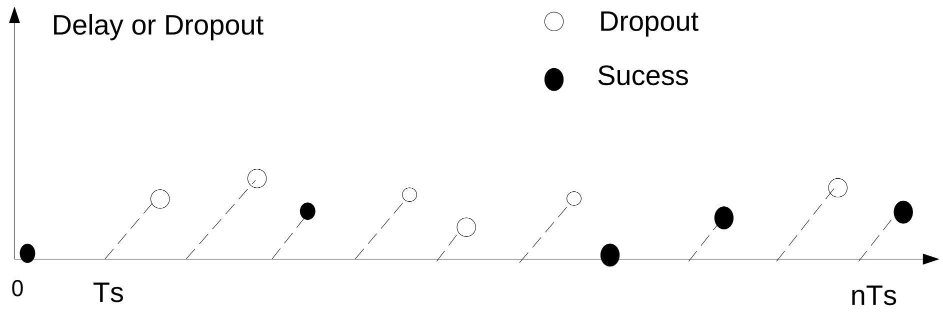

Figure 3: The illustration of uncertain time delay and packet dropout of Networked Control Systems.

packet containing the most recent state (output) samples. If the communication bus is idle, then the packet will be transmitted to the controller. Otherwise, if the bus is busy, then the output node will wait for some time, say $\varpi < T_s$, and try again. After several attempts or when newer sampled data become available, if the transmission still can not be completed, then the packet is discarded, which is also considered as a packet dropout. On the other hand, the controller and actuator are event driven and work in a simpler way. The controller, as a receiver, has a receiving buffer which contains the most recently received data packet from the sensors (the overflow of the buffer may be dealt with as packet dropouts). The controller reads the buffer periodically at a higher frequency than the sampling frequency, say every $\frac{T_s}{N}$ for some integer N large enough. Whenever there are new data in the buffer, the controller will calculate the new control signal and transmit it to the actuator. Upon the arrival of the new control signal, the actuator updates the output of the Zero-Order-Hold (ZOH) to the new value.

Based on the above assumptions, a typical time delay and packet dropout pattern is shown in Figure 3. In this figure, the small bullet, $\bullet$, stands for the packet being transmitted successfully from the sensor to the controller's receiving buffer, maybe with some delay, and being read by the controller, at some time $t = kT_s + h\frac{T_s}{N}$ (k and h are integers). The new control signal is sent to the actuator and the actuator holds this new value until the next update control signal comes. The symbol, $\circ$, denotes the packet being dropped, due to error, bus being busy, conflict or buffer overflow etc.

3 Switched System Models

In this section, we will consider the sampled-data model of the plant. Because we do not assume the synchronization between the sampler and the digital controller, the control signal is no longer of constant value within a sampling period. Therefore the control signal within a sampling period has to be divided into subintervals corresponding to the controller's reading buffer period, $T = \frac{T_s}{N}$. Within each subinterval, the control signal is constant under the assumptions of the previous section. Hence the continuous-time plant may be discretized into the following sampled-data systems:

$$x[k+1] = Ax[k] + [B\ A_T B\ \cdots A_T^{N-1} B] \begin{bmatrix} u^1[k] \\ u^2[k] \\ \vdots \\ u^N[k] \end{bmatrix} + Ed[k] \qquad (2)$$

where $A = e^{A^c T_s}$, $A_T = e^{A^c \frac{T_s}{N}}$, $B = \int_0^{\frac{T_s}{N}} e^{A^c \eta} B^c d\eta$ and $E = \int_0^{T_s} e^{A^c \eta} E^c d\eta$. Note that for linear time-invariant plant and constant-periodic sampling, the matrices A, A_T, B and E are constant. In addition, if the sampling period T_s is small enough and/or N is large enough then one could approximate A_T as an identity matrix and simplify the representation.

3.1 Modeling Uncertain Access Delay

During each sampling period, there are several different cases that may arise.

First, if there is no delay, namely $\tau = 0$, $u^1[k] = u^2[k] = \cdots = u^N[k] = u[k]$, then the state transition equation (2) for this case can be written as:

$$\begin{aligned} x[k+1] &= Ax[k] + [B\ A_T B\ \cdots A_T^{N-1} B] \begin{bmatrix} u[k] \\ u[k] \\ \vdots \\ u[k] \end{bmatrix} + Ed[k] \\ &= Ax[k] + \sum_{i=0}^{N-1} A_T^i B u[k] + Ed[k] \end{aligned}$$

Secondly, if the delay $\tau = h \times T$, where $T = \frac{T_s}{N}$, and $h = 1, 2, \cdots, d_{max}$[1], then $u^1[k] = u^2[k] = \cdots = u^h[k] = u[k-1]$, $u^{h+1}[k] = u^{h+2}[k] = \cdots = u^N[k] = u[k]$, and (2) can be written as:

$$\begin{aligned} x[k+1] &= Ax[k] + [B\ A_T B\ \cdots A_T^{N-1} B] \begin{bmatrix} u[k-1] \\ \vdots \\ u[k-1] \\ u[k] \\ \vdots \\ u[k] \end{bmatrix} + Ed[k] \\ &= Ax[k] + \sum_{i=0}^{h-1} A_T^i B u[k-1] + \sum_{i=h}^{N-1} A_T^i B u[k] + Ed[k] \end{aligned}$$

Note that $h = 0$ implies $\tau = 0$, which corresponds to the previous "no delay" case.

[1]The value of d_{max} is determined as the least integer greater than the positive scalar $\frac{\tau_{max}}{T}$, where τ_{max} stands for the maximum access delay.

Let us assume that the controller uses just the time-invariant linear feedback control law, $u[k] = Kx[k]$, which may be obtained as the solution of a LQR problem without considering the network induced effects. Then, we may plug in the $u[k] = Kx[k]$ and get

$$\begin{aligned} x[k+1] &= Ax[k] + \sum_{i=0}^{h-1} A_T^i BKx[k-1] + \sum_{i=h}^{N-1} A_T^i BKx[k] + Ed[k] \\ &= (A + \sum_{i=h}^{N-1} A_T^i BK)x[k] + \sum_{i=0}^{h-1} A_T^i BKx[k-1] + Ed[k] \end{aligned}$$

If we let $\hat{x}[k] = \begin{bmatrix} x[k-1] \\ x[k] \end{bmatrix}$, then the above equations can be written as:

$$\begin{aligned} \hat{x}[k+1] &= \begin{bmatrix} x[k] \\ x[k+1] \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ \sum_{i=0}^{h-1} A_T^i BK & A + \sum_{i=h}^{N-1} A_T^i BK \end{bmatrix} \begin{bmatrix} x[k-1] \\ x[k] \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} d[k] \end{aligned}$$

where $h = 0, 1, 2, \cdots, d_{max}$. And the controlled output $z[k]$ is given by

$$z[k] = \begin{bmatrix} 0 & C \end{bmatrix} \hat{x}[k]$$

where $C = C^c$.

Finally, if a packet-dropout happens, which may be due to a corrupted packet or sending it out with delay greater than τ_{max}, then the actuator will implement the previous control signal, i.e. $u^1[k] = u^2[k] = \cdots = u^N[k] = u[k-1]$. Therefore, the state transition equation (2) for this case can be written as:

$$\begin{aligned} x[k+1] &= Ax[k] + [B \ A_T B \ \cdots A_T^{N-1} B] \begin{bmatrix} u[k-1] \\ u[k-1] \\ \vdots \\ u[k-1] \end{bmatrix} + Ed[k] \\ &= Ax[k] + \sum_{i=0}^{N-1} A_T^i Bu[k-1] + Ed[k] \\ &= Ax[k] + \sum_{i=0}^{N-1} A_T^i BKx[k-1] + Ed[k] \end{aligned}$$

Using the same variable transformation as in the above case, we get

$$\begin{aligned} \hat{x}[k+1] &= \begin{bmatrix} x[k] \\ x[k+1] \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ \sum_{i=0}^{N-1} A_T^i BK & A \end{bmatrix} \begin{bmatrix} x[k-1] \\ x[k] \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} d[k] \end{aligned}$$

The controlled output $z[k]$ is given by

$$z[k] = \begin{bmatrix} 0 & C \end{bmatrix} \hat{x}[k]$$

where $C = C^c$.

3.2 Switched System Model

In the following, we will model the uncertain multiple successive packet dropouts and formulate the above NCSs as a class of discrete-time switched systems.

Motivated by the above analysis of NCSs, we introduce a family of discrete-time linear systems described by the following difference equations.

$$x[k+1] = A_q x[k] + E_q d[k], \quad k \in \mathbb{Z}^+ \tag{3}$$

where $x[k] \in \mathbb{R}^n$ is the state variable, and the disturbance input $d[k]$ is contained in $\mathcal{D} \subset \mathbb{R}^r$. $A_q \in \mathbb{R}^{n\times n}$ and $E_q \in \mathbb{R}^{n\times r}$ are constant matrices indexed by $q \in Q$, where the finite set $Q = \{q_1, q_2, \cdots, q_n\}$ is called the set of *modes*.

Combine the family of discrete-time uncertain linear systems (3) with a class of piecewise constant functions of time $\sigma : \mathbb{Z}^+ \to Q$. Then we can define the following time-varying system as a discrete-time switched linear system

$$x[k+1] = A_{\sigma[k]} x[k] + E_{\sigma[k]} d[k], \quad k \in \mathbb{Z}^+ \tag{4}$$

The sequence $\sigma[k]$ is usually called a *switching signal*.

Associated with the switched system (4), a controlled output $z[k]$ is considered.

$$z[k] = C_{\sigma[k]} x[k]$$

where $C_{\sigma[k]} \in \mathbb{R}^{p\times n}$ and $z[k] \in \mathbb{R}^p$.

For the NCS we considered in this paper, we may formulate it as a switched system with $d_{max} + 2$ different modes, which can be expressed as follows.

$$\left\{ \begin{array}{rcl} \hat{x}[k+1] & = & A_h \hat{x}[k] + E_h d[k] \\ z[k] & = & C_h \hat{x}[k] \end{array} \right. \tag{5}$$

where $A_h = \begin{bmatrix} 0 & I \\ \sum_{i=0}^{h-1} A_T^i BK & A + \sum_{i=h}^{N-1} A_T^i BK \end{bmatrix}$, $E_h = \begin{bmatrix} 0 \\ E \end{bmatrix}$ and $C_h = \begin{bmatrix} 0 & C \end{bmatrix}$ for $h = 0, 1, 2, \cdots d_{max}, N$. And the set of modes Q is given by $Q = \{0, 1, 2, \cdots d_{max}, N\}$. Note that $h = 0$ implies $\tau = 0$, which corresponds to the "no delay" case, while $h = N$ corresponds to the "packet dropout" case.

In our previous work [14], it was assumed that there is an upper bound on the maximum number of successive packet dropouts, e.g., at most four packets dropped in a row. Under this assumption, NCSs under bounded uncertain access delay and packet dropout were modeled as switched linear systems with arbitrary switching. The asymptotic stability and persistent disturbance attenuation properties of the NCSs were studied in the switched system framework, and a necessary and sufficient condition was given for the NCSs' asymptotic stability in [14]. However, assuming an absolute upper bound on the maximum number of packets dropped in a row could be conservative in certain cases. In this paper, we provided an alternative way to model NCSs as switched systems. Instead of incorporating all possible delay-dropout patterns so as to relax the switching signal to be arbitrary, we specified a subclass of the switching signal by restricting the occurring frequency and the number of dropped and seriously-delayed packets in the time average sense. In particular, it was shown that there exist bounds on the delay and packet dropout rate and percentage, below which the NCSs' stability and $\mathcal{L}_2$ disturbance attenuation properties may be preserved to a desired level. These bounds were identified based on multiple Lyapunov functions incorporated with average dwell time scheme.

4 Stability Analysis

Considering the switched system model for NCSs (5), it is reasonable to assume that, for the cases of no delay ($h = 0$) or small delay ($h \leq h_0$), the corresponding state matrix A_h's are Schur stable, while, for the cases of large delay ($h > h_0$) or packet dropout ($h = N$), the A_h's are not Schur stable. Therefore, in this paper it is assumed that the first r, corresponding to h_0, of all the $d_{max}+2$ matrices in $\{A_h\}$ are Schur stable, while the rest matrices are not Schur stable, where $r \leq d_{max} + 2$ and $h \in Q = \{0, 1, 2, \cdots, d_{max}, N\}$. In the sequel, for simplicity of notation, we will index the switched NCS model with i, for $i \in Q = \{0, 1, 2, \cdots, d_{max}, N\}$, and replace $\hat{x}$ as x. In this section, we set $d[k] = 0$ in (5) for the purpose to study its stability.

It is known that for Schur stable systems $x[k+1] = A_i x[k]$, there always exist positive scalars $\lambda_1 < 1$ and h_i's, $i \leq r$ such that $\|A_i^k\| \leq h_i \lambda_1^k$ for any $k \geq 1$[2]. Note that for any Schur unstable system $x[k+1] = A_i x[k]$ ($i > r$), there always exist a constant $0 < \sigma < 1$ making the system $x[k+1] = \sigma A_i x[k]$ Schur stable. Hence we may assume that there exist positive scalars $\lambda_2 \geq 1$ and h_i's, $i > r$ such that $\|A_i^k\| \leq h_i \lambda_2^k$ for any $k \geq 1$. Therefore, we get

$$\|A_i^k\| \leq \begin{cases} h_i \lambda_1^k & i \leq r \\ h_i \lambda_2^k & i > r \end{cases} \tag{6}$$

Following [24], we introduce the notations as below. Denote $h = \max_i\{h_i\}$. For any switching signal $\sigma(k)$ and any $k_2 > k_1 > 0$, let $N_\sigma(k_1, k_2)$ denote the number of switchings of $\sigma(k)$ on the interval $[k_1, k_2)$. Let $K_i(k_1, k_2)$ denote the total period that the i-th subsystem is activated during $[k_1, k_2)$. Define $K^-(k_1, k_2) = \sum_{i \leq r,\ i \in Q} K_i(k_1, k_2)$, which stands for the total activation period of the Schur stable subsystems. On the other hand, $K^+(k_1, k_2) = \sum_{i > r,\ i \in Q} K_i(k_1, k_2)$ denotes the total activation period of the Schur unstable subsystems. We have $K^-(k_1, k_2) + K^+(k_1, k_2) = k_2 - k_1$.

For given $N_0 \geq 0$, τ_a, let $\mathcal{S}_a(\tau_a)$ denote the set of all switching signals satisfying

$$N_\sigma(0, k) \leq N_0 + \frac{k}{\tau_a} \tag{7}$$

where the constant τ_a is called the *average dwell time* and N_0 the *chatter bound*. The idea is that there may exist consecutive switching separated by less than τ_a, but the average time interval between consecutive switchings is not less than τ_a. Note that the concept of average dwell time between subsystems was originally proposed for continuous-time switched systems in [10]. With these assumptions and notations, we may apply the techniques and

[2]The vector/matrix norm considered here is the l^2 norm and its induced matrix norm.

results developed in [24] to the NCSs and get the following theorem for globally exponential stability. The proof of the theorem is not difficult by using the technique of Theorem 3 in [24], and thus is omitted here.

Theorem 1 *For any given $\lambda \in (\lambda_1, 1)$, the NCS (5) is globally exponentially stable with stability degree λ if there exists a finite constant τ_a^* and $\lambda^* \in (\lambda_1, \lambda)$ such that the $K^+(0,k)$ and $N_\sigma(0,k)$ satisfy the following two conditions*

1. $\inf_{k>0} \frac{K^-(0,k)}{K^+(0,k)} \geq \frac{\ln\lambda_2 - \ln\lambda^*}{\ln\lambda^* - \ln\lambda_1}$ *holds for some scalar $\lambda^* \in (\lambda_1, \lambda)$;*

2. *The average dwell time is not smaller than τ_a^*, i.e. $N_\sigma(0,k) \leq N_0 + \frac{k}{\tau_a^*}$, where $\tau_a^* = \frac{\ln h}{\ln\lambda - \ln\lambda^*}$, and N_0 may be specified arbitrarily.*

Remark 1 The first condition implies that if we expect the entire system to have decay rate λ, we should restrict the total number of lost packets and large delay packets in the sense that on average $K^+(0,k)$ has an upper-bound, $K^+(0,k) \leq \frac{\ln\lambda^* - \ln\lambda_1}{\ln\lambda_2 - \ln\lambda_1} k$. That means the percentage of dropped and seriously delayed packets should be below certain bound, given by $\frac{\ln\lambda^* - \ln\lambda_1}{\ln\lambda_2 - \ln\lambda_1}$.

Remark 2 The main point of the second condition can be described as follows. Although the first condition may be satisfied, which means that on average the packet lost is limited and the total number of large delayed packet is bounded, in the worst case the packet dropout and large access delay happen in a burst fashion. For such worst case, the NCSs may fail to achieve the decay rate. The second condition restricts the frequency of the packet dropout and large delayed packet, and to make sure the above worst case can not happen.

Remark 3 The above theorem says that the NCSs' stability, with most of the packets arriving in a timely fashion, does not degenerate seriously, which is reasonable.

5 Disturbance Attenuation Properties

In this section, we will study the disturbance attenuation property for the NCSs (5). Note that the $\mathcal{L}_2$ gain property of discrete-time switched systems was studied in [24] under the assumption that all subsystems were Schur stable. In this section, we will extend the $\mathcal{L}_2$ gain property of discrete-time switched system to the case that not all subsystems are Schur stable. The techniques used in this section are similar to those in [23] for continuous-time switched systems.

Following the assumptions in [24], the initial state is assumed to be the origin, $x[0] = 0$. And we assume that the Schur stable subsystems achieve an $\mathcal{L}_2$ gain smaller than γ_0. It is known that there exist a positive scalar $\lambda_- < 1$ and a set of positive definite matrices P_i, for $i \leq r$ and $i \in Q$, such that

$$A_i^T P_i A_i - \lambda_-^2 P_i + C_i^T C_i + A_i^T P_i E_i (\gamma_0^2 I - E_i^T P_i E_i)^{-1} E_i^T P_i A_i < 0$$

holds [7]. Observing that for Schur unstable subsystems, there always exist a constant $0 < \sigma < 1$, such that the subsystems $(\sigma A_i, E_i, \sigma C_i)$ can achieve the $\mathcal{L}_2$ gain level γ_0. Therefore, we assume that for Schur unstable subsystems there exist a positive scalar $\lambda_+ \geq 1$ and a set of positive definite matrices P_i, for $i > r$ and $i \in Q$, such that

$$A_i^T P_i A_i - \lambda_+^2 P_i + C_i^T C_i + A_i^T P_i E_i (\gamma_0^2 I - E_i^T P_i E_i)^{-1} E_i^T P_i A_i < 0$$

Using the solution P_i's, we define the following *piecewise Lyapunov function* candidate

$$V(k) = V_{\sigma[k]}(x) = x^T[k] P_{\sigma[k]} x[k] \tag{8}$$

for the switched system, where $P_{\sigma[k]}$ is switched among the solution P_i's in accordance with the piecewise constant switching signal $\sigma[k]$. It can be shown as in [24] that there always exist constant scalars $\alpha_1, \alpha_2 > 0$, for example, $\alpha_1 = \inf_{i \in Q} \lambda_m(P_i)$, $\alpha_2 = \sup_{i \in Q} \lambda_M(P_i)$, such that

$$\alpha_1 \|x\|^2 \leq V_i(x) \leq \alpha_2 \|x\|^2, \ \forall x \in \mathbb{R}^n, \ \forall i \in Q \tag{9}$$

Here $\lambda_M(P_i)$ and $\lambda_m(P_i)$ denotes the largest and smallest eigenvalue of P_i respectively. There exist a constant scalar $\mu \geq 1$ such that

$$V_i(x) \leq \mu V_j(x), \ \forall x \in \mathbb{R}^n, \ \forall i, j \in Q \tag{10}$$

A conservative choice is $\mu = \sup_{k,l \in Q} \frac{\lambda_M(P_k)}{\lambda_m(P_l)}$.

Following the steps in [24], for each $V_i(x) = x^T[k] P_i x[k]$ along the solutions of the corresponding subsystem, we may obtain that

$$\begin{array}{rl} & V_i(x[k+1]) - V_i(x[k]) \\ \leq & \left\{ \begin{array}{l} -(1-\lambda_-^2) V_i(x[k]) - z^T[k] z[k] + \gamma_0 d^T[k] d[k] \\ -(1-\lambda_+^2) V_i(x[k]) - z^T[k] z[k] + \gamma_0 d^T[k] d[k] \end{array} \right. \end{array}$$

For a piecewise constant switching signal $\sigma[k]$ and any given integer $k > 0$, we let $k_1 < \cdots < k_i$ $(i \geq 1)$ denote the switching points of $\sigma[k]$ over the interval $[0, k)$. Then, using the above difference inequalities, we obtain

$$V(k) \leq \left\{ \begin{array}{l} \lambda_-^{2(k-k_i)} V(k_i) - \sum_{j=k_i}^{k-1} \lambda_-^{2(k-1-j)} \Gamma(j) \\ \lambda_+^{2(k-k_i)} V(k_i) - \sum_{j=k_i}^{k-1} \lambda_+^{2(k-1-j)} \Gamma(j) \end{array} \right.$$

where $\Gamma(j) = z^T[j] z[j] - \gamma_0^2 d^T[j] d[j]$. Since $V(k_i) \leq \mu V(k_i^-)$ holds on every switching point k_i, we obtain by induction that

$$\begin{array}{rcl} V(k) & \leq & \mu^{N_\sigma(0,k)} \lambda_-^{2K^-(0,k)} \lambda_+^{2K^+(0,k)} V(0) \\ & & \displaystyle - \sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)} \lambda_-^{2K^-(j,k-1)} \lambda_+^{2K^+(j,k-1)} \Gamma(j) \\ & = & \displaystyle - \sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)} \lambda_-^{2K^-(j,k-1)} \lambda_+^{2K^+(j,k-1)} \Gamma(j) \end{array}$$

The last equality is because of the zero initial state assumption $x[0]=0$.

We assume that on any interval $[k_1, k_2)$ the total activation periods of the unstable subsystems satisfies $K^+(k_1, k_2) \leq \frac{\ln\lambda^* - \ln\lambda_-}{\ln\lambda_+ - \ln\lambda_-}(k_2 - k_1)$, or equivalently

$$\frac{K^-(k_1, k_2)}{K^+(k_1, k_2)} \geq \frac{\ln\lambda_+ - \ln\lambda^*}{\ln\lambda^* - \ln\lambda_-} \tag{11}$$

holds for some scalar $\lambda^* \in (\lambda_-, 1)$ and $\forall k_2 > k_1 \geq 0$. Then we get

$$\begin{aligned} & \lambda_-^{2K^-(j,k-1)}\lambda_+^{2K^+(j,k-1)} \\ \leq \;& (\lambda^*)^{2K^-(j,k-1)+2K^+(j,k-1)} = (\lambda^*)^{2(k-1-j)} \end{aligned}$$

Therefore, we get

$$V(k) \leq -\sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)}(\lambda^*)^{2(k-1-j)}\Gamma(j) \tag{12}$$

When $\mu = 1$, we get from $V(k) \geq 0$ and (12) that

$$\sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)}(\lambda^*)^{2(k-1-j)}\Gamma(j) \leq 0 \tag{13}$$

We sum (13) from $k=1$ to $k=+\infty$ to obtain

$$\begin{aligned} & \sum_{k=1}^{+\infty}(\sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)}(\lambda^*)^{2(k-1-j)}\Gamma(j)) \\ = \;& \sum_{j=1}^{+\infty}\Gamma(j)(\sum_{k=j+1}^{+\infty} \mu^{N_\sigma(j,k-1)}(\lambda^*)^{2(k-1-j)}) \\ = \;& (1-(\lambda^*)^2)^{-1}\sum_{j=1}^{+\infty}\Gamma(j) \leq 0 \end{aligned}$$

which means

$$\sum_{j=0}^{+\infty} z^T[j]z[j] \leq \gamma_0^2 \sum_{j=0}^{+\infty} d^T[j]d[j] \tag{14}$$

Therefore, $\mathcal{L}_2$ gain γ_0 is achieved for the switched system, namely the NCS (5).

For the case $\mu > 1$, we multiply both sides of (12) by $\mu^{-N_\sigma(0,k-1)}$ to get

$$\sum_{j=0}^{k-1} \mu^{-N_\sigma(0,j)}(\lambda^*)^{2(k-1-j)}z^T[j]z[j] \leq \gamma_0^2 \sum_{j=0}^{k-1} \mu^{-N_\sigma(0,j)}(\lambda^*)^{2(k-1-j)}d^T[j]d[j] \tag{15}$$

Now, we choose a positive scalar λ larger than 1 to consider the following average dwell time condition: for any positive integer $j > 0$,

$$N_\sigma(0, j) \leq \frac{j}{\tau_a^*}, \quad \tau_a^* = \frac{\ln\mu}{2\ln\lambda} \tag{16}$$

Therefore $\mu^{-N_\sigma(0,j)} > \lambda^{-2j}$ holds for any $j > 0$, where $\lambda = \mu^{(2\tau_a^*)^{-1}}$. Then, from (15) we obtain

$$\sum_{j=0}^{k-1} \lambda^{-2j} (\lambda^*)^{2(k-1-j)} z^T[j] z[j] \leq \gamma_0^2 \sum_{j=0}^{k-1} (\lambda^*)^{2(k-1-j)} d^T[j] d[j]$$

Similarly, we sum both sides of the above inequality from $k = 1$ to $k = +\infty$ to get

$$(1 - (\lambda^*)^2)^{-1} \sum_{j=1}^{+\infty} \lambda^{-2j} z^T[j] z[j] \leq (1 - (\lambda^*)^2)^{-1} \gamma_0^2 \sum_{j=1}^{+\infty} d^T[j] d[j]$$

and thus

$$\sum_{j=1}^{+\infty} \lambda^{-2j} z^T[j] z[j] \leq \gamma_0^2 \sum_{j=1}^{+\infty} d^T[j] d[j] \tag{17}$$

holds for any $d[k] \in \mathcal{L}_2[0, +\infty)$. Following the notation in [24], we say that a *weighted* $\mathcal{L}_2$ *gain* γ_0 is achieved. In summary, we prove the following theorem.

Theorem 2 *The NCS (5) achieves a weighted $\mathcal{L}_2$ gain γ_0 if the $K^+(k_1, k_2)$ satisfies (11) and $N_\sigma(0, k)$ satisfy the condition of (16).*

Remark 4 Similarly, the condition (11) restricts the number of the packet dropout and large delayed packet, while the condition (16) restricts the happening frequency of them. Both of the conditions are given in the sense of average over time.

6 Conclusions

In this paper, we modeled a class of NCSs under uncertain access delay and packet dropout as discrete-time switched linear systems. The stability and disturbance attenuation issues for such NCSs were studied in the framework of switched systems. It was shown that the asymptotic stability and disturbance attenuation level might be preserved for the NCSs under certain bounds on the amount and rate of the dropped and large delayed packets. Although we only consider state feedback control law here, the techniques and results developed here can be easily extended to the case of static output feedback control law. It should be pointed out that the conditions concerning the delay and packet dropout rate for the preservation of the NCSs' stability and $\mathcal{H}^\infty$ disturbance attenuation properties were based on Lyapunov theory. Therefore, the conditions are sufficient only and maybe conservative for some cases.

We believe that switched system approaches to NCSs are promising, as many research topics like networked continuous-controller design, controller and scheduling policy co-design could be pursued in the switched system framework. For example, in [16], a stability and $\mathcal{L}_2$ performance preserving network bandwidth management policy was proposed based on switched systems approaches. The potential of dealing with NCSs as switched systems comes from the existence of solid theoretic results in the field of switched systems, jump linear systems etc. Interested readers may refer to [19, 15] for surveys on the most recent progress on switched linear systems,

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