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Robust Controlled Invariant Sets for a class of Uncertain Hybrid Systems [1]

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1 Introduction

Invariant set theory has been widely studied in the literature, see for example [1, 2] and reference therein. [1] gives a compressive review of the invariant set theory. [2] brings together some of the main ideas in set invariance theory and places them in a general, nonlinear setting. A similar concept, maximal safety set, has been studied in the literature of hybrid systems. The authors of [7] consider a class of discrete time hybrid systems with piecewise linear time-invariant flow function and polyhedral constraints. The invariant sets for piecewise affine systems have also been studied in [3] based on convex optimization techniques and linear matrix inequalities.

However, if the effect of the uncertainty in the model is not taken into account, then it is possible that a controller could drive the system into an unsafe region. A small disturbance or fault could then cause the system to fail. This paper concentrates on the robust controlled invariant sets for a class of uncertain hybrid systems. Our goal is to determine whether a given region in the state space is robust controlled invariant and to compute the maximal robust controlled invariant subset under structured dynamic uncertainty and disturbances.

2 Model

The discussion in this paper assumes the following uncertain, discrete-time hybrid dynamical systems:

Definition 2.1 *Consider the discrete-time* Uncertain Piecewise Linear Hybrid Systems *defined by*

$$x(t+1) = \tilde{A}_{q(t)}x(t) + \tilde{B}_{q(t)}u(t) + E_{q(t)}d(t) \quad (2.1)$$
$$q(t+1) = \delta(q(t), \pi(x(t)), \sigma_c(t), \sigma_u(t)) \quad (2.2)$$

where $q \in Q = \{q_1, q_2, \cdots, q_s\}$ and Q is the collection of discrete states (modes); $x \in X \subset \mathbb{R}^n$ and X stands for the continuous state space, the continuous control $u \in \mathcal{U} \subset \mathbb{R}^m$, and the continuous disturbance $d \in \mathcal{D} \subset \mathbb{R}^p$, where $\mathcal{U}$, $\mathcal{D}$ are bounded convex polyhedral sets; and

- *$\tilde{A}_q \in \mathbb{R}^{n\times n}$, $\tilde{B}_q \in \mathbb{R}^{n\times m}$, and $E_q \in \mathbb{R}^{n\times p}$ are the system matrices for the discrete state q. Assume polytopic uncertainty, that is $[\tilde{A}_q, \tilde{B}_q] = \sum_{i=1}^{N_q} \lambda_i [A_q^i, B_q^i]$, where $\lambda_i \geq 0, \sum_{i=1}^{N_q} \lambda_i = 1$;*
- *$\pi : X \to X/E_\pi$ partitions the continuous state space $\mathbb{R}^n$ into polyhedral equivalence classes.*
- *$q(t+1) \in act(\pi(x(t)))$, where $act : X/E_\pi \to 2^Q$ defines the active mode set,*
- *$\delta : Q \times X/E_\pi \times \Sigma_c \times \Sigma_u \to Q$ is the discrete state transition function. Here $\sigma_c \in \Sigma_c$ denotes a controllable event and Σ_u the collection of uncontrollable events.*
- *The* guard *$G(q, q')$ of the transition (q, q') is defined as the set of all states (q, x) such that $q' \in act(\pi(x(t)))$ and there exist controllable event $\sigma_c \in \Sigma_c$ such that $q' = \delta(q, \pi(x), \sigma_c, \sigma_u)$ for every uncontrollable event $\sigma_u \in \Sigma_u$.*

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Assume that exact state measurement (q, x) is available. An *admissible control input* (or *law*) is one which satisfies the input constraints $(\Sigma_c, \mathcal{U})$. The elements of an allowable disturbance sequence are contained in $(\Sigma_u, \mathcal{D})$.

3 Geometric condition for invariance

Given a set $\Omega \subset Q\times X$ and an initial state $(q_0, x_0) \in \Omega$, it is of interest to determine whether there exist admissible control laws such that the evolution of the system will remain inside the set for all time, despite the presence of structured dynamic uncertainty and disturbances.

Definition 3.1 *The set $\Omega \subset Q \times X$ is* robust controlled invariant *for the uncertain hybrid systems of Definition 2.1 if and only if $\forall(q_0, x_0) \in \Omega$, $\forall(\sigma_u, d(t)) \in \Sigma_u \times \mathcal{D}$ and $\forall[\tilde{A}_{q(t)}, \tilde{B}_{q(t)}] \in Conv[A^i_{q(t)}, B^i_{q(t)}]$ there always exist admissible control input $(\sigma_c, u(t)) \in \Sigma_c \times \mathcal{U}$, such that the system evolution satisfies $(q(t), x(t)) \in \Omega$, $\forall t \geq 0$.*

A natural question is how to check whether for a given set $\Omega \subset Q \times X$ is robust controlled invariant or not. To answer this question, we first study the robust (one-step) predecessor operator, $pre(\cdot)$, for the uncertain hybrid systems. It is defined as the set of states in $Q\times X$ for which an admissible control input exists which will guarantee that the system will be driven to Ω in one step, for all allowable disturbances and dynamic uncertainties. The following is an important, well-known geometric condition[1] for a set to be control invariant.

Theorem 3.1 *The set $\Omega \subseteq \mathbb{R}^n$ is a robust controlled invariant set* if and only if $\Omega \subseteq pre(\Omega)$.

It follows immediately that the set Ω is robust control invariant if and only if $pre(\Omega) \cap \Omega = \Omega$, since $\Omega \subseteq pre(\Omega) \Leftrightarrow pre(\Omega) \cap \Omega = \Omega$. Testing for invariance need to: compute $pre(\Omega)$, which can be efficiently done by the predecessor operator algorithm described in [5, 6]; test whether $\Omega \subseteq pre(\Omega)$, this can be done by a feasibility of a linear programming problem[4]. So this condition can be efficiently tested by solving a finite number of linear programming problems that depends on the number of regions and discrete states of the system.

4 Maximal Robust Controlled Invariant Set

We have answered how to check the robust controlled invariance for a given set $\Omega \subset Q \times X$. And in general, a given set Ω is not robust controlled invariant. However, some subsets of Ω are likely to be robust controlled invariant. In addition, it follows immediately from the definition that the union of two robust controlled invariant sets is robust controlled invariant. So we have the following definition.

Definition 4.1 *The set $\tilde{\mathcal{C}}_\infty(\Omega)$ is the maximal robust positively invariant set contained in $\Omega \subset Q \times X$ for the uncertain hybrid systems of Definition 2.1 if and only if $\tilde{\mathcal{C}}_\infty(\Omega)$ is robust controlled invariant and contains all the robust controlled invariant sets contained in Ω.*

It can be shown that the maximal robust positively invariant set is unique. Then the next question is how to find the maximal robust controlled invariant set $\tilde{\mathcal{C}}_\infty(\Omega)$. We have the following algorithm for computing the maximal robust control invariant set.

INPUT: $\Omega = (q, P)$, $\tilde{\mathcal{C}}_0(\Omega) = X$, $\tilde{\mathcal{C}}_1(\Omega) = \Omega$, $i = 1$;
while $\tilde{\mathcal{C}}_i(\Omega) \neq \tilde{\mathcal{C}}_{i-1}(\Omega)$
  $\tilde{\mathcal{C}}_{i+1}(\Omega) = pre(\tilde{\mathcal{C}}_i(\Omega)) \cap \Omega$
  if $\tilde{\mathcal{C}}_{i+1}(\Omega) = \emptyset$
    Terminate and **OUTPUT**: $\tilde{\mathcal{C}}_\infty(\Omega) = \emptyset$.
  end if
  $i = i + 1$
end while
OUTPUT: $\tilde{\mathcal{C}}_\infty(\Omega) = \tilde{\mathcal{C}}_i(\Omega)$

Remark: The i-step robust admissible set $\tilde{\mathcal{C}}_i(\Omega)$ have properties: $\tilde{\mathcal{C}}_{i+1}(\Omega) \subseteq \tilde{\mathcal{C}}_i(\Omega)$, $\tilde{\mathcal{C}}_i(\Omega) = \bigcap_{k=1}^{i} \tilde{\mathcal{C}}_k(\Omega)$.

5 Maximum Permissive Control Law

Another interesting problem is to construct a (memoryless) *control law*, $c : Q \times X \to 2^{\Sigma_c \times \mathcal{U}}$, that robustly drives the system to guarantee that the states remain within some region (assume proper initial conditions) despite the uncertainties and disturbances, while satisfying some certain input/output constraints. If such control law exists and is *non-blocking*, i.e. $c(q, x) \neq \emptyset$ for all $(q, x) \in Q \times X$, then we say the control law c, solves the robust controlled invariance problem. Many control laws may be able to solve a particular problem. It is often preferred to find a control law that imposes fewer restrictions on the inputs allowed. The *least restrictive* (or *maximum permissive*) [7] control law is the maximal element among all the control laws that solve a given robust controlled invariance problem in the partial order defined by $\preceq$, where $c_1 \preceq c_2$ if $\forall (q, x) \in Q \times X$, $c_1(q, x) \subseteq c_2(q, x)$.

For a given region $\Omega \subset Q \times X$, if there exists an unique nonempty maximal robust controlled invariant subset $\tilde{\mathcal{C}}_\infty(\Omega)$, then there exists a unique non-blocking least restrictive control law that solves the robust controlled invariance problem for $\tilde{\mathcal{C}}_\infty(\Omega)$ with proper initial conditions. The least restrictive controller can be given as follows:

$$c(q,x) = \begin{cases} \{(\sigma_c, u) | \forall \sigma_u \in \Sigma_u, d \in \mathcal{D}, \\ \forall [\tilde{A}_q, \tilde{B}_q] \in Conv[A_q^i, B_q^i], & (q,x) \in \tilde{\mathcal{C}}_\infty(\Omega) \\ \tilde{A}_q x + \tilde{B}_q u + E_q d \in \tilde{\mathcal{C}}_\infty(\Omega)\}, \\ \\ \Sigma_c \times \mathcal{U}, & (q,x) \notin \tilde{\mathcal{C}}_\infty(\Omega) \end{cases}$$

6 Conclusion

In this paper, we put our group's recent progress in the analysis and synthesis of uncertain piecewise linear hybrid systems into the framework of invariant set theory. We develop an implementable procedure to check the robust controlled invariance and compute the maximal robust controlled invariant subset under structured dynamic uncertainty and disturbances. A formulation of least restrictive control law that solves a given robust controlled invariance problem is presented. In general, the maximal robust control invariant set can not be determined in finite number of steps. So the above algorithm and the lest restrictive control law synthesis is *semi-decidable*[7], because the termination of the algorithm is not guaranteed. In [4] two termination conditions were proposed to formulate a constructive algorithm.

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