Practical Stabilization of Integrator Switched Systems[1]

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Abstract

In this paper, practical stabilization problems for integrator switched systems are studied. In such class of switched systems, no subsystem has an equilibrium. However, the system can still exhibit interesting behaviors around a given point under appropriate switching laws. Such behaviors are similar to those of a conventional stable system near an equilibrium. We introduce some practical stability notions to define such behaviors. In particular, a necessary and sufficient condition for practical stabilizability of such systems is given. Moreover, for practically stabilizable systems, we develop a minimum dwell time switching law which can easily be implemented. Finally, as an application, we apply the switching law to a batch process example.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law orchestrating the active subsystem at each time instant. Many results on stability analysis and stabilization of switched systems have been reported in the literature (see [1, 6] and the references therein). Most of the available results are on switched systems whose subsystems share a common equilibrium. Methods based on single or multiple Lyapunov functions have been reported for the stability analysis and design of such systems. Methods based on geometric properties of the subsystem vector fields have also been reported [8].

In our recent research, we observed that the assumption that all subsystems share a common equilibrium may not hold for all switched systems and may limit the applicability of stability results. When subsystems have different equilibria or no equilibrium, a system can still exhibit interesting behaviors under appropriate switching laws. Such behaviors are similar to those of a conventional stable system near an equilibrium point. In this paper, we introduce some practical stability notions to define such behaviors for switched systems. Such notions are extensions of the traditional practical stability concepts [3, 4].

In this paper, we focus on practical stabilization problems for a simple yet important class of switched systems — integrator switched systems. Many real-world processes including chemical processes [5, 7] can be modeled as such systems. After introducing some practical stability notions, we propose a necessary and sufficient condition for the practical stabilizability of such systems (Theorem 3.1). Additional feasible ways of checking the condition are also proposed. Moreover, for practically stabilizable systems, we develop a minimum dwell time switching law which achieves ε-practically asymptotic stability. The switching law is then applied to a three tank problem in chemical batch process to illustrate its effectiveness.

2 Practical Stability Notions for Switched Systems

2.1 Switched Systems

A *switched system* is a dynamic system which consists of subsystems

$$\dot{x} = f_i(x), \quad f_i : \mathbb{R}^n \to \mathbb{R}^n, \quad i \in I = \{1, \cdots, M\} \tag{2.1}$$

and a switching law orchestrating the active subsystem at each instant. The state trajectory of such a system is determined by the initial state and the timed sequence of active subsystems. A *switching sequence* defined below regulates the timed sequence of active subsystems.

Definition 2.1 (Switching Sequence) *A* switching sequence *σ in $[t_0, t_f]$ is defined as*

$$\sigma = \big((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)\big) \tag{2.2}$$

where $0 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_K \le t_f$, and $i_k \in I$ for $k = 0, 1, \cdots, K$. We also define $\Sigma_{[t_0,t_f]} = \{$switching sequence σ's in $[t_0, t_f]\}$. ☐

σ indicates that subsystem i_k is active in $[t_k, t_{k+1})$ (subsystem i_K in $[t_K, t_f]$). For a switched system to be well-behaved, we only consider **nonZeno** sequences which switch at most a finite number of times in any finite interval $[t_0, t_f]$, though different sequences may have different numbers of switchings. The feature distinguishing a switched system from a general hybrid system is that its continuous state does not exhibit jumps at switching instants.

Switching sequences as defined above are usually generated by switching laws which are defined below.

Definition 2.2 (Switching Law) *For switched system (2.1), a switching law $\mathcal{S}$ is defined to be a mapping $\mathcal{S} : \mathbb{R}^n \times \mathbb{R} \to \bigcup_{t_0} \Sigma_{[t_0,\infty)}$ that specifies a switching sequence $\sigma = \sigma(x_0, t_0) \in \Sigma_{[t_0,\infty)}$ for any initial point x_0 and any initial time t_0.* ☐

Remark 2.1 $\mathcal{S}$ is quite often described by some rules or algorithms rather than mathematical formulae. Such rules or algorithms describe how to generate a switching sequence given x_0 and t_0. In this paper, we will specify our switching laws using such descriptions. ☐

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Remark 2.2 Sometimes we are only interested in the behavior of a switched system in a finite time duration $[t_0, t_f]$. In such cases, we can use the same definition of switching law as the one above but only pay attention to the subsequences of $\sigma(x_0, t_0)$ in $[t_0, t_f]$. □

2.2 Some Practical Stability Notions

In most of the literature results on stability analysis and stabilization of switched systems, it is assumed that a common equilibrium exists for all subsystems. However, this assumption may not be true and may limit the applicability of switched systems. In fact, when subsystems have different equilibria or no equilibrium, a system can still exhibit interesting behaviors around a given point under appropriate switching laws. The behaviors are similar to those of a conventional stable system near an equilibrium. The following example illustrates such behaviors.

Example 2.1 Consider a switched system consisting of: subsystem 1: $\dot{x} = [-3,\ 2.5]^T$; subsystem 2: $\dot{x} = [-2.5,\ -3]^T$; subsystem 3: $\dot{x} = [3,\ -2.5]^T$; subsystem 4: $\dot{x} = [2.5,\ 3]^T$. If we apply the switching rule which makes subsystem 1 active in quadrant I, subsystem 2 active in quadrant II, subsystem 3 active in quadrant III, and subsystem 4 active in quadrant IV, the system will exhibit "convergent behaviors" around the origin. Figure 1 shows a trajectory from $x_0 = [2,\ 1]^T$ under this switching rule in a finite time duration. □

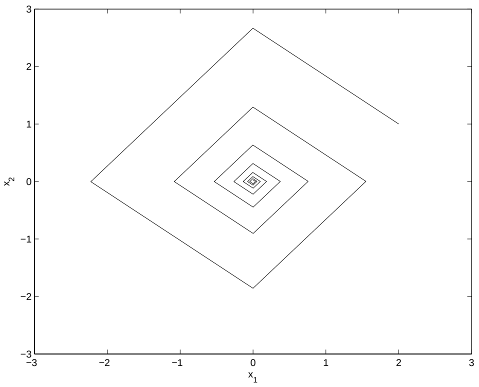

Figure 1: A sample trajectory starting from $x_0 = [2,\ 1]^T$ for Example 2.1.

In Example 2.1, under the given switching rule, the origin exhibits behaviors similar to those of an asymptotically stable system. However, as the trajectory becomes closer and closer to the origin, the system needs to switch faster and faster. This violates the nonZenoness requirement for valid switching sequences. In practice, a lower bound for the time between switchings will usually be imposed that prevents Zenoness. Such a lower bound is called the *minimum dwell time* [2] and its value may be different given different application objectives. If we incorporate a minimum dwell time into the switching rule in Example 2.1, trajectories starting from any point in $\mathbb{R}^2$ will be attracted toward the origin and eventually oscillate near the origin within certain bound.

The concept of bringing the trajectory to be within a given bound is quite useful in practice. For example, in temperature control systems, usually we are more interested in keeping the temperature within certain bounds, rather than stabilizing the system asymptotically to a set-point. In fact, such concept has been formally termed practical stability in [3, 4] for ordinary differential equations. In the following, we adapt and expand some practical stability notions to switched systems and formally define the notion of practical stabilizability. Without loss of generality, we only discuss the case of the origin below.

Definition 2.3 (ε-Practical Stability) *Assume a switching law S is given for the switched system (2.1). Given $\varepsilon > 0$, the system is said to be* ε-practically stable around the origin under the switching law S *if there exists $\delta = \delta(\varepsilon) > 0$ such that $x(t) < \varepsilon$ whenever $x(0) = x_0$ satisfies $\|x_0\| < \delta$.* □

Definition 2.4 (ε-Attractivity) *Assume a switching law S is given for the switched system (2.1). Given $\varepsilon > 0$, the origin is said to be* ε-attractive *if there exists $\eta = \eta(\varepsilon) > 0$ such that for every $x(0) = x_0$ satisfying $\|x_0\| < \eta$, there exists $T = T(x_0) \geq 0$ such that $\|x(t)\| < \varepsilon$ for any $t \geq T$. Moreover, the origin is said to be* globally ε-attractive *if η can be chosen to be ∞.* □

Definition 2.5 (ε-Practically Asymptotic Stability) *Assume a switching law S is given for the switched system (2.1). Given $\varepsilon > 0$, the system is said to be* ε-practically asymptotically stable around the origin under the switching law S *if it is ε-practically stable and the origin is ε-attractive. The system is said to be* globally ε-practically asymptotically stable *if it is ε-practically stable and the origin is globally ε-attractive.* □

Remark 2.3 In the definition of ε-practically asymptotic stability, we require not only ε-attractivity but also ε-practical stability. ε-attractivity does not imply ε-practical stability. It is possible that for any $\delta < \varepsilon$, a trajectory exists that starts at $x(0)$ with $\|x(0)\| < \delta$ and violates $\|x(t)\| < \varepsilon$ for some time and finally settles down with $\|x(t)\| < \varepsilon$. This still satisfies ε-attractivity; however, ε-practical stability is not satisfied. □

Definition 2.6 (Practical Stabilizability) *The switched system (2.1) is said to be* practically stabilizable *if for any $\varepsilon > 0$, there exists a switching law $S = S(\varepsilon)$ such that the system is ε-practically asymptotically stable around the origin under S. It is said to be* globally practically stabilizable *if for any $\varepsilon > 0$, $S = S(\varepsilon)$ exists such that the system is globally ε-practically asymptotically stable around the origin under S.* □

Remark 2.4 In the definition of practical stabilizability, ε can be varied as opposed to the fixed ε in the previous several definitions. Hence a practically stabilizable system has the property that, for any given bounded ball centered at the origin, a valid switching law can be found that drives the system trajectory into the ball and then keeps it in the ball. □

3 Practical Stabilization Results for Integrator Switched Systems

In the sequel, we will focus on a special class of switched systems — integrator switched systems, which consist of subsystems

$$\dot{x} = a_i,\ i \in I = \{1, \cdots, M\} \tag{3.1}$$

where $a_i \in \mathbb{R}^n$ ($a_i \neq 0$), $i \in I$ are constant vectors. Such class of systems receives particular attention due to the following reasons. First, they can model many real world processes, such as batch processes [5, 7]. Second, the simple structure of such systems makes possible rigorous analysis that leads to nice theoretical and practical results. Third, the complete exploration of such systems is the first step toward the study of practical stability of general nonlinear switched systems.

3.1 Results on Practical Stabilizability

For an integrator switched system, practical stabilizability is equivalent to globally practical stabilizability. In the following, with the help of some convex analysis notions and results (see Appendix A of [9]), we propose some necessary and sufficient conditions for globally practical stabilizability of system (3.1). The main result is Theorem 3.1 which provides a necessary and sufficient condition for practical stabilizability. Lemma 3.1 provides a feasible way of verifying the condition in Theorem 3.1. Then Theorem 3.2 and three corollaries are proposed that illustrate some implications of the necessary and sufficient condition and emphasize more on systems with $n+1$ subsystems in $\mathbb{R}^n$. Due to the space limitation, we do not provide the proofs of these results here. Interested readers are referred to Appendix B of [9]) for all the proofs.

Theorem 3.1 (Necessary and Sufficient Condition) *An integrator switched system (3.1) in $\mathbb{R}^n$ is globally practically stabilizable if and only if $C = \mathbb{R}^n$, where C is the convex cone $C = \{\sum_{i=1}^{M} \lambda_i a_i | \lambda_1 \geq 0, \cdots, \lambda_M \geq 0\}$.*

Proof: See Appendix B of [9]. □

In order to apply Theorem 3.1, we need to verify the validity of the condition $C = \mathbb{R}^n$. Exhaustively checking whether $x \in C$ for any $x \in \mathbb{R}^n$ is impossible due to the infinite number of x to check. The lemma below provides a necessary and sufficient condition which is equivalent to $C = \mathbb{R}^n$ and computationally feasible to verify.

Lemma 3.1 *$C = \mathbb{R}^n$ if and only if there exists a subset $\{a_{i_1}, \cdots, a_{i_l}\}$ of $\{a_1, \cdots, a_M\}$ that satisfies the following conditions:*

(a). $\text{span}\{a_{i_1}, \cdots, a_{i_l}\} = \mathbb{R}^n$ *and*

(b). there exist $\lambda_j > 0$, $j = 1, \cdots, l$, such that $\sum_{j=1}^{l} \lambda_j a_{i_j} = 0$.

Proof: See Appendix B of [9]. □

Remark 3.1 Lemma 3.1 provides a feasible way of checking whether $C = \mathbb{R}^n$. By exhaustively checking all possible subsets of $\{a_1, \cdots, a_M\}$ for the validity of conditions (a) and (b), we can determine whether a given system is practically stabilizable. Because there are at most 2^M subsets and we only need to check condition (b) for the 0 point, the computation can be done within finite time and therefore is feasible. □

Furthermore, from Lemma 3.1, we can immediately conclude that the number of subsystems in a globally practically stabilizable system should be no less than $n+1$. The following theorem confirms it.

Theorem 3.2

(a). If an integrator switched system (3.1) in $\mathbb{R}^n$ is globally practically stabilizable, then there are at least $n+1$ subsystems.

(b). Moreover, there exists an integrator switched system consisting of $n+1$ subsystems which is globally practically stabilizable.

Proof: See Appendix B of [9]. □

The case of $n+1$ subsystems that form a globally practically stabilizable system is important, because in many stabilizable systems such $n+1$ subsystems do exist. The following three corollaries related to such case can be inferred from the above theorems and lemma.

Corollary 3.1 *An integrator switched system (3.1) in $\mathbb{R}^n$ consisting $n+1$ subsystems with vector fields $a_1, \cdots, a_{n+1}$ is globally practically stabilizable if and only if* $\text{span}\{a_1, \cdots, a_{n+1}\} = \mathbb{R}^n$ *and there exist $\lambda_i > 0$, $i = 1, \cdots, n+1$ such that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$.*

Proof: See Appendix B of [9]. □

Corollary 3.2 *An integrator switched system (3.1) in $\mathbb{R}^n$ with $n+1$ subsystems with vector fields $a_1, \cdots, a_{n+1}$ is globally practically stabilizable if and only if any n vectors in the set $\{a_1, \cdots, a_{n+1}\}$ are linearly independent and there exist $\lambda_i > 0$, $i = 1, \cdots, n+1$ not all zero such that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$.*

Proof: See Appendix B of [9]. □

In many cases, we can find $n+1$ subsystems which can be used for determining the practical stabilizability of the system. The following corollary provides a sufficient condition for doing so.

Corollary 3.3 (A Sufficient Condition) *An integrator switched system (3.1) in $\mathbb{R}^n$ with M ($M \geq n+1$) subsystems is globally practically stabilizable if there exists a subset of $n+1$ subsystems which, if regarded as a switched system with $n+1$ subsystems, is globally practically stabilizable.*

Proof: See Appendix B of [9]. □

3.2 A Minimum Dwell Time Switching Law

Now we construct a switching law that is easy to implement and makes a system globally ε-practically asymptotically stable if it is determined to be globally practically stabilizable by the conditions proposed above. We will mainly focus on integrator switched systems in $\mathbb{R}^n$ with $n+1$ subsystems that are globally practically stabilizable and propose a valid minimum dwell time switching law. As mentioned in Section 3.1, the case of $n+1$ subsystems is important because in many stabilizable systems such $n+1$ subsystems do exist. Hence the switching law proposed here can actually be applied to many switched systems with more than $n+1$ subsystems.

Let us first illustrate the idea of our switching law by the following example.

Example 3.1 Consider a switched system in $\mathbb{R}^2$ consisting of: subsystem 1: $\dot{x} = a_1 = [1,\ 0.5]^T$; subsystem 2: $\dot{x} = a_2 = [-1,\ 1.5]^T$; subsystem 3: $\dot{x} = a_3 = [-0.5,\ -1]^T$ (see figure 2(a)). It can be verified by Corollary 3.2 that it is globally practically stabilizable.

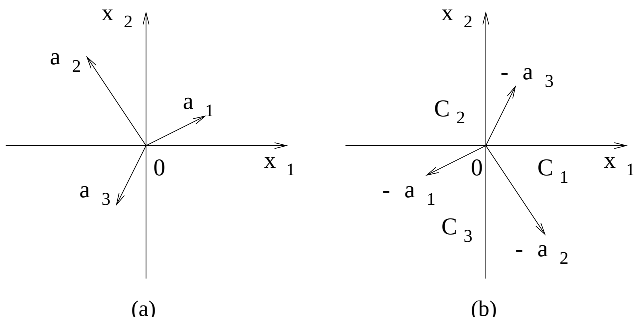

Figure 2: Example 3.1: (a) Vector fields a_1, a_2, a_3. (b) Convex cones C_1, C_2, C_3.

For a given $\varepsilon > 0$, we now develop a valid switching law to achieve ε- practically asymptotic stability. We first denote by C_1 the convex cone generated by the vectors $-a_2, -a_3$; by C_2 the convex cone generated by $-a_1, -a_3$; by C_3 the convex cone generated by $-a_1, -a_2$ (see figure 2(b)). Note that C_1, C_2, and C_3 have mutually disjoint interiors and $C_1 \cup C_2 \cup C_3 = \mathbb{R}^2$.

A minimum dwell time switching law: Let subsystem 2 be active in $\mathrm{Int}(C_1)$, subsystem 3 be active in $\mathrm{Int}(C_2)$, subsystem 1 be active in $\mathrm{Int}(C_3)$. When the state is on the common boundary of any two convex cones, choose the active subsystem to be the one corresponding to the convex cone that the trajectory has the potential to enter next, if the system still evolves according to the current active subsystem. For example, if x evolves in C_1 (following subsystem 2) and intersects the ray in the same direction as $-a_3$, then subsystem 3 will become active. Moreover, in order to eliminate the Zenoness phenomenon near the origin, besides the above rules, we also impose a minimum dwell time τ (i.e., the minimum time duration that any subsystem must be active before the system can switch again).

The choice of a minimum dwell time τ: In general, the smaller the τ is, the smaller the ε can be, so that the system can be made ε-practically asymptotically stable. As $\tau \to 0$, we find that ε can also go to 0. However when $\tau = 0$, Zenoness problem will occur, therefore τ cannot be infinitely small either. For this example in $\mathbb{R}^2$, some geometric observations suggest that we can choose a τ satisfying the following inequality

$$\begin{aligned}\tau \leq \min\Big\{ & \tfrac{1}{\|a_1\|}\big(\varepsilon - \tfrac{\delta}{\sin\theta_{12}}\big), \tfrac{1}{\|a_2\|}\big(\varepsilon - \tfrac{\delta}{\sin\theta_{23}}\big), \\ & \tfrac{1}{\|a_3\|}\big(\varepsilon - \tfrac{\delta}{\sin\theta_{31}}\big), \tfrac{\delta}{\|a_1\|}, \tfrac{\delta}{\|a_2\|}, \tfrac{\delta}{\|a_3\|}\Big\},\end{aligned} \tag{3.2}$$

where θ_{12} is the angle extended by l_1 and l_2 ($0 < \theta_{12} < \pi$). Similar definitions apply for θ_{23} and θ_{31}. The δ in (3.2) corresponds to the δ in Definition 2.3 and can be chosen to be a value that satisfies

$$\delta < \min\{\varepsilon \sin\theta_{12}, \varepsilon \sin\theta_{23}, \varepsilon \sin\theta_{31}\}. \tag{3.3}$$

Note that the details of the derivation of (3.2) and (3.3) are given in Appendix.

Equipped with the switching law and (3.2), we return to our example. We choose $\varepsilon = 0.3$, and $\delta = 0.1$ which satisfies (3.3), it can then be determined from (3.2) that $\tau \leq 0.0555$ will lead to a valid switching law that makes the system ε-practically asymptotically stable. Figure 3 shows a trajectory starting from $[1,\ 1]^T$ with $\tau = 0.05$. Figure 4 shows $x_1(t)$ and $x_2(t)$. Note that when time becomes large, the maximum deviation from 0 is -0.1165 for x_1, and 0.075 for x_2. So the state is actually within a ball with radius 0.1386 which is smaller than 0.3. The requirement is therefore satisfied. □

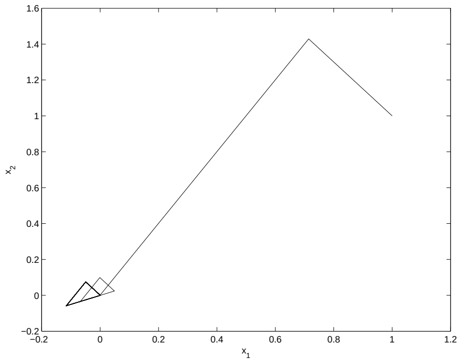

Figure 3: Example 3.1: A trajectory starting from $[1,\ 1]^T$.

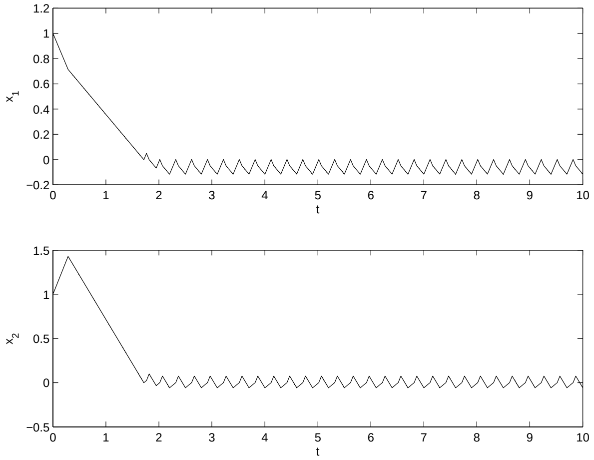

Figure 4: Example 3.1: $x_1(t)$ and $x_2(t)$.

The switching law proposed in Example 3.1 can be extended to the case of practically stabilizable systems in $\mathbb{R}^n$ with $n+1$ subsystems. We denote by C_k the convex cone generated by the vectors $-a_1, \cdots, -a_{k-1}, -a_{k+1}, \cdots, -a_{n+1}$ (if $k = n+1$ then regard subsystem 1 as subsystem $k+1$) for all $1 \leq k \leq n+1$. It can be shown that $C_1, \cdots, C_{n+1}$ have mutually disjoint interiors and $C_1 \cup \cdots \cup C_{n+1} = \mathbb{R}^n$.

A minimum dwell time switching law: Let subsystem $k+1$ be active whenever the state is in $\text{Int}(C_k)$, $1 \leq k \leq n+1$. When the state is on the common boundary of convex cones, we choose the active subsystem to be the one corresponding to the convex cone that the trajectory has the potential to enter next, if the system still evolves according to the current active subsystem. In order to eliminate the Zenoness phenomenon near the origin, besides the above rules, we also impose a minimum dwell time τ.

The choice of a minimum dwell time τ*:* For systems in $\mathbb{R}^n$, geometric observations as those in Example 3.1 are currently still under research, since direct extensions of results $\mathbb{R}^2$ into $\mathbb{R}^n$ are not readily available. However, given any $\varepsilon > 0$, the above switching law will make the system ε-practically asymptotically stable if τ is chosen to be small enough. In practice, we usually specify an ε and then reduce the value of τ and test the resulting trajectory until the ε-practically asymptotic stability is achieved.

4 A Three Tank Example

Now we apply the results developed in Section 3 to a chemical batch process example.

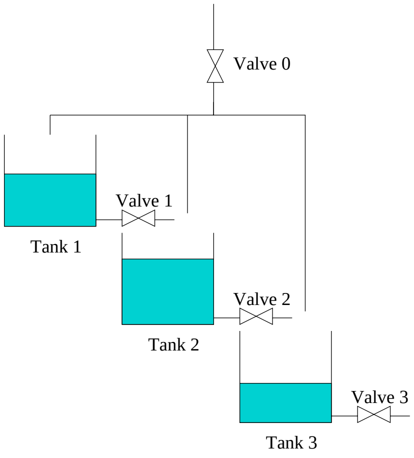

Figure 5: Example 4.1: The three tanks system.

Example 4.1 (A Three Tank Example) Consider the three tanks system in figure 5. All tanks are identical and all flows cause the tank-levels to rise or decrease by 0.1 unit/sec. There are four allowable operating modes: Mode 1: Valve 0 on, Valves 1,2,3 off, the corresponding dynamics $\dot{x} = a_1 = [0.1,\ 0.1,\ 0.1]^T$; Mode 2: Valve 1 on, Valves 0,2,3 off, the dynamics $\dot{x} = a_2 = [-0.1,\ 0.1,\ 0]^T$; Mode 3: Valve 2 on, Valves 0,1,3 off, the dynamics $\dot{x} = a_3 = [0,\ -0.1,\ 0.1]^T$; Mode 4: Valve 3 on, Valves 0,1,2 off, the dynamics $\dot{x} = a_4 = [0,\ 0,\ -0.1]^T$. We want to develop a switching law such that the water levels in the tanks are driven toward the desired value $[80,\ 50,\ 70]^T$ and each tank level is then kept within $[-2,+2]$ range around the desired level.

Using Corollary 3.2, we can show this system with 4 subsystems in $\mathbb{R}^3$ is practically stabilizable. We can choose $\varepsilon = 2$ and apply the switching law proposed in Section 3.2 to make the system ε-practically asymptotically stable around the point $[80,\ 50,\ 70]^T$ (although the point is not the origin, but with state shift, the stabilization result can be applied). We choose $\tau = 5$ sec. Figure 6 shows the three tank levels starting from $[90,\ 45,\ 75]^T$. When time becomes large, the maximum deviations from the desired point are 0.4999 for x_1, 0.9999 for x_2, and 1.4999 for x_3. They satisfy the requirements. $\square$

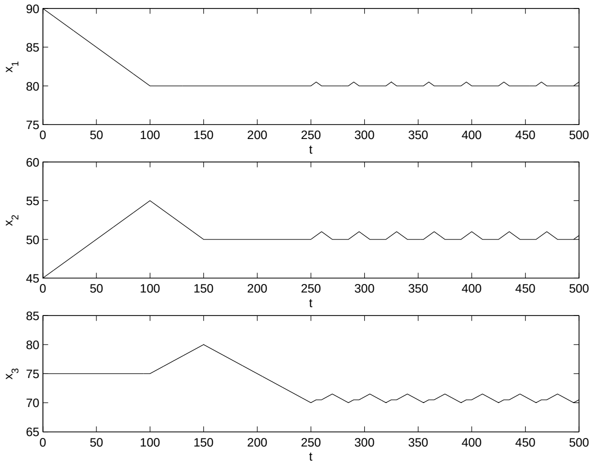

Figure 6: Example 4.1: The three tank levels starting from $[90,\ 45,\ 75]^T$.

5 Conclusion

This paper reports some results for practical stabilization problems of integrator switched systems. Some practical stability notions are introduced, and a necessary and sufficient condition for practical stabilizability of integrator switched systems is then given. Moreover, a minimum dwell time switching law for practically stabilizable systems in $\mathbb{R}^n$ consisting of $n+1$ subsystems is proposed that can be used to achieve ε-practically asymptotic stability. A more detailed version of the paper can be found in [9]. The research in this paper is a first step toward the studies of general nonlinear subsystems. Future research includes the estimation of bound for minimum dwell time for systems in $\mathbb{R}^n$, and extensions of the results to switched systems with nonintegrator subsystems.

Appendix: Some Geometric Observations for Choosing τ in Example 3.1

Given an $\varepsilon > 0$, we can choose τ based on the following reasonings. Figure 7 helps our reasonings below.

First, consider ε-practical stability. From Definition 2.3, we need to have a closed ball $B[0;\delta]$ such that any trajectory starting in this ball will remain in the open ball $B(0;\varepsilon)$. Figure 7 depicts the two balls. l_i's are the rays corresponding to $-a_i$'s. Assume that subsystem 3 is active and the state is in $B[0,\delta] \cap C_2$. Also assume that the points P_1, Q_1 are on the line tangent to $B[0;\delta]$ and parallel to l_3, P_1 is on l_1, and $|P_1Q_1| = \|a_3\|\tau$. Moreover, assume the points 0, P_1, Q_1, and R_1 form a parallelogram. Now we note that for

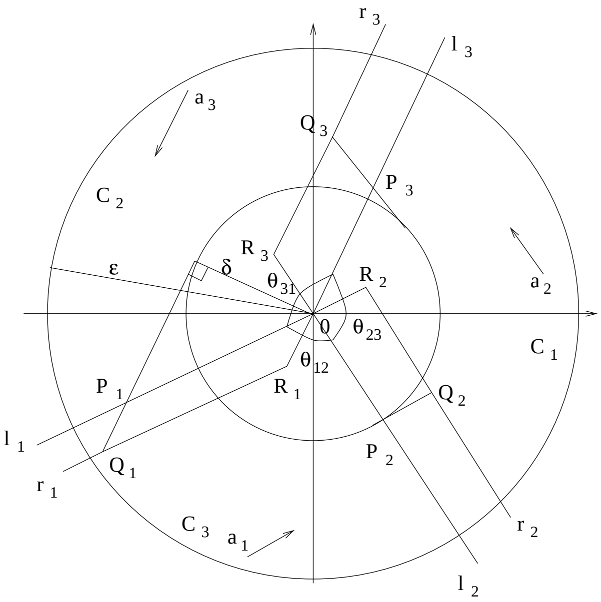

Figure 7: Geometric observations for choosing τ for Example 3.1.

any point in $B[0;\delta]\cap C_2$, when subsystem 3 is active and the system follows the minimum dwell time switching law proposed in Example 3.1, the trajectory will either intersect l_1 and switch to subsystem 1 immediately (when the time elapsed is no less than τ), or it will enter $0P_1Q_1R_1$ and then switch to subsystem 1 (when time elapsed is equal to τ). The importance of $0P_1Q_1R_1$ lies in the fact that for trajectories starting from $B[0;\delta]\cap C_2$ and following subsystem 3, all trajectories will switch to subsystem 1 in $0P_1Q_1R_1$. As long as the line segment $0R_1$ is in $B[0;\delta]$, by following subsystem 1, the trajectory will eventually intersect OR_1 and hence be in $B[0;\delta]\cap C_3$. Similar arguments can be applied to show that the trajectories starting in $B[0;\delta]\cap C_3$ will switch in the parallelogram $0P_2Q_2R_2$ and then enter into $B[0;\delta]\cap C_1$; and the trajectories starting in $B[0;\delta]\cap C_1$ will switch in the parallelogram $0P_3Q_3R_3$ and then enter into $B[0;\delta]\cap C_2$. Now in order to achieve ε-practical stability, a sufficient condition is to require that the farthest point Q_1 of the parallelogram $0P_1Q_1R_1$ be inside $B(0;\varepsilon)$. A sufficient condition for this is $|0P_1|+|P_1Q_1|\le\varepsilon$ which can also be written as

$$||a_3||\tau+\frac{\delta}{\sin\theta_{31}}\le\varepsilon, \tag{A.1}$$

where θ_{31} is the angle extended by l_3 and l_1 ($0<\theta_{31}<\pi$). Also note from our above discussion, we require that $0R_1$ be in $B[0;\delta]$, which is equivalent to

$$||a_3||\tau\le\delta. \tag{A.2}$$

Similarly, we can obtain the inequalities

$$\begin{aligned} ||a_1||\tau+\frac{\delta}{\sin\theta_{12}} &\le \varepsilon, & \text{(A.3)}\\ ||a_1||\tau &\le \delta, & \text{(A.4)}\\ ||a_2||\tau+\frac{\delta}{\sin\theta_{23}} &\le \varepsilon, & \text{(A.5)}\\ ||a_2||\tau &\le \delta. & \text{(A.6)} \end{aligned}$$

From (A.1)-(A.6), we find that if we choose

$$\tau\le\min\Big\{\tfrac{1}{||a_1||}(\varepsilon-\tfrac{\delta}{\sin\theta_{12}}),\tfrac{1}{||a_2||}(\varepsilon-\tfrac{\delta}{\sin\theta_{23}}), \tfrac{1}{||a_3||}(\varepsilon-\tfrac{\delta}{\sin\theta_{31}}),\tfrac{\delta}{||a_1||},\tfrac{\delta}{||a_2||},\tfrac{\delta}{||a_3||}\Big\}, \tag{A.7}$$

then the switching law with τ satisfying (A.7) will lead to ε-practical stability. Note that the δ in (A.7) corresponds to the δ in Definition 2.3 and can be chosen by the designer, however it must satisfy the following condition

$$\delta<\min\{\varepsilon\sin\theta_{12},\varepsilon\sin\theta_{23},\varepsilon\sin\theta_{31}\}, \tag{A.8}$$

so that the τ in (A.1), (A.3), (A.4) can take positive value. Besides the constraint (A.8), we can freely choose δ to achieve different bounds for τ.

We claim that the switching law in Example 3.1 with τ satisfying (A.7) also achieves ε-attractiveness. This is because any trajectory starting in C_2 following subsystem 3 will enter into the band formed by l_1, $0R_1$, and the ray r_1 which emits from R_1 and is in the direction of R_1Q_1 (see figure 7) and then switch to subsystem 1. Therefore, after one switching from subsystem 3 to 1, all trajectories starting in C_2 can then intersect $0R_1$, which is in $B[0;\delta]$. Then by the above arguments for ε-practical stability, the trajectory will always be in $B(0,\varepsilon)$.

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